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Generalized Helm Model for Transverse Electroexcitation of Nuclear Levels

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Helm's model for nuclear transition charge densities, assuming these to extend over a smeared-out shell at the nuclear radius, has been found useful in describing Coulomb-type electroexcitation of individual nuclear levels. We have extended the model by obtaining corresponding nuclear transition current densities (satisfying the continuity equation) and magnetization densities, which permit calculation of transverse matrix elements, and hence offer a description of electroexcitation of single levels at large electron scattering angles. The model yields a good numerical fit to the 15.1-MeV level in ^{12}C and to other examples. In addition, the relation between the M, T and the B notation for electroexcitation matrix elements is established, and the Isabelle-Bishop expansion in powers of the momentum transfer is rederived.

I. INTRODUCTION

THE principal difference between electro- and photoexcitation of a nuclear level, and the main advantage of the former method, consists in the fact that electroexcitation admits a variable momentum transfer q, while in photoexcitation, one is restricted to $q=\omega$; ω being the excitation energy of the level. Electron scattering, therefore, permits an exploration of the relevant nuclear form factors as a function of q, posing a stringent test for nuclear models which must be able to describe this functional dependence. The shell model usually provides[1–3] an acceptable q dependence, but often gives an absolute magnitude of the cross section too large by a factor of two or more; moreover, it requires a considerable calculational effort and provides little physical insight into the results. It seems desirable, therefore, to establish a simple approximate model, whose predictions can easily be applied to a variety of experimental results which one would like to survey; whose parameters (including multipolarities) can readily be adjusted for obtaining a reasonable fit to the data; and which, by its broadly chosen assumptions, provides an over-all view of the physical mechanism. For the exctroexcitation of giant resonance states, a model of this kind is given by the generalized Goldhaber-Teller model.[4–6] Here, the states are assumed to represent oscillations of any two of the four nucleon fluids (including spin) against the two others; one obtains transition form factors in terms of the ground-state form factors, from a transition density which is a gradient of the ground-state charge density, i.e., which has its largest values about the nuclear radius.

The many states that are rapidly being discovered in nuclei below the giant resonance, using large-angle elec-

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tron scattering,[7–10] can presumably not be described throughout by the generalized Goldhaber-Teller model[11]. For the Coulomb part of the electroexcitation cross section, which dominates the scattering at all angles except close to 0° or 180°, there exists a model due to Helm[12,13] which is more schematic, and hence more generally applicable to the excitation of individual nuclear levels. This model assumes that the transition charge density is concentrated about the nuclear radius also, but is smeared out by a Gaussian convolution. It has shown itself exceedingly useful for the interpretation of angular distributions of inelastically scattered electrons,[12–17] in many cases permitting a unique determination of the transition multipolarities, with parameters taken over from the elastic scattering data. Since, however, the transverse electric and magnetic parts of the electron-scattering cross section, which become prominent at large backward-scattering angles, lead to a much greater variety of excited states than the Coulomb part, most electro-excitation experiments now concentrate on scattering angles at[10,18] or near[7–9,19] 180°. In order to provide an interpretation for transversely excited states seen in such experiments, we have generalized the Helm model to permit a calculation of the transverse matrix elements. The magnetization density we have assumed to be concentrated near the nuclear surface also. For the nuclear-charge current density, we obtained an expression by solving in a unique fashion the continuity equation containing the Helm charge distribution. The resulting transverse matrix elements contain two level parameters (besides the universal nuclear radius and surface thickness) which can be fixed by fitting to the low-q limit of the experiments, and then provide a parametrization of the data to all higher q values.

We have further, for the sake of future reference, written down the conversion between the M, T notation[1] and the B notation[20] (which are both being used independently in the literature) for the electroexcitation matrix elements, and have rederived their expansion[7,21] in powers of q, explicitly stating the definition[22] of the expansion coefficients, both in general and in terms of our generalized Helm model. Finally, some illustrative examples are discussed, among others the[2,3,23] 15.1-MeV $M1$ state of ^{12}C, and a satisfactory description of the data is found to be provided by our model.

II. GENERAL PRELIMINARIES

In first Born approximation, the differential inelastic electron scattering cross section is given by[1]

$$\frac{d\sigma}{d\Omega}=\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}\Big\{V_l(\vartheta)\sum_{L=0}^{\infty}|\hat{J}_0^{-1}\langle J\|M_L(q)\|J_0\rangle|^2 +V_t(\vartheta)\sum_{L=1}^{\infty}\big(|\hat{J}_0^{-1}\langle J\|T_L^e(q)\|J_0\rangle|^2 +|\hat{J}_0^{-1}\langle J\|T_L^m(q)\|J_0\rangle|^2\big)\Big\}\,,\quad(1)$$

where $\mathbf{k}_1(\mathbf{k}_2)$ are the initial (final) electron momenta (we neglect the electron mass), $\alpha=e^2/4\pi\hbar c=1/137$, $\mathbf{q}=\mathbf{k}_1-\mathbf{k}_2$ the momentum transfer, $\Delta^2=\mathbf{q}^2-\omega^2$, $\omega=k_1-k_2$ the nuclear excitation energy, J_0M_0 and JM the initial and final nuclear spins and their z components, $\hat{J}_0=(2J_0+1)^{1/2}$, and ϑ the electron scattering angle. The kinematical factors V_l, V_t are given by

$$V_l(\vartheta)=\frac{\Delta^4}{q^4}2k_1k_2\cos^2\tfrac{1}{2}\vartheta\,,\quad(2a)$$

$$V_t(\vartheta)=\frac{2k_1k_2}{q^2}\sin^2\tfrac{1}{2}\vartheta\left[(k_1+k_2)^2-2k_1k_2\cos^2\tfrac{1}{2}\vartheta\right],\quad(2b)$$

with $V_l(180°)=0$. The cross section contains a longitudinal (or Coulomb) matrix element of the operator

$$M_{LM}(q)=\int d^3r\,\rho(\mathbf{r})j_L(qr)Y_{LM}(\hat{r})\quad(3a)$$

determined by the nuclear charge distribution $e\rho(\mathbf{r})$, as well as a transverse electric and a transverse magnetic matrix element, which originate from the nuclear charge current $e\mathbf{j}(\mathbf{r})$ and magnetization density $e\boldsymbol{\mu}(\mathbf{r})$:

$$T_{LM}^{e,m}(q)=T_{LM}^{e,mj}(q)+T_{LM}^{e,m\mu}(q)\,,\quad(3b)$$

$$T_{LM}^{ej}(q)=q^{-1}\int d^3r\,\mathbf{j}\cdot\nabla\times j_L(qr)\,\mathbf{Y}_{LL}^M(\hat{r})\,,\quad(3c)$$

$$T_{LM}^{mj}(q)=\int d^3r\,\mathbf{j}\cdot j_L(qr)\,\mathbf{Y}_{LL}^M(\hat{r})\,,\quad(3d)$$

$$T_{LM}^{e\mu}(q)=q\int d^3r\,\boldsymbol{\mu}\cdot j_L(qr)\,\mathbf{Y}_{LL}^M(\hat{r})\,,\quad(3e)$$

$$T_{LM}^{m\mu}(q)=\int d^3r\,\boldsymbol{\mu}\cdot\nabla\times j_L(qr)\,\mathbf{Y}_{LL}^M(\hat{r})\,.\quad(3f)$$

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The reduced matrix elements are defined by the Wigner-Eckart theorem:

$$\langle JM|\mathfrak{M}_{LM_L}|J_0M_0\rangle = \hat{J}^{-1}(J_0M_0,LM_L|JM)\langle J\|\mathfrak{M}_L\|J_0\rangle. \quad (4)$$

There exists in the literature[7,20,21] an alternative way of writing the cross section, namely (for the Lth multipole)

$$\left(\frac{d\sigma}{d\Omega}\right)_L=\alpha\frac{q^{2L}}{k_1^2}\frac{4\pi(L+1)}{L[(2L+1)!!]^2}\left\{\frac{L}{L+1}B(CL,q)v_l(\vartheta)+[B(EL,q)+B(ML,q)]v_t(\vartheta)\right\}, \quad (5)$$

and one can establish the following connections: For the kinematical factors, one has

$$V_{l,t}(\vartheta)=2k_1k_2(1-\cos\vartheta)^2v_{l,t}(\vartheta), \quad (6)$$

and for the reduced matrix elements, which are defined by

$$B(iL,q)=\sum_{M_LM}|\langle JM|\mathfrak{M}_{LM_L}{}^i|J_0M_0\rangle|^2 =\hat{J}_0{}^{-2}|\langle J\|\mathfrak{M}_L{}^i\|J_0\rangle|^2, \quad (7)$$

($i=C,E,M$), one has

$$\mathfrak{M}_{LM}{}^C=e(4\pi)^{-1/2}q^{-L}(2L+1)!!M_{LM}, \quad (8a)$$

$$\mathfrak{M}_{LM}{}^E=e(4\pi)^{-1/2}q^{-L}[L/(L+1)]^{1/2}\times(2L+1)!!T_{LM}{}^e, \quad (8b)$$

$$\mathfrak{M}_{LM}{}^M=-ie(4\pi)^{-1/2}q^{-L}[L/(L+1)]^{1/2}\times(2L+1)!!T_{LM}{}^m, \quad (8c)$$

so that finally,

$$B(CL,q)=\alpha\hat{J}_0{}^{-2}[(2L+1)!!/q^L]^2|\langle J\|M_L\|J_0\rangle|^2, \quad (9a)$$

$$B(EL,q)=\alpha\hat{J}_0{}^{-2}[L/(L+1)][(2L+1)!!/q^L]^2\times|\langle J\|T_L{}^e\|J_0\rangle|^2, \quad (9b)$$

$$B(ML,q)=\alpha\hat{J}_0{}^{-2}[L/(L+1)][(2L+1)!!/q^L]^2\times|\langle J\|T_L{}^m\|J_0\rangle|^2. \quad (9c)$$

Many experiments were done at small values of the momentum transfer, and it becomes useful to perform an expansion of the matrix elements[7,21] in powers of q^2 (by expanding the spherical Bessel function). If we introduce the notation[22]

$$\langle J\|r^{L+l}\|J_0\rangle_L=\langle J\|\int d^3r\rho(\mathbf{r})r^{L+l}Y_L(\hat{r})\|J_0\rangle, \quad (10a)$$

$$\langle J\|r^{L+l}\|J_0\rangle_L{}^{ej}=R\langle J\|\int d^3r\mathbf{j}\cdot\nabla\times r^{L+l}\mathbf{Y}_{LL}(\hat{r})\|J_0\rangle, \quad (10b)$$

$$\langle J\|r^{L+l}\|J_0\rangle_L{}^{e\mu}=R^{-1}\langle J\|\int d^3rr^{L+l}\boldsymbol{\mu}\cdot\mathbf{Y}_{LL}(\hat{r})\|J_0\rangle, \quad (10c)$$

$$\langle J\|r^{L+l}\|J_0\rangle_L{}^m=\langle J\|\int d^3r\times[r^{L+l}\mathbf{j}\cdot\mathbf{Y}_{LL}(\hat{r})+\boldsymbol{\mu}\cdot\nabla\times r^{L+l}\mathbf{Y}_{LL}(\hat{r})]\|J_0\rangle, \quad (10d)$$

[where in Eqs. (10b), (10c), the charge radius R of the nucleus has been introduced as a factor in order to make all the reduced matrix elements have dimension r^{L+l}], then the expansion is found as follows: For the Coulomb part, we have

$$\left\{\frac{B(CL,q)}{B(CL,0)}\right\}^{1/2}=1-\frac{q^2R_C{}^2}{2(2L+3)}+\frac{q^4R_C{}^{*4}}{8(2L+3)(2L+5)}-\cdots, \quad (11a)$$

using "transition radii" defined by

$$R_C{}^2=\langle J\|r^{L+2}\|J_0\rangle_L/\langle J\|r^L\|J_0\rangle_L, \quad (11b)$$

$$R_C{}^{*4}=\langle J\|r^{L+4}\|J_0\rangle_L/\langle J\|r^L\|J_0\rangle_L, \quad (11c)$$

and where the matrix element at zero momentum transfer [which to a good accuracy is equal to the matrix element $B(CL{\cdot}\omega)$ at $q=\omega$] is given by

$$B(CL,0)=\alpha\hat{J}_0{}^{-2}|\langle J\|r^L\|J_0\rangle_L|^2. \quad (11d)$$

For monopoles, $L=0$, the series expansion of Eq. (11a) starts with the q^2 term only, since from the orthogonality of initial and final nuclear wave functions,

$$\langle J\|\int d^3r\rho(\mathbf{r})\|J_0\rangle\equiv0.$$

The preceding equations should then be replaced by

$$\left\{\frac{q^{-4}B(C0,q)}{\lim_{q\to0}[q^{-4}B(C0,q)]}\right\}^{1/2}=1-\frac{q^2R_C{}^2}{6}+\cdots, \quad (11a')$$

with

$$R_C{}^2=\tfrac{3}{10}\langle J\|r^4\|J_0\rangle_0/\langle J\|r^2\|J_0\rangle_0, \quad (11b')$$

and

$$\lim_{q\to0}[q^{-4}B(C0,q)]=(\alpha/36\hat{J}_0{}^2)|\langle J\|r^2\|J_0\rangle_0|^2. \quad (11d')$$

For the transverse electric part, we find, using a notation of transition radii similar to Spamer's,[7]

$$\left\{\frac{q^2B(EL,q)}{\lim_{q\to0}[q^2B(EL,q)]}\right\}^{1/2}=1-\frac{q^2}{2(2L+3)}\frac{L+3}{L+1}R_E{}^2+\frac{q^4}{8(2L+3)(2L+5)}\frac{L+5}{L+1}R_E{}^{*4}-\cdots, \quad (12a)$$

$$\frac{L+3}{L+1}R_E{}^2=\frac{\langle J\|r^{L+2}\|J_0\rangle_L{}^{ej}-2(2L+3)R^2\langle J\|r^L\|J_0\rangle_L{}^{e\mu}}{\langle J\|r^L\|J_0\rangle_L{}^{ej}}, \quad (12b)$$

$$\frac{L+5}{L+1}R_E{}^{4*}=\frac{\langle J\|r^{L+4}\|J_0\rangle_L{}^{ej}-4(2L+5)R^2\langle J\|r^{L+2}\|J_0\rangle_L{}^{e\mu}}{\langle J\|r^L\|J_0\rangle_L{}^{ej}}, \quad (12c)$$

and where

$$\lim_{q\to 0}[q^2B(EL,q)]=\frac{\alpha}{R^2}\frac{L}{L+1}\hat{J}_0^{-2}|\langle J\|r^L\|J_0\rangle_L{}^{ej}|^2$$

$$\cong\omega^2B(EL,\omega). \tag{12d}$$

Finally, we have for the transverse magnetic part:

$$\left\{\frac{B(ML,q)}{B(ML,0)}\right\}^{1/2}=1-\frac{q^2}{2(2L+3)}\frac{L+3}{L+1}R_M{}^2$$

$$+\frac{q^4}{8(2L+3)(2L+5)}\frac{L+5}{L+1}R_M{}^{*4}-\cdots, \tag{13a}$$

with

$$\frac{L+3}{L+1}R_M{}^2=\frac{\langle J\|r^{L+2}\|J_0\rangle_L{}^m}{\langle J\|r^L\|J_0\rangle_L{}^m}, \tag{13b}$$

$$\frac{L+5}{L+1}R_M{}^{*4}=\frac{\langle J\|r^{L+4}\|J_0\rangle_L{}^m}{\langle J\|r^L\|J_0\rangle_L{}^m}, \tag{13c}$$

and where $B(ML,\omega)\cong$

$$B(ML,0)=\alpha[L/(L+1)]\hat{J}_0^{-2}|\langle J\|r^L\|J_0\rangle_L{}^m|^2. \tag{13d}$$

If one considers the first terms in the expansion only, and uses the approximate equalities at $q=\omega$, one can show easily that

$$B(EL,\omega)\cong B(CL,\omega), \tag{14a}$$

or more accurately,

$$\langle J\|r^L\|J_0\rangle_L{}^{ej}=[(L+1)/L]^{1/2}\omega R\langle J\|r^L\|J_0\rangle_L, \tag{14b}$$

(an exact relation), which also implies

$$\langle J\|T_L{}^e(\omega)\|J_0\rangle\cong[(L+1)/L]^{1/2}\langle J\|M_L(\omega)\|J_0\rangle. \tag{14c}$$

Use has been made here of the equation of continuity,

$$\nabla\cdot\mathbf{j}=-\dot{\rho}=i\omega\rho. \tag{14d}$$

The relations (14) are known as Siegert's theorem; they furnish a connection between the Coulomb and the transverse electric (charge current part only) matrix element, which is correct even if $\mathbf{j}$ contains meson exchange currents, as is the case in an actual nucleus. Note that the transverse matrix elements at $q=\omega$ describe photonuclear processes.[1,23]

Finally, it may be useful to quote the radiative width[7] to the ground state of the level J in terms of the electroexcitation B's for $q\to\omega$:

$$\Gamma=8\pi\sum_{L=1}^{\infty}\frac{L+1}{L}((2L+1)!!)^{-2}\omega^{2L+1}\left(\frac{\hat{J}_0}{\hat{J}}\right)^2$$

$$\times[B(EL,\omega)+B(ML,\omega)], \tag{15a}$$

as well as the photoabsorption cross section[1] integrated over its width:

$$\int\sigma(k)dk=(2\pi)^3\sum_{L=1}^{\infty}\frac{L+1}{L}[(2L+1)!!]^{-2}\omega^{2L-1}$$

$$\times[B(EL,\omega)+B(ML,\omega)]. \tag{15b}$$

Some of the foregoing can be found at different places in the literature. It was thought to be useful for reference purposes to have all this material written up in one place, the exact meanings of the parameters, such as $R_C{}^2$, $R_E{}^2$, etc., defined, and the conversions between the M, T, and the B notation stated.

III. FORMULATION OF THE MODEL

Helm[12] has given an expression for the Coulomb matrix element by assuming the transition charge density to be concentrated on a shell about the nuclear radius, in the form of a δ function smeared out by a convolution,

$$\rho(\mathbf{r})=\int\rho_0(\mathbf{r}-\mathbf{r}')\rho_1(\mathbf{r}')d^3r'. \tag{16}$$

The charged shell is given by

$$\rho_0(\mathbf{r})=Z\delta(\mathbf{r}-\mathbf{R}), \tag{17a}$$

(to be averaged over all directions of $\mathbf{R}$), with $\mathbf{R}$ a position vector of length equal to the charge radius, over whose direction we will average later; the smearing is taken as

$$\rho_1(\mathbf{r})=(2\pi g^2)^{-3/2}\exp(-r^2/2g^2), \tag{17b}$$

with some surface thickness g. By the convolution theorem, the Fourier transform of $\rho(\mathbf{r})$ is just the product of the Fourier transforms of ρ_0 and ρ_1,

$$F(q)\equiv\int d^3re^{i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r})=f_0(\mathbf{q})f_1(q), \tag{18}$$

$$f_0(\mathbf{q})=Ze^{i\mathbf{q}\cdot\mathbf{R}},\quad f_1(q)=\exp(-g^2q^2/2). \tag{19}$$

The Coulomb matrix element (3a) can easily be shown to be given by

$$M_{LM}(q)=(4\pi i^L)^{-1}\int F(\mathbf{q})Y_{LM}(\hat{q})d\hat{q}, \tag{20}$$

where $\hat{q}=\mathbf{q}/q$, and inserting from Eq. (18), one has

$$M_{LM}(q)=Zf_1(q)j_L(qR)Y_{LM}(\hat{R}). \tag{21}$$

This gives for the reduced matrix element

$$\hat{J}_0^{-1}\langle J\|M_L(q)\|J_0\rangle=\beta_L{}^{J_0J}f_1(q)j_L(qR), \tag{22}$$

with a parameter to be fitted to experiment,

$$\beta_L{}^{J_0J}=\hat{J}_0^{-1}Z\langle J\|Y_L(\hat{R})\|J_0\rangle_{\text{av}}, \tag{23}$$

av standing for an average over the directions $\hat{R}$. This is the result of Helm.

We now extend this procedure to the calculation of $T_{LM}{}^{e,m}$; for this purpose, we need corresponding expressions for $\mathbf{j}(\mathbf{r})$ and $\boldsymbol{\mu}(\mathbf{r})$. The current can be found by solving the continuity equation (14d), which we show can be done, if certain reasonable assumptions are made [see Eqs. (27) below], in a unique way. First, we note that with $\rho(\mathbf{r})$ written as in Eq. (16), we also have

$$\mathbf{j}(\mathbf{r})=\int \mathbf{j}_0(\mathbf{r}-\mathbf{r}')\rho_1(\mathbf{r}')d^3r', \tag{24}$$

which follows from Eq. (14d) by taking Fourier transforms. At the same time, $\mathbf{j}_0(\mathbf{r})$ must satisfy

$$\nabla\cdot\mathbf{j}_0(\mathbf{r})=i\omega\rho_0(\mathbf{r}), \tag{25}$$

and using Eq. (17a), we have

$$\nabla\cdot\mathbf{j}_0(\mathbf{r})=i\omega Zr^{-2}\delta(r-R)\sum_{LM} Y_{LM}{}^*(\hat{R})Y_{LM}(\hat{r}). \tag{26}$$

The basic assumption will be that $\mathbf{j}_0$ is of the form

$$\mathbf{j}_0(\mathbf{r})=\rho_2(r)\mathbf{v}, \tag{27a}$$

with

$$\begin{aligned}\rho_2(r)&=3Z/4\pi R^3 \quad r<R,\\ &=0 \qquad\qquad\;\; r>R,\end{aligned} \tag{27b}$$

and with the flow of nuclear matter being irrotational,

$$\nabla\times\mathbf{v}=0, \tag{27c}$$

which is an assumption that is commonly made.[1,24] In Eq. (26), we have to treat the monopole separately from the higher multipoles,

$$\mathbf{j}_0(\mathbf{r})=\mathbf{j}_0{}^{(0)}(\mathbf{r})+\mathbf{j}_0{}^{(1)}(\mathbf{r}), \tag{28}$$

where $\mathbf{j}_0{}^{(0)}$ satisfies

$$\begin{aligned}\nabla\cdot\mathbf{j}_0{}^{(0)}&=i\omega Z\delta(r-R)/4\pi r^2\\ &=-\tfrac{1}{3}i\omega r(d\rho_2/dr).\end{aligned} \tag{29}$$

It was shown in a previous paper[25] that for the monopole, conservation of matter requires a modification of the right-hand side expression, namely

$$\nabla\cdot\mathbf{j}_0{}^{(0)}=-\tfrac{1}{3}i\omega\frac{1}{r^2}\frac{d}{dr}[r^3\rho_2(r)]. \tag{30}$$

Further, only a radial velocity is possible (which is curl-free), and the solution is uniquely

$$\mathbf{j}_0{}^{(0)}(\mathbf{r})=-\tfrac{1}{3}i\omega\mathbf{r}\rho_2(r). \tag{31}$$

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For the higher multipoles $L\geqslant 1$, one makes the ansatz

$$\begin{aligned}j_0{}^{(1)}(\mathbf{r})=\rho_2(r)&\sum_{L\geq 1,M} Y_{LM}{}^*(\hat{R})\\ &\times[Af(r)\mathbf{Y}_{L,L+1}{}^M(\hat{r})+Bg(r)\mathbf{Y}_{L,L-1}{}^M(\hat{r})],\end{aligned} \tag{32}$$

being also a multipole expansion in $\hat{R}$, as required by Eq. (26). The only other multipole term in $\hat{r}$, $\mathbf{Y}_{LL}{}^M(\hat{r})$, is then ruled out by parity. The corresponding velocity is essentially the square bracket in Eq. (32), and if one requires its curl to vanish, one obtains one equation for A and B (taking f, g to be known for the moment); two further equations arise from inserting Eq. (32) into Eq. (26) and comparing coefficients of ρ_2 and ρ_2'. These three equations for two unknowns are soluble only if the rank of the 3×3 matrix of coefficients (including the column of absolute terms) is less than three, and it turns out that this is not the case unless one of the unknowns vanishes. If $B=0$, one finds further that

$$f(r)=Cr^{-L-2},$$

which gives a singularity at $r=0$ and is thus not permissible. With $A=0$, one finds

$$g(r)=Cr^{L-1},$$

which is the permissible case. The expansion of the current thus contains

$$r^{L-1}\mathbf{Y}_{L,L-1}{}^M(\hat{r})$$

only; this choice has also been made in our previous paper[26] on giant multipoles in the generalized Goldhaber-Teller model. The final solution is then

$$\begin{aligned}\mathbf{j}_0{}^{(1)}(\mathbf{r})=-i\omega R\frac{4\pi}{3}\rho_2(r)&\sum_{L\geq 1,M}\left(\frac{2L+1}{L}\right)^{1/2}\left(\frac{r}{R}\right)^{L-1}\\ &\times Y_{LM}{}^*(\hat{R})\mathbf{Y}_{L,L-1}{}^M(\hat{r}).\end{aligned} \tag{33}$$

As for the magnetization density, this can simply be assumed to be of the form

$$\boldsymbol{\mu}(\mathbf{r})=\boldsymbol{\mu}_0\int\bar{\rho}_0(\mathbf{r}-\mathbf{r}')\bar{\rho}_1(\mathbf{r}')d^3r', \tag{34}$$

with $\boldsymbol{\mu}_0$ some constant vector containing the nucleon spins and anomalous moments, and

$$\bar{\rho}_0(\mathbf{r})=A\delta(\mathbf{r}-\bar{\mathbf{R}}), \tag{35a}$$

$$\bar{\rho}_1(\mathbf{r})=(2\pi\bar{g}^2)^{-3/2}\exp(-r^2/2\bar{g}^2)\,; \tag{35b}$$

a different nuclear radius $\bar{R}$ and surface thickness $\bar{g}$ will in general be needed here, since the magnetization density is based on the nucleon distribution rather than on the proton distribution, and it is well known that these two differ.

[26] R. Raphael, H. Überall, and C. Werntz, Phys. Rev. **152**, 899 (1966).

With the current and magnetization density of Eqs. (24) and (34), one may then calculate the transverse matrix elements (3c)-(3f) in a straightforward manner, and one can factor out again the functions $f_1(q)$ and

$$\bar{f}_1(q)=\exp(-\bar{g}^2q^2/2)\,, \tag{36}$$

respectively, by using the relation

$$j_L(qr)Y_{LM}(\hat{r})=(4\pi i^L)^{-1}\int e^{i\mathbf{q}\cdot\mathbf{r}}Y_{LM}(\hat{q})d\hat{q}\,. \tag{37}$$

The monopole does not contribute to any transverse matrix elements. We find

$$T_{LM}{}^{mj}(q)\equiv 0\,, \tag{38}$$

because of the absence of the term $\mathbf{Y}_{LL}{}^M(\hat{r})$ in the current. This operator also vanishes in the Goldhaber-Teller model.[5] For the other reduced matrix elements, one obtains

$$\hat{J}_0^{-1}\langle J\|T_L{}^e(q)\|J_0\rangle=\left(\frac{L+1}{L}\right)^{1/2}\beta_L{}^{J_0J}\frac{\omega}{q}f_1(q)j_L(qR)+\gamma_{L0}{}^{J_0J}\frac{q}{2m}\bar{f}_1(q)j_L(q\bar{R})\,, \tag{39a}$$

$$\hat{J}_0^{-1}\langle J\|T_L{}^m(q)\|J_0\rangle=-\frac{q}{2m}\bar{f}_1(q)\left\{\left(\frac{L}{2L+1}\right)^{1/2}\gamma_{L+}{}^{J_0J}\times j_{L+1}(q\bar{R})+\left(\frac{L+1}{2L+1}\right)^{1/2}\gamma_{L-}{}^{J_0J}j_{L-1}(q\bar{R})\right\}\,, \tag{39b}$$

with m=nucleon mass, where we introduced the parameters

$$\gamma_{L\lambda}{}^{J_0J}=2mi^\lambda A\hat{J}_0^{-1}\langle J\|\boldsymbol{\mu}_0\cdot\mathbf{Y}_{L,L+\lambda}(\hat{R})\|J_0\rangle_{\mathrm{av}}\,, \tag{40}$$

($\lambda=0,\pm1$). Since the reduced matrix elements, when multiplied by i^L, are real,[27] and since different multipoles do not interfere, all our (dimensionless) parameters β_L and $\gamma_{L\lambda}$ may be taken real.

Equations (39) represent our generalization of the original Helm model of Eq. (22). Both transverse matrix elements depend on two parameters which can be obtained by fitting to the "photon point" at $q=\omega$ (this determines β_L or γ_{L-}, respectively), and to the experimental slope of the data versus q^2 (this will fix γ_{L0} or γ_{L+}, respectively), whereas R, $\bar{R}$, g, and $\bar{g}$ can be taken from the known ground-state charge and matter distribution; anyway, the latter four parameters should not vary from level to level, whereas β_L and $\gamma_{L\lambda}$ are characteristic for a given level. The expressions (39) then predict the q dependence of the form factors for higher values of q. In the transverse electric case, however, one can do better since the first parameter β_L, due to the Siegert theorem (see below), is the same one appearing already in the Coulomb matrix element, Eq. (22), and can be independently obtained there as one can experimentally separate[7] $B(CL,q)$ from $B(EL,q)$. The second term containing γ_{L0} is called "electric spin-flip transition" and has no longitudinal counterpart.

The same expansions in powers of q^2 as in Eqs. (11), (12), and (13) can also be made for the matrix elements of the Helm model. One obtains then, e.g., the connections

$$\beta_L{}^{J_0J}=\hat{J}_0^{-1}R^{-L}\langle J\|r^L\|J_0\rangle_L\,, \tag{41a}$$

$$R_C{}^2=R^2+(2L+3)g^2\,; \tag{41b}$$

$$\beta_L{}^{J_0J}=\hat{J}_0^{-1}R^{-L}(L/(L+1))^{1/2}(\omega R)^{-1}\times\langle J\|r^L\|J_0\rangle_L{}^{ej}\,, \tag{42a}$$

$$\frac{L+3}{L+1}R_E{}^2=R^2+(2L+3)g^2-\frac{2L+3}{m\omega}\left(\frac{L}{L+1}\right)^{1/2}\left(\frac{\bar{R}}{R}\right)^L\frac{\gamma_{L0}{}^{J_0J}}{\beta_L{}^{J_0J}}\,; \tag{42b}$$

$$\gamma_{L-}{}^{J_0J}=-\hat{J}_0^{-1}\bar{R}^{-L}2m\bar{R}((L+1)(2L+1))^{-1/2}\times\langle J\|r^L\|J_0\rangle_L{}^m\,, \tag{43a}$$

$$\frac{L+3}{L+1}R_M{}^2=\frac{2L+3}{2L+1}\left\{\bar{R}^2\left[1-\frac{2}{2L+3}\left(\frac{L}{L+1}\right)^{1/2}\frac{\gamma_{L+}{}^{J_0J}}{\gamma_{L-}{}^{J_0J}}\right]+(2L+1)\bar{g}^2\right\}\,. \tag{43b}$$

The right-hand sides of Eqs. (41a), (42a) are identical from the Siegert theorem, Eq. (14b). This explains why the same parameter β_L appears in the longitudinal and transverse electric form factors of the generalized Helm model.

The following radiative widths are predicted by the Helm model:

$$\Gamma_e=8\pi\alpha\frac{L+1}{L[(2L+1)!!]^2}\omega^{2L+1}R^{2L}\frac{\hat{J}_0^2}{\hat{J}^2}(\beta_L{}^{J_0J})^2\,, \tag{44a}$$

$$\Gamma_m=8\pi\alpha\frac{L+1}{L[(2L+1)!!]^2}\omega^{2L+1}\bar{R}^{2L}L(2L+1)\times\left(\frac{1}{2m\bar{R}}\right)^2\frac{\hat{J}_0^2}{\hat{J}^2}(\gamma_{L-}{}^{J_0J})^2\,. \tag{44b}$$

IV. APPLICATION TO TRANSITIONS IN ^{12}C

As illustrative examples, we consider two transitions in ^{12}C—the $M1$ transition from the ground state to the 15.1 MeV 1^+ level and the $E2$ transition to the 16.1 MeV 2^+ level. The former, of course, is of special interest, since the original Helm model does not apply to magnetic transitions.

[27] R. H. Pratt, J. D. Walecka, and T. A. Griffy, Nucl. Phys. 64, 677 (1965).

A. 15.1 MeV M1 Transition in ^{12}C

From Eq. (1), we see that for an $M1$ transition from a zero spin initial state, we can write the cross section

$$\frac{d\sigma}{d\Omega}\Big/\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}V_t(\vartheta)q^2=q^{-2}|\langle 1^+\|T_1{}^m\|0^+\rangle|^2. \quad (45a)$$

This form is a specially convenient one for analyzing the data. From Eq. (39b), we see that for $q<\bar{R}^{-1}$,

$$q^{-2}|\langle 1^+\|T_1{}^m\|0^+\rangle|^2$$
$$=\frac{\gamma_{1-}{}^2}{6m^2}-\left(\frac{\gamma_{1-}{}^2}{6m^2}\bar{g}^2+\frac{\bar{R}^2}{45\sqrt{2}}\frac{\gamma_{1+}\gamma_{1-}}{m^2}\right)q^2+O(q^4). \quad (45b)$$

Hence if we plot

$$\frac{d\sigma}{d\Omega}\Big/\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}V_t(\vartheta)q^2$$

versus q^2, the data points, for low q^2, should lie along a straight line whose intercept with $q^2=0$ gives the magnitude of γ_{1^-}, and whose slope yields the magnitude of γ_{1^+} and its sign relative to γ_{1^-}.[28]

Using data taken from Dudelzak and Taylor,[29] Goldemberg *et al.*,[30] Barber *et al.*,[31] and Schmid and Scholz,[32] we have plotted in Fig. 1 the q^2 dependence of the transverse $M1$ form factor for the 15.1-MeV level in ^{12}C. The values of $\bar{g}^2$ and $\bar{R}$ used in obtaining the theoretical curve are taken from optical-model fits to elastic proton-nucleus scattering.[33] It is encouraging that one can fit the experimental points, even at larger values of q^2, so well.

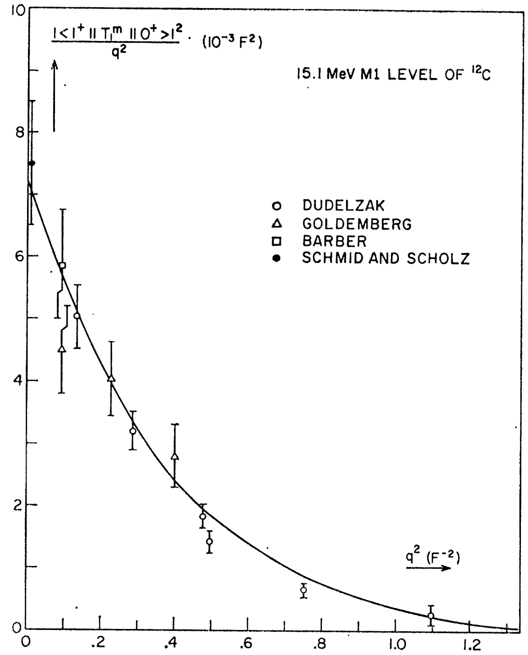

FIG. 1. Momentum transfer dependence of the transverse $M1$ form factor for the 15.1 MeV 1^+ level in ^{12}C, fitted by the generalized Helm model with parameters $\bar{g}^2=1.04F^2$, $\bar{R}=1.25A^{1/3}F$, $|\gamma_1-|=0.995$, $|\gamma_{1+}|=1.37$; γ_{1-} and γ_{1+} are relatively positive.

B. 16.1 MeV E2 Transition in ^{12}C

For $E2$ transitions from a zero-spin ground state, it is convenient to consider

$$\frac{d\sigma}{d\Omega}\Big/\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}V_l(\vartheta)q^2$$
$$=q^{-2}|\langle 2^+\|M_2\|0^+\rangle|^2+[V_t(\vartheta)/V_l(\vartheta)]q^{-2}$$
$$\times\{|\langle 2^+\|T_2{}^{ej}+T_2{}^{e\mu}\|0^+\rangle|^2\}. \quad (46a)$$

To a good approximation,[34]

$$\frac{8\pi\alpha^2}{\Delta^4}\frac{k_2}{k_1}V_l(\vartheta)=4\pi\left(\frac{\alpha^2}{4k_1{}^2}\frac{\cos^2\frac{1}{2}\vartheta}{\sin^4\frac{1}{2}\vartheta}\right)=4\pi\sigma_M, \quad (46b)$$

where σ_M is the proton Mott cross section. Thus the left-hand side of Eq. (46a) may be written

$$\frac{d\sigma}{d\Omega}\Big/\frac{k_2}{k_1}\frac{8\pi\alpha^2}{\Delta^4}V_l(\vartheta)q^2=\frac{d\sigma}{d\Omega}\Big/4\pi\sigma_M q^2. \quad (46c)$$

As is easily seen from Eqs. (22) and (39a), for low momentum transfer, the right-hand side of Eq. (46a), as a function of q^2, is a straight line whose slope and intercept at $q^2=0$ determine the magnitudes and relative sign of β_2 and γ_{20}.

If, as is the case for the data of Dudelzak and Taylor,[29] there are no experimental points at these low values of q^2, the data may be treated somewhat differently. We write

$$\frac{d\sigma}{d\Omega}=4\pi\sigma_M f_1{}^2 j_2{}^2(qR)\left\{\left(1+\frac{3}{2}\frac{\omega^2}{q^2}\frac{V_t}{V_l}\right)\beta_2{}^2+\frac{V_t}{V_l}\left[\sqrt{6}\frac{\bar{f}_1}{f_1}\frac{j_2(q\bar{R})}{j_2(qR)}\omega\beta_2\gamma_{20}+\left(\frac{\bar{f}_1}{f_1}\frac{j_2(q\bar{R})}{j_2(qR)}\right)^2\gamma_{20}{}^2q^2\right]\right\}. \quad (46d)$$

[28] In the spirit of the model, for a given nucleus, we take $\bar{g}$ and $\bar{R}$ to be given fixed parameters which may be determined, e.g., from elastic scattering data. The form factor is a sensitive function of these parameters—decreasing $\bar{g}$ and $\bar{R}$ by 10% would raise the theoretical curve in Fig. 1 far above the data points. Since $\bar{g}$ and $\bar{R}$ are not to be tampered with, $\gamma_{1\pm}$ have well-defined values. The fact that a good fit to the data is obtained with reasonable values of $\bar{g}$ and $\bar{R}$ is an indication of the validity of the model.

[29] B. Dudelzak and R. E. Taylor, J. Phys. Radium **22**, 544 (1961).

[30] J. Goldemberg, W. C. Barber, F. H. Lewis, Jr., and J. D. Walecka, Phys. Rev. **134**, B1022 (1964).

[31] W. C. Barber, F. Berthold, G. Fricke, and F. E. Gudden, Phys. Rev. **120**, 2081 (1960).

[32] H. Schmid and W. Scholz, Z. Physik **175**, 430 (1963).

[33] A. E. Glassgold, Rev. Mod. Phys. **30**, 419 (1958).

[34] The approximation consists of neglecting ω^2 relative to $\Delta^2\equiv 4k_1k_2\sin^2(\vartheta/2)$.

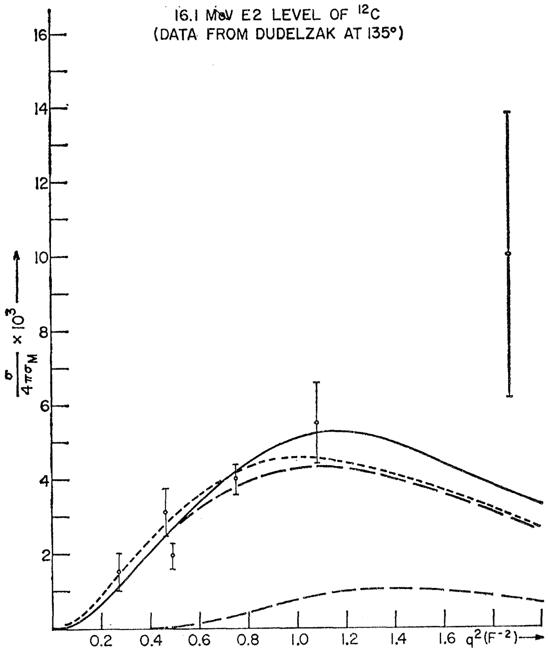

FIG. 2. Cross section versus q^2 normalized by the proton Mott cross section σ_M for the 16.1 MeV 2^+ level in ^{12}C at $\vartheta=135°$, fitted by the generalized Helm model with parameters $\bar{g}^2=1.04\mathrm{F}^2$, $\bar{R}=1.25A^{1/3}\mathrm{F}$. $g^2=0.81\mathrm{F}^2$, $R=1.10\ A^{1/3}\mathrm{F}$, $\beta_2{}^2=0.14$, $\gamma_{20}=0.952$ (solid and broken curves), $\gamma_{20}=0$ (dashed curve). The upper broken curve is the longitudinal contribution, the lower one the transverse contribution. The dashed curve is the cross section obtained when no magnetic contribution is included ($T_2{}^{e\mu}=0$).

Since the ratio $\bar{f}_1 j_2(q\bar{R})/f_1 j_2(qR)$ is a slowly varying function of q and ω^2/q^2 is small, if we plot $(d\sigma/d\Omega)/4\pi\sigma_M f_1{}^2(q) j_2{}^2(qR)$ versus q^2, the experimental points can be fitted with relative ease, and the magnitudes and relative sign of β_2 and γ_{20} determined.

In Fig. 2, we plot $(d\sigma/d\Omega)/4\pi\sigma_M$ as a function of q^2. One sees that the fit is quite good except for one point at $q^2=1.87f^{-2}$. This point, however, cannot be given very much weight—the cross section at such a high value of q^2 is very difficult to determine and the experimental errors involved are very large.[35] The solid curve is the fit obtained with the generalized Helm model. The upper and lower broken curves are $|\langle 2^+\|M_2\|0^+\rangle|^2$ and $(V_t/V_l)|\langle 2^+\|T_2{}^{ej}+T_2{}^{e\mu}\|0^+\rangle|^2$, respectively. The dashed curve is that obtained when we assume no magnetic contribution ($\gamma_{20}=0$). One sees that the longitudinal contribution is the dominant one at the scattering angle $\vartheta=135°$ used in Ref. 29. The small value of the transverse contribution at low q^2 is due to destructive interference between $T_2{}^{ej}$ and $T_2{}^{e\mu}$. At larger q^2, $T_2{}^{e\mu}$ predominates over $T_2{}^{ej}$.

Because of the dominance of the longitudinal contribution, the fit obtained in Fig. 2 presents of course a considerably weaker argument for the validity of our generalization of the Helm model than does Fig. 1.

It might be remarked that it has been customary when using the Helm model for a determination of the parameter β, to assume that the transverse contribution is zero. This is obviously inconsistent with Siegert's theorem, Eq. (14c). The values of β thus obtained will always be somewhat too high. [Reference 29 finds in this way a radiative transition width $\Gamma=1.6$ eV, or from Eq. (44a) a value $\beta_2=0.475$, against the result $\beta_2=0.375$ from our fit.]

V. SUMMARY

We have generalized the schematic model of Helm to provide us with expressions for the transverse form factors of inelastic electron scattering. They depend on the nuclear radius R, $\bar{R}$ and surface thickness g, $\bar{g}$ for charge and matter distribution, respectively, which should be similar for all levels, and the parameters β_L, $\gamma_{L0,\pm}$ characteristic for each level. It is hoped that the model will prove useful for describing, in a semiquantitative fashion, the q dependence of the transverse excitation strength of nuclear levels determined in inelastic electron scattering at large backward angles, pending a more careful analysis based on some detailed nuclear model. The inherent simplicity of our results should make them handy for a rapid analysis of experimental data.

ACKNOWLEDGMENTS

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[35] Indeed when Dudelzak and Taylor attempted to interpret their results in the framework of the theory of R. H. Dalitz and D. R. Yennie [Phys. Rev. **105**, 1598 (1957)] they did not include this point on their curve. It would be interesting to have better data at these higher values of q^2 in order to check whether the cross section decreases as predicted by the model.