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Nucleon-Nucleon Scattering from One-Boson-Exchange Potentials. II. Inclusion of all Momentum-Dependent Terms through Order p^2*

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The nucleon-nucleon interaction in relative angular momentum states $l \geq 1$ is described over the energy range 25–350 MeV by a potential used in conjunction with the Schrödinger equation. The potential is a superposition of pole terms obtained from single exchanges of $\omega, \rho, \pi, \eta, \sigma_0$, and σ_1 mesons and, rather than taking the usual static limit, all terms of order k^2/M^2 are retained. Because of these momentum-dependent terms, no cutoff is necessary in $l \geq 1$ states. The meson coupling constants and the masses of the scalar mesons (σ_0 and σ_1) are the parameters which are adjusted to fit the experimentally determined phase parameters. A comparison with the data is given, showing a quantitative fit (X^2/datum$=1.6$) when the model is augmented with reasonable values of the S-wave parameters.

I. INTRODUCTION

THIS is one of a series of papers treating the S matrix for the 0- to 350-MeV nucleon-nucleon interaction in terms of ω, ρ, π, σ_0, and σ_1 poles in the cross channel, suitably unitarized by calling the sum a potential, inserting the Fourier transform thereof into the Schrödinger equation, and solving for the phase shifts. These phase shifts are compared with the experimental phase shifts found by Arndt and MacGregor, partial wave by partial wave, and the pole parameters are adjusted until a best fit is obtained. A comparison with the scattering data is also made. The pole parameters which we adjust are the nucleon-antinucleon meson coupling constants g_ω, f_ω, g_ρ, f_ρ (where g and f signify Dirac and Pauli coupling, respectively), $g_\eta^2, g_\pi^2, g_{\sigma_0}^2$, and $g_{\sigma_1}^2$. The σ_0 and σ_1 have quantum numbers T, $J^P=0$, 0^+ and $1,0^+$. These are unfounded experimentally, at least at present, but their presence is crucial to the pole fits. The masses m_{σ_0} and m_{σ_1} are searched along with the coupling constants.

In an earlier paper, which we shall call Paper I,[1] a set of pole parameters was found which fit the experimental phase shifts reasonably well for states of orbital angular momentum $l \geq 1$. It was found, however, that the potential had to be cut off at short distances to eliminate an r^{-3} divergence. This problem was treated by setting the potential equal to zero everywhere within 0.6 F. In that same work one term of order p^2 was neglected in the potential, for convenience in numerical solution. This term occurred in the central (non-spin-dependent) part of the scalar and vector one-boson potentials, and was expected to be compensated for by a slight change in the coupling constants. In the present work we included the p^2 term and arrived at a good fit to the data with about the same pole parameters, so apparently the neglect of the p^2 term in Paper I was indeed not too critical. However, we found in the present work that by including the p^2 term, cutoff was no longer required; the p^2 term is positive (repulsive) in both N-N isotopic spin

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[1] R. A. Bryan and B. L. Scott, Phys. Rev. 135, B434 (1964). The equations for R_C, R_{12}, and R_{LS} given in Sec. II of that paper are incorrect. They should read, in the notation of that paper,

$$R_C(f/g)=1+\mu(1+\tfrac{1}{2}\mu^2)^{-1}(f/g),$$
$$R_{12}(f/g)=[1+(2/\mu)(f/g)]^2,$$
$$R_{LS}(f/g)=1+(8/3\mu)(f/g),$$

where $\mu=mv/M$. Fortunately, the numerical error is slight and our conclusions remain valid. These remarks apply also to R. A. Bryan, C. R. Dismukes, and W. Ramsay, Ref. 5.

states and reduces the degree of singularity in the potential enough for the Schrödinger equation to be solved.

Extensive reference to other pole models in the literature has been given in Paper I and, more recently, in a review article by one of us (R.A.B.).[2] Particular reference to dispersion-theory treatments may also be found in articles by Arndt, MacGregor, and Bryan.[3] For the most part we have not attempted to include higher-symmetry schemes in this list because of the tentative nature of these proposals. However, one such scheme seems appropriate to mention, because it is so closely related to all pole models. This is the five-vector model of Green and co-workers,[4] wherein the ω and σ_0 mesonic fields are considered components of a single field with five elements. It couples to the nucleon as follows; $\mathcal{L}^{\text{int}} \sim g\bar{\psi}\gamma_\nu\omega_\nu\psi$, $\nu=1, 2, 3, 4, 5$, with $\omega_5=\sigma_0$. The interesting point here is that with this Lagrangian the leading term in the ω and σ_0 pole-term contributions automatically cancel; in all lower symmetry models the σ_0 has to be introduced explicitly to bring about this cancellation. In practice, Green finds that he must split the ω and σ_0 masses to fit the data, in agreement with our own findings.

II. CALCULATIONS

A. One-Boson-Exchange Potentials (OBEP)

The one-boson-exchange potentials used in our calculations are defined such that the potential, for a given meson exchange, is taken to be equal, in momentum space, to the exact relativistic one-meson-exchange contribution to the S matrix in field theory. To determine the potential in coordinate space, one takes the Fourier transform of the momentum-space expression. The above definition has been given explicitly in Ref. 5 and elsewhere.[6] A somewhat different definition of the potential is given by Wong.[7] We list below the OBEP for scalar, vector, and pseudoscalar meson exchange, and the interaction Lagrangians which define them. Note that a definition of f different from that of Ref. 1 is used here.

These potentials are actually given here only through order k^2/M^2, where k is the magnitude of the 3-momentum of any nucleon. However, the error incurred by dropping terms of order k^4/M^4 and higher is small in most cases, of the order of a few percent. The only cases where the error is higher are those of the familiar one-pion-exchange potential (error may reach 15% at 300 MeV) and the f^2 term of the one-ρ meson-exchange potential. The expressions given below agree with the (corrected) OBEP listed by Hoshizaki, Lin, and Machida[6] to order k^2/M^2. (Note that the tensor and spin-orbit potentials have no terms of order lower than k^2 in momentum space.)

1. Vector Meson

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}\bar{\Psi}[g\gamma^\mu\phi_\mu^{(V)}+(f/4M)\sigma^{\mu\nu}(\partial_\nu\phi_\mu^{(V)}-\partial_\mu\phi_\nu^{(V)})]\Psi\,,$$

$$V^{(V)}=g^2\frac{e^{-mr}}{r}+(g^2+gf)\frac{m^2}{2M^2}\frac{e^{-mr}}{r}-g^2\frac{1}{2M^2}\left(\nabla^2\frac{e^{-mr}}{r}+\frac{e^{-mr}}{r}\nabla^2\right)\quad\text{(new } p^2 \text{ term)}$$
$$-(g+f)^2\frac{m^2}{4M^2}\left(\frac{1}{3}+\frac{1}{mr}+\frac{1}{m^2r^2}\right)\frac{e^{-mr}}{r}S_{12}+(g+f)^2\frac{m^2}{6M^2}\frac{e^{-mr}}{r}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+(3g^2+4gf)\frac{1}{2M^2}\frac{1}{r}\frac{d}{dr}\left(\frac{e^{-mr}}{r}\right)\mathbf{L}\cdot\mathbf{S}$$
$$-(g^2+gf)\frac{4\pi}{2M^2}\delta^{(3)}(\mathbf{r})-(g+f)^2\frac{4\pi}{6M^2}\delta^{(3)}(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,.$$

2. Scalar Meson

$$\mathcal{L}^{\text{int}}=(4\pi)^{1/2}g\bar{\Psi}\Psi\phi^{(S)}\,,$$

$$V^{(S)}=g^2\left[\left(-1+\frac{m^2}{4M^2}\right)\frac{e^{-mr}}{r}-\frac{1}{2M^2}\times\left(\nabla^2\frac{e^{-mr}}{r}+\frac{e^{-mr}}{r}\nabla^2\right)\quad\text{(new } p^2 \text{ term)}\right.$$
$$\left.+\frac{1}{2M^2}\frac{1}{r}\frac{d}{dr}\left(\frac{e^{-mr}}{r}\right)\mathbf{L}\cdot\mathbf{S}-\frac{4\pi}{4M^2}\delta^{(3)}(\mathbf{r})\right].$$

[2] R. A. Bryan, in *Proceedings of the International Conference on Nuclear Physics Gatlinburg, Tennessee, 1966* (Academic Press Inc., New York, 1967).

[3] R. A. Bryan and R. A. Arndt, Phys. Rev. **150**, 1299 (1966); see also R. A. Arndt, R. A. Bryan, and M. H. MacGregor, *ibid.* **152**, 1490 (1966); Phys. Letters **21**, 314 (1966).

[4] A. S. Green, T. Sawada, and R. D. Sharma, *Iosbaric Spin in Nuclear Physics* (Academic Press Inc., New York, 1966); A. E. S. Green and R. D. Sharma, Phys. Rev. Letters **14**, 380 (1965), and references therein.

[5] R. A. Bryan, C. R. Dismukes, and W. Ramsay, Nucl. Phys. **45**, 353 (1963).

[6] N. Hoshizake, I. Lin, and S. Machida, Progr. Theoret. Phys. (Kyoto) **26**, 680 (1961).

[7] D. Y. Wong, Nucl. Phys. **55**, 212 (1964). Wong's definition of the potential requires that the first Born approximation for the scattering amplitude be equal to the field-theoretic single-particle-exchange result, while the definition used here and previously (Refs. 1, 5, and 6) equates the Born approximation for the S matrix of the potential to the field-theory result. The difference between the two is a purely kinematical factor of M/E. We prefer the definition used in our paper, because we do not believe that this purely kinematical factor should participate in the complex unitarization performed by the Schrödinger equation. However, our own calculations show that the differences in the coupling constants occasioned by the two definitions are small, perhaps surprisingly so.

3. Pseudoscalar Meson

$$\mathcal{L}^{\rm int}=(4\pi)^{1/2}g\bar{\Psi}\gamma^5\Psi\phi^{(P)},$$

$$V^{(P)}=g^2\left[\frac{m^2}{12M^2}\frac{e^{-mr}}{r}\sigma_1\cdot\sigma_2+\frac{m^2}{4M^2}\left(\frac{1}{m^2r^2}+\frac{1}{mr}+\frac{1}{3}\right)\times\frac{e^{-mr}}{r}S_{12}-\frac{4\pi}{12M^2}\delta^{(3)}(\mathbf{r})\sigma_1\cdot\sigma_2\right].$$

In these equations, m is the mass of the meson in question. Other symbols are defined as in I. As is usual we have set $\hbar=1=c$.

B. Schrödinger-Equation Solution

One sums the OBEP due to the several mesons and inserts this into the Schrödinger equation. Thus

$$V=\sum_\nu V^{(\nu)},\qquad \nu=\rho,\,\omega,\,\pi,\,\eta,\,\sigma_0,\,\sigma_1,\qquad(2.1)$$

and

$$-(1/M)\nabla^2\psi+V\psi=(k^2/M)\psi.$$

To eliminate the ∇^2 derivative in the potential one may conveniently use a transformation on the wave function suggested by Green.[8] Set

$$V(r)=V_0(r)-(1/M)[\nabla^2\phi(r)+\phi(r)\nabla^2],$$

where V_0 contains all the terms except the momentum-dependent term; ϕ can be determined by comparison with the potentials. First we shall consider only the uncoupled states. Let

$$\psi=r^{-1}u_{lsj}(r)\mathcal{Y}_{lsj}(\hat{r},\sigma),$$

where $\mathcal{Y}_{lsj}$ is the generalized spherical harmonic for spin, orbital, and total angular momentum s, l, and j, respectively. Then the radial wave function satisfies the equation

$$(1+2\phi)u_l''+2\phi'u_l'+[k^2-MV_{0,l}-(1+2\phi)l(l+1)r^{-2}+\phi'']u_l=0,$$

where the j and s indices have been suppressed for simplicity.

$V_{0,l}$ stands for V_0 properly evaluated for the angular-momentum state in question. Now if we set

$$u_l=(1+2\phi)^{-1/2}v_l,$$

then v_l obeys an ordinary radial equation

$$v_l''-l(l+1)r^{-2}v_l+k^2v_l=MW_lv_l,\qquad(2.2)$$

where

$$W_l=\frac{V_{0,l}}{1+2\phi}-\left(\frac{\phi'}{1+2\phi}\right)^2\frac{1}{M}+\frac{2\phi}{1+2\phi}\frac{k^2}{M}.$$

Furthermore, one may use v_l directly to obtain the scattering phase shift, since this is deduced by matching the wave function to the free particle solution at some distance well beyond the range of forces, and at that radius $v_l=u_l$ since ϕ approaches zero asymptotically.

The coupled partial differential equations can be handled through a straightforward generalization of the foregoing. Thus, for total angular momentum j, Eqs. (2.2) and (2.3) generalize to

$$\frac{1}{M}\begin{bmatrix}\dfrac{d^2}{dr^2}-\dfrac{(j-1)j}{r^2}+k^2 & 0\\ 0 & \dfrac{d^2}{dr^2}-\dfrac{(j+1)(j+2)}{r^2}+k^2\end{bmatrix}\begin{pmatrix}v_{j-1}\\ v_{j+1}\end{pmatrix}=\frac{1}{1+2\phi}\begin{pmatrix}V_{0,\,j-1} & V_{0,T}\\ V_{0,T} & V_{0,\,j+1}\end{pmatrix}\begin{pmatrix}v_{j-1}\\ v_{j+1}\end{pmatrix}+\left[\frac{2\phi}{1+2\phi}\frac{k^2}{M}-\left(\frac{\phi'}{1+2\phi}\right)^2\frac{1}{M}\right]\begin{pmatrix}v_{j-1}\\ v_{j+1}\end{pmatrix},$$

where $V_{0,T}$ is entirely given by the S_{12} contributions of the potential. The spin and total angular momentum symbols s and j have been suppressed for simplicity.

III. EMPIRICAL FIT TO THE 25- TO 350-MeV N-N DATA

We adjusted the masses of the proposed scalar mesons and all meson-nucleon coupling constants until we found a best fit to the experimental P- and D-wave phase shifts. We did not attempt to fit the $^1S_0$ and $^3S_1$ phase shifts, nor ϵ_1, because these parameters depend strongly on the nature of the potential at distances of less than 1 F, and we thought that the one-boson-exchange model could not reasonably be extended to such short distances without introducing additional phenomenological parameters.

Phase shifts were calculated by solving the Schrödinger equation for each set of pole parameters and matching to the experimental phase shifts at three energies (50, 142, and 310 MeV). A Minneapolis-Honeywell 800 medium-speed computer was used for the calculations. Because the Schrödinger-equation solutions were time-consuming, we did not attempt to fit F and higher wave phase shifts. However, it had been our experience that any pole model which fit the P and D waves and had a reasonable value for g_π^2, fit the F and higher waves as well. This turned out to be true.

A momentum cutoff was not needed, because the p^2-dependent term in the potential reduced the singularities in the equivalent potentials, W_l [Eq. (2.2)], to order r^{-2} at the origin, and in those l states where the singularities were negative (attractive), they did not exceed the centrifugal barrier in magnitude.

The pole parameters were searched for by minimiz-

[8] A. M. Green, Nucl. Phys. 33, 218 (1963).

TABLE I. Meson-nucleon coupling constants and meson masses which yield the fit to experimental phase shifts graphed in Fig. 1, and the fit to the experimental scattering measurements graphed in Fig. 2. The quantities contained within parentheses were not searched for, but rather fixed beforehand.

Meson	T,J^P	Mass (MeV)	g^2	f/g
π	$1,0^-$	(138.7)	12.5	...
η	$0,0^-$	(548.7)	10.6	...
σ_1	$1,0^+$	770	5.8	...
σ_0	$0,0^+$	590	9.9	...
ρ	$1,1^-$	(763)	1.36	3.82
ω	$0,1^-$	(782.8)	19.1	0.0

ing the quantity

$$\sum \frac{[\delta_i-\delta_i^{(\text{expt})}]^2}{[\Delta\delta_i^{(\text{expt})}]^2},$$

where the δ_i are theoretical phase shifts, the $\delta_i^{(\text{expt})}$ experimental phase shifts, and the $\Delta\delta_i^{(\text{expt})}$ errors quoted for the experimental phase shifts; i indexes both the type of phase shift and the energy. For the experimental values we took the phase-shift solutions of Arndt and MacGregor[9] at 50, 142, and 310 MeV, for $l=1$ and 2 states. Arndt and MacGregor also supply solutions at 25, 95, and 210 MeV, but these were not included in the search because of lack of machine time.

The pole parameters were varied two at a time. After a large number of trials we arrived at the set listed in Table I. These yield a pretty good fit to the phase shifts, as may be seen from Fig. 1. In this figure the theoretical phase shifts are plotted as solid lines. The experimental phase shifts are plotted as error bars. The predicted F phase shifts (not shown) also agree well with the Arndt-MacGregor phase shifts. These and the P- and D-state phase shifts are listed in Table II.

Of course, fitting experimental phase shifts is not the same thing as fitting the experimental scattering observables, since, in the former case, contributions to χ^2 arising from correlations between the phase shifts are neglected. We were therefore interested in testing the theoretical phase shifts against the data. To this end we used a code (MIDPOP) available at Livermore which computes the scattering observables, σ, P, D, A, R, etc., from phase shifts and then computes χ^2 through comparison with data. If θ_j and $\theta_j^{(\text{expt})}$ are the theoretical and experimental observables, and $\Delta\theta_j^{(\text{expt})}$ are the experimental errors, then

$$\chi^2=\sum_j \frac{[\theta_j-\theta_j^{(\text{expt})}]^2}{[\Delta\theta_j^{(\text{expt})}]^2}. \tag{3.1}$$

The index j refers to both energy and angle.

Since $\delta(^1S_0)$, $\delta(^3S_1)$, and ϵ_1 were not provided by the potential model, these were searched for. (The code also provides this facility.) In addition, the normalization χ of the observables at each energy was treated as a datum with an experimentally determined error. The code then supplied the values of χ, $\delta(^1S_0)$, $\delta(^3S_1)$, and ϵ_1, which minimizes χ^2. These values of χ are given on the graphs of Fig. 2. All the higher partial wave phase shifts were computed theoretically. P- through F-phase shifts were computed from the potential model. The $l=4$ and higher phase shifts were determined from the relativistic one-pion-exchange term alone ($g_\pi^2=13$). For experiment we used the 704 data selected by Arndt and MacGregor[9] in the energy intervals near 25, 50, 95, 142, 210, and 330 MeV. The calculated χ^2 for the model was 1150, or 1.6 per datum. A representative sampling of the fits to the data is given in Fig. 2.

TABLE II. Nuclear bar phase shifts predicted by the one-boson-exchange-potential model using the meson parameters listed in Table I, in degrees.

Phase shift \ T_{lab} (MeV)	25	50	95	142	210	310
$\delta(^1P_1)$	−6.31	−9.96	−14.89	−19.46	−25.62	−33.95
$\delta(^3P_0)$	8.07	11.30	10.25	6.06	−1.25	−11.92
$\delta(^3P_1)$	−4.78	−8.30	−13.07	−17.27	−22.79	−30.25
$\delta(^3P_2)$	2.25	5.52	10.55	13.79	15.63	15.00
ϵ_2	−0.72	−1.62	−2.63	−3.07	−3.07	−2.49
$\delta(^1D_2)$	0.57	1.36	2.89	4.52	6.64	8.62
$\delta(^3D_1)$	−2.61	−6.27	−11.51	−15.42	−19.28	−22.57
$\delta(^3D_2)$	3.32	8.01	15.16	20.28	24.38	26.07
$\delta(^3D_3)$	−0.01	0.06	0.50	1.07	1.63	1.53
ϵ_3	0.50	1.51	3.08	4.35	5.58	6.55
$\delta(^1F_3)$	−0.38	−1.03	−1.99	−2.71	−3.51	−4.63
$\delta(^3F_2)$	0.09	0.29	0.64	0.90	1.01	0.49
$\delta(^3F_3)$	−0.20	−0.62	−1.40	−2.12	−3.06	−4.42
$\delta(^3F_4)$	0.01	0.08	0.31	0.67	1.32	2.36
ϵ_4	−0.04	−0.17	−0.46	−0.75	−1.12	−1.54
$\delta(^3G_3)$	−0.05	−0.22	−0.79	−1.53	−2.65	−4.20
$\delta(^3H_4)$	0.01	0.02	0.08	0.17	0.30	0.46

[9] R. A. Arndt and M. H. MacGregor (private communication). For a somewhat more recent version of these phase-shift solutions, see R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).

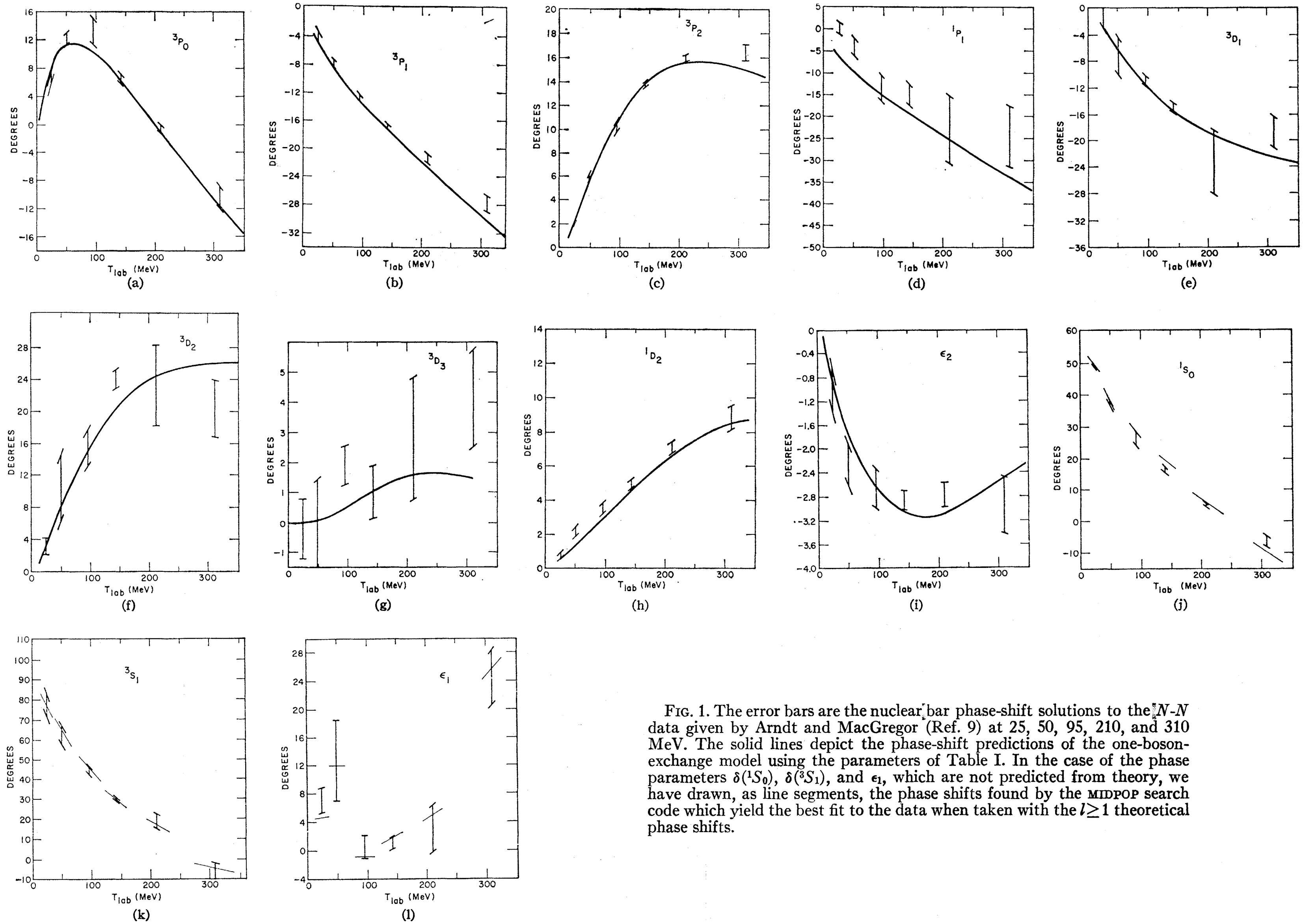

FIG. 1. The error bars are the nuclear-bar phase-shift solutions to the N-N data given by Arndt and MacGregor (Ref. 9) at 25, 50, 95, 210, and 310 MeV. The solid lines depict the phase-shift predictions of the one-boson-exchange model using the parameters of Table I. In the case of the phase parameters $\delta(^1S_0)$, $\delta(^3S_1)$, and ϵ_1, which are not predicted from theory, we have drawn, as line segments, the phase shifts found by the MIDPOP search code which yield the best fit to the data when taken with the $l \geq 1$ theoretical phase shifts.

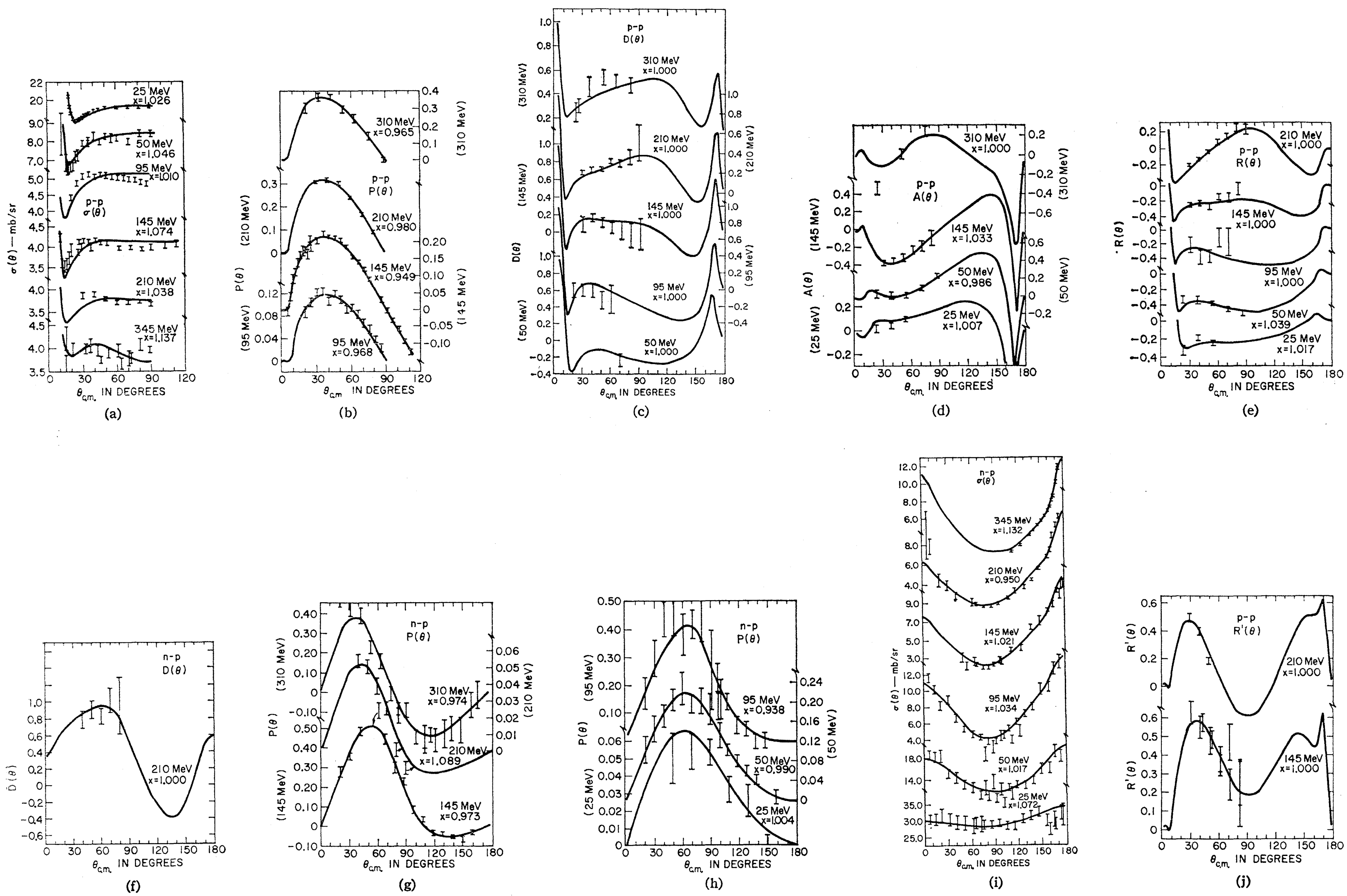

FIG. 2. Some of the p-p and n-p scattering data selected by Arndt and MacGregor over the 25–350 MeV energy range, and theoretical predictions (solid lines) for these data. $T_{\rm lab}$ is indicated on each plot. The MIDPOP code automatically renormalizes the data with this renormalization factor treated as an independent datum. The factor by which the published data have been multiplied is denoted by x, e.g., $x=1.026$, etc.

TABLE III. The $\delta(^1S_0)$, $\delta(^3S_1)$, and ϵ_1 nuclear bar phase parameters which, when used with the potential model phase shifts of Table II, and OPEC$(g_\pi^2=13)$ for all higher partial waves, minimize χ^2 for the 704 p-p and n-p data selected by Arndt and MacGregor.[a]

Phase shift (Degrees) \ $T_{\rm lab}$ (MeV)	25	50	95	142	210	310
$\delta(^1S_0)$	49.13	38.10	28.42	18.92	6.26	−9.96
$\delta(^3S_1)$	76.32	64.43	46.06	29.06	16.82	−5.37
ϵ_1	4.68	11.77	−2.93	1.98	4.81	25.12

[a] See Ref. 9.

The χ^2 fit to the data is better than we had expected from just fitting phase shifts. It is more representative of what one might have expected from adjusting the constants to fit the data. The plots of the fits to the data look quite acceptable over the full-energy range, 25–350 MeV.

The values of $\delta(^1S_0)$, $\delta(^3S_1)$, and ϵ_1 obtained by the MIDPOP search that were used in making these predictions of the data are plotted as line segments in Fig. 1 and tabulated in Table III. Line segments are shown since the phase shifts were assumed to have energy derivatives the same as those in the Arndt-MacGregor analysis in each of the six energy bands. Note that the phase shifts are in good agreement with the Arndt-MacGregor phase shifts, i.e., the phases obtained by the MIDPOP search did not take on unphysical values to effect a good fit to the data.

IV. ONE-BOSON-EXCHANGE POTENTIAL MODEL

The coupling constants and masses of the present one-boson-exchange potential are not very different from those of our earlier model, Paper I, in which the p^2 term in the central potential was neglected and all potentials were cut off within 0.6 F. These procedures were believed to be wrong by some, who expected that inclusion of the p^2 term would require substantial changes in the model. This turned out not to be the case. However, inclusion of the p^2 terms did have the favorable effect of making the nonrelativistic approximation uniform to order p^2 and eliminating the need for cutoff in $l\geq 1$ states.

The parameters which differ most, percentagewise, are g_ρ^2 (up from 0.68 to 1.36) and g_η^2 (up from 7.0 to 10.6). The other constants agree within 15%. The change in g_η^2 is not too significant, because the η Born term is weak compared to the scalar and vector meson Born terms.

The values of most of the coupling constants seem reasonable. g_π^2 at 12.5 is somewhat below the expected value of $\sim$14, but this is in part due to the use of a nonrelativistic approximation to the pion-pole term; $g_\pi^2(M^2/E^2)$ has been replaced by $(g_\pi^2)_{\rm NR}=(M^2/E^2)\times(g_\pi^2)_{\rm Rel}\sim$12–14 over the 0–320 MeV range. (In this connection one should note that although we are keeping all terms of order k^2/M^2 in the potentials, for the pion the leading term is already of this order.)

The search gave $g_\rho^2=0.4$ and $f_\rho/g_\rho=3.8$. The latter ratio agrees well with the value expected from the nucleon electromagnetic form-factor data, which is $f_\rho/g_\rho=4\pm1$. However, g_ρ^2 seems a little large. Sakurai[10] has recently reported on eight different ways in which g_ρ^2 may be determined with the resulting values lying in the range 0.5–0.7. In this range, neither g_ρ^2 nor f_ρ/g_ρ is so well determined by the N-N data as the combination $(f_\rho+g_\rho)^2$ (the constant which weights the tensor potential and the spin-spin potential). If one takes $f_\rho/g_\rho=5.5$ while holding $(f_\rho+g_\rho)^2$ constant, then one would obtain $g_\rho^2=0.7$, in much better agreement with Sakurai's work.

The value for g_ω^2 obtained from the search, viz., 19, seems unusually large. This constitutes one reason for not believing literally the one-boson-exchange potential prescription for unitarizing the pole terms. We do not doubt that in the OBEP framework such a large vector meson coupling constant is needed to fit the data—it is required to match the phenomenological spin-orbit potentials[11]—but 19 is rather large compared with $SU(3)$ predictions. Assuming pure F-type coupling of the vector meson octet to the baryon octet, $SU(3)$ predicts $g^2(\omega_8)=3g_\rho^2$, where ω_8 signifies the isotopic singlet member of the unmixed octet. Thus $g^2(\omega_8)$ is predicted to be $3(1.36)\sim4$. Of course, the physical ω is a mixture of ω_8 and ω_0 (unitary singlet) vector mesons, but even so, $g_\omega^2\sim19$ seems rather large. Recently, another estimate of the g_ω^2/g_ρ^2 ratio has been suggested based on the nonet scheme. In this work, Sugawara and Von Hippel[12] find that the ϕ can be decoupled from the nucleon and if one assumes mainly F-type coupling to the vector octet, then $g_\omega^2=9g_\rho^2\sim12$. This number, while much closer to our value than the $SU(3)$ prediction, is still substantially below the value obtained from the search, 19.

The Pauli-Dirac coupling constant ratio f_ω/g_ω is quite in accord with experiment. Its value as given by the search was very close to zero, and was finally set iden-

[10] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966); P. Signell and J. W. Durso, *ibid.* **18**, 185 (1967).

[11] This is studied extensively in Paper I. We do not believe that our large value for g_ω^2 is due to numerical error. R. S. McKean [Phys. Rev. **125**, 1399 (1962)] arrives at $g_\omega^2\sim25$ in fitting phenomenological spin-orbit potentials.

[12] H. Sugawara and Frank von Hippel, Phys. Rev. **145**, 1331 (1966).

tically to zero; it is to be compared with the nucleon electromagnetic form-factor prediction $f_\omega/g_\omega \sim 0 \pm 0.2$.

The η-coupling constant given by the search was 11. This does not agree with the $SU(3)$ prediction, $g_\eta^2 \cong 2$, assuming an F/D ratio of $\frac{1}{3}$. However, the large value for g_η^2 may be traced to the fact that g_ω^2 was found to be 19. A large value for g_η^2 is then required in order that the net isoscalar tensor potential be zero, as required by experiment. This is explained in more detail in Paper I. Thus, the fact that g_η^2 violates the $SU(3)$ prediction may be because of the over-large value for g_ω^2.

Little can be said about the scalar meson coupling constants, since there is no strong evidence that either the σ_0 or σ_1 really exists. Possibly these are only empirical approximations to 2π and 5π S-wave contributions to N-N scattering, respectively. On the other hand, there is some evidence[13] for a resonance with the σ_0 quantum numbers near 700 MeV, and more recently,[14] near 1100 MeV, so that the idea of a $T=0$, $J=0^+$ resonance is perhaps not so strange. Also, in the most recent tabulation of mesons,[15] a listing can be found at 1003 MeV with $T=1$, $J=0^+$, quantum numbers appropriate for the σ_1. However, the status of this entry is unclear at the present time. If the σ_0 belongs to an $SU(3)$ octet, then a σ_1 follows automatically as the $T=1$ member of the octet. Of course, the σ_0 might only be a unitary singlet.

It is important to know just how well determined are the values of the parameters which have been considered above. This is a rather difficult question to answer precisely, mainly because of the sometimes strong correlation between various parameters. [It has already been mentioned that $(f_\rho+g_\rho)^2$ is more well determined than is either g_ρ or f_ρ.] Another difficulty is that an increase in a parameter may cause a much larger change in the χ^2 than the corresponding decrease. This is a reflection of the fact that the χ^2 surface is a quite complicated function of the parameters, which may make extrapolations dangerous. It is possible, however, to obtain some qualitative feeling for the sensitivity of χ^2 to the various parameters. For this purpose we have estimated the change in the values of the parameters of Table I necessary to increase the χ^2 by 10%. These changes were estimated to be $\Delta g_{\sigma_0}{}^2=0.1$, $\Delta g_{\sigma_1}{}^2=0.2$, $\Delta g_\omega{}^2=0.3$, $\Delta g_\pi{}^2=0.4$, $\Delta g_\eta{}^2=2.0$, $\Delta g_\rho{}^2=0.5$, $\Delta f_\rho/g_\rho=0.8$, and $\Delta f_\omega/g_\omega=0.006$.

It is much more difficult to determine the precision of the determination of the σ_0 and σ_1 masses because of the very great amount of computer time which a thorough analysis would entail. Our search on the masses does indicate that reasonably good fits to the phase parameters could be found with the σ_0 mass between 550 and 610 MeV and the σ_1 mass between 700 and 800 MeV.

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[13] M. Feldman, W. Frati, J. Halpern, A. Kanofsky, M. Nussbaum, S. Richert, and P. Yamin, Phys. Rev. Letters **14**, 869 (1965); V. Hagopian, W. Selove, J. Alitti, J. P. Baton, M. Neveu-Rene, R. Gessaroli, and A. Romano, *ibid.* **14**, 1077 (1965).

[14] D. J. Crennell, G. R. Kalbfleisch, K. W. Lai, J. M. Scarr, T. G. Schumann, I. O. Skillicorn, and M. S. Webster, Phys. Rev. Letters **16**, 1025 (1966).

[15] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).