verse magnetic field correction to the Franz-Keldysh terms, would predict $\Delta \approx (\mathcal{E}_g - \hbar\omega_0)H_{\rm eff}^2/E^2$.] In Fig. 8 we plot the experimental values of Δ, obtained for four values of E and $\mathcal{E}_g - \hbar\omega_0$ and for various values of H, versus the corresponding theoretical values of Δ calculated from Eq. (16); here we used $\mathcal{E}_g/2m^*c^2 = 4.5\times10^{-3}$. Figure 8 essentially illustrates the approximate dependence of $\Delta_{\rm expt}$ on H^2/E^2. It also shows that Eq. (16) predicts the correct order of magnitude for Δ.

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Theory of Tunneling, Including Photon-Assisted Tunneling, in Semiconductors in Crossed and Parallel Electric and Magnetic Fields

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Tunneling phenomena in crossed electric and magnetic fields cannot be properly described using the one-band effective-mass approximation. For this purpose, the two-band Hamiltonian is solved in the presence of crossed electric and magnetic fields and also in parallel fields. For crossed fields, two types of solutions are obtained. The first, for $E < H(\mathcal{E}_g/2m^*c^2)^{1/2}$, are of the harmonic-oscillator type, with quantized energy levels. In this region there is no interband tunneling in a pure material. In the region of high electric field, where $E > H(\mathcal{E}_g/2m^*c^2)^{1/2}$, the solutions are of the electric-field type with a continuous energy spectrum. The analogy of this model to the motion of free classical relativistic electrons in crossed fields is discussed. WKB solutions in the region of high electric field are used to calculate tunneling (Zener) current and photon-assisted tunneling [Franz-Keldysh (FK) effect]. The Hamiltonian is solved to obtain quasistationary solutions, neglecting a term which acts as the perturbation causing the Zener tunneling. The tunneling integrals are computed by the method of steepest descent. The results are nearly identical to those of Aronov and Pikus, obtained by a different method. In general, the magnetic field decreases both Zener and FK tunneling. The result for Zener tunneling predicts that the current will depend on E and H approximately as $\exp(-H^2/E^3)$, in good agreement with experiment. In the FK effect, for photon energies close to that of the gap, the ratio of the absorption in crossed fields to that at $H=0$ varies with frequency and field approximately as $\exp[-(\mathcal{E}_g-\hbar\omega)^{5/2}H^2/E^3]$. This is confirmed experimentally, both in the frequency and field dependence, by Reine, Vrehen, and Lax. Thus the main features of electron tunneling in crossed fields can be explained by a WKB treatment which in addition provides a good physical picture of the tunneling process. In order to provide a unified picture of tunneling in crossed and parallel fields, we also obtain expressions for Zener and FK tunneling in parallel fields using WKB solutions to the two-band model. The results for FK tunneling are very similar to those of the preceding paper, in that the electron motion separates into quantized motion transverse to the magnetic field and nonquantized motion parallel to both fields. The magnetic field reduces the tunneling by increasing the effective energy gap by the energy of the transverse motion. Our expressions reduce to those of the preceding paper in the limit of large $\mathcal{E}_g$. The ratio of the absorption in parallel fields to that at $H=0$ varies approximately as $\exp[-(\mathcal{E}_g-\hbar\omega)^{1/2}H/E]$. The result for Zener tunneling predicts $\exp(-H/E)$ behavior, in contrast to $\exp(-H^2/E^3)$ for crossed fields. The model in the preceding paper does not predict any Zener tunneling.

I. INTRODUCTION

THE effect of a strong electric field on a semiconductor is to induce a tunneling (Zener) current.[1,2] In addition, a strong electric field induces photon-assisted tunneling [the Franz-Keldysh (FK) effect], that is, the absorption of photons of energy less than the gap.[3] When a magnetic field is applied as well, the resulting motion depends on whether the magnetic field is parallel or perpendicular to the electric field. Parallel fields act independently to produce two simultaneous motions of the electrons: quantized magnetic-type motion transverse to both fields and nonquantized

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electric-type motion along the fields. For interband tunneling in parallel fields the effective energy gap is increased by the energy of the transverse motion in each band. The case of photon-assisted tunneling in this configuration is studied experimentally and theoretically by Reine, Vrehen, and Lax (RVL) in the preceding paper.[4] A different treatment of the parallel-field case is included in the present paper.

The effect of a magnetic field perpendicular to the electric field is more complex in that the magnetic and electric motions are no longer independent. Also, it has been shown recently by Praddaude[5] and by Zawadzki and Lax (ZL)[6] that in crossed fields where the electric field is relatively large, the effective-mass approximation is no longer valid. The problem must be treated as one of two coupled bands, the two-band model,[7] in order to take proper account of the effect of nonparabolicity of the bands. In this model the problem is analogous to classical relativistic motion in crossed fields, with the energy $2m_0c^2$ replaced by $\mathcal{E}_g$ and with m_0 replaced by m^*, that is, with the velocity of light c replaced by $(\mathcal{E}_g/2m^*)^{1/2}$. For Ec/H greater than this quantity, the problem can be transformed by a suitable Lorentz transformation into that of motion in an electric field alone. In this case the quantum-mechanical solutions are nonbound and nonquantized. On the other hand, for Ec/H smaller than $(\mathcal{E}_g/2m^*)^{1/2}$, the problem can be transformed to one of motion in a magnetic field alone, with bound, quantized solutions.

Haering and Adams[8] treated the problem of tunneling in crossed fields in the one-band effective-mass approximation using a WKB treatment. Their expressions are only valid for Ec/H small, that is, where the one-band effective-mass approximation holds. In this region the solutions are bound states with no Zener tunneling allowed between valence states and conduction states because of the requirement of energy conservation. However, Haering and Adams treat the problem of tunneling across a junction rather than within a pure material, so that the bands on opposite sides of the overlap in energy allowing tunneling with energy conserved.

We have treated this problem using a similar WKB treatment, but using the two-band model to obtain expressions for Zener current and photon-assisted tunneling which are valid in the region of large electric field. Recently, Aronov and Pikus (AP)[9] have obtained nearly identical expressions using a different technique, namely, canonical transformations analogous to those used for the Dirac equation. Their treatment includes the spin of the electron, which, however, does not seem to affect the tunneling results significantly. We believe that the two methods give such similar results because both treatments use the method of steepest descent to perform several integrations, making their treatment, in principle more exact, equivalent to our WKB approximation. The results of both theories are supported, at least qualitatively, by the experiments of the preceding paper.[4] The WKB method has the merit of providing a good physical picture of the tunneling process in crossed fields.

In order to give a unified picture of the effects of parallel and crossed fields, we have treated the parallel-field case, as well as the crossed-field case, in the two-band approximation. The results for photon-assisted tunneling are very similar to those of RVL given in the preceding paper,[4] and reduce to their expressions in the parabolic approximation. In the two-band model the effects of parallel fields still separate into a magnetic plus an electric motion, as in the parabolic model. The only essential difference is that the energy gap for tunneling is given by $\mathcal{E}_n=[\mathcal{E}_g^2+4\mathcal{E}_g(n+\frac{1}{2})\hbar\omega_c]^{1/2}$ instead of $\mathcal{E}_g+2(n+\frac{1}{2})\hbar\omega_c$. We also calculate Zener tunneling in parallel fields, which can not be done in the one-band approximation of RVL without explicitly taking into account the coupling between the two bands.

II. THE TWO-BAND EQUATION

To describe tunneling in crossed and parallel fields we use a two-band model which has been found to give a good approximation for the small-gap semiconductors such as the III-V and IV-VI intermetallic compounds. The two-band equation of ZL describing a Bloch electron in the presence of crossed electric and magnetic fields has been derived, somewhat artificially, neglecting the symmetry character (s, p, etc.) of the bands involved. We note in Appendix A that explicit solutions may be obtained for a p-like valence band and an s-like conduction band (such as in InSb). The resulting equation for the s band is the same as that of ZL, showing that for this problem the three degenerate p-like bands act as one spherical band. In order to obtain a single valence-band solution we retain the simplifying assumption of ZL. We work in the Kohn-Luttinger representation, in which the electric potential $e\mathbf{E}\cdot\mathbf{r}$ is diagonal. Neglecting spin and the diagonal free-mass term $P^2/2m_0$, the $\mathbf{k}\cdot\mathbf{p}$ equation for the two spherical bands is

$$\mathcal{H}\psi=\begin{bmatrix} -eEx+\frac{1}{2}\mathcal{E}_g & \pi_{21}\cdot\mathbf{P} \\ \pi_{12}\cdot\mathbf{P} & -eEx-\frac{1}{2}\mathcal{E}_g \end{bmatrix}\begin{bmatrix} f_2 \\ f_1 \end{bmatrix}=\mathcal{E}\begin{bmatrix} f_2 \\ f_1 \end{bmatrix}, \quad (1)$$

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where, in the Landau gauge,

$$\begin{aligned} &\mathbf{P}=\mathbf{p}+e\mathbf{A}/c\,, \quad \mathbf{E}=(-E,0,0)\,, \\ &\mathbf{A}=(0,Hx,0)\,, \quad \text{crossed fields}\,, \\ &\mathbf{A}=(0,0,Hy)\,, \quad \text{parallel fields}. \end{aligned} \tag{2}$$

The electric field is along the x axis, and the magnetic field is along the x axis for parallel fields, the z axis for crossed fields. With spin included, this equation for crossed fields is equivalent to that solved by AP, and can be written in terms which display its similarity to the Dirac equation.[10] In the diagonalizing coordinate system, $\pi_{12}{}^{\alpha}\pi_{21}{}^{\beta}=\delta_{\alpha\beta}\mathcal{E}_g/2m^*$, thus defining the effective mass for both bands.

For a given energy $\mathcal{E}$ this matrix equation determines conduction and valence-band solutions ψ^c and ψ^v given by linear combinations of the envelope functions f multiplied by the band-edge Bloch functions. As in the relativistic case, for energies close to the band edges, each ψ has a large and a small component, with the large component multiplying the band-edge Bloch function for the band involved. For each band we solve by substitution for the large component, f_2 for the conduction band, f_1 for the valence band, then use one of the equations to compute the small component from the large one. For the conduction band we have, for energy $\mathcal{E}_c$,

$$\left[(-eEx+\tfrac{1}{2}\mathcal{E}_g-\mathcal{E}_c)+\frac{\mathcal{E}_gP^2}{2m^*(eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E}_c)}\right]f_2{}^c+\frac{i\hbar\mathcal{E}_geEP_xf_2{}^c}{2m^*(eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E}_c)^2}\equiv(\mathcal{H}_0{}^c-\mathcal{E}_c)f_2{}^c+\mathcal{H}_2{}'f_2{}^c=0\,, \tag{3a}$$

and for the small component

$$f_1{}^c=(\boldsymbol{\pi}_{12}\cdot\mathbf{P}f_2{}^c)/(eEx+\tfrac{1}{2}\mathcal{E}_g+\mathcal{E}_c)\,. \tag{3b}$$

For the valence band, for energy $\mathcal{E}_v$, we find

$$\left[(-eEx-\tfrac{1}{2}\mathcal{E}_g-\mathcal{E}_v)+\frac{\mathcal{E}_gP^2}{2m^*(eEx-\frac{1}{2}\mathcal{E}_g+\mathcal{E}_v)}\right]f_1{}^v+\frac{i\hbar\mathcal{E}_geEP_xf_1{}^v}{2m^*(eEx-\frac{1}{2}\mathcal{E}_g+\mathcal{E}_v)^2}\equiv(\mathcal{H}_0{}^v-\mathcal{E}_v)f_1{}^v+\mathcal{H}_1{}'f_1{}^v=0\,, \tag{4a}$$

and for the small component,

$$f_2{}^v=(\boldsymbol{\pi}_{21}\cdot\mathbf{P}f_1{}^v)/(eEx-\tfrac{1}{2}\mathcal{E}_g+\mathcal{E}_v)\,. \tag{4b}$$

The singular terms $\mathcal{H}_2{}'f_2{}^c$ and $\mathcal{H}_1{}'f_1{}^v$ result from the fact that the kinetic energy $\boldsymbol{\pi}\cdot\mathbf{P}$ and the electric potential $e\mathbf{E}\cdot\mathbf{r}$ do not commute, and are referred to by ZL as effective spin-orbit terms. AP obtain a similar term for crossed fields in the electric-field region but not in the magnetic-field region, although our equation indicates that the term is present as long as there is a nonzero electric field. However, the term causes no transitions for crossed fields in the magnetic-field region, although it has interband matrix elements. In this region the energy levels are quantized and there are no conduction- and valence-band states with the same energy between which Zener tunneling can take place.

We show in Sec. III that as long as we adopt the above method of solving for the large component for each band, the singular terms are small for the region of solution for each band and thus can be neglected. In fact, these terms become the perturbation which cause the Zener tunneling, although we assume they are negligible in the photon-assisted case. Thus we are dealing with quasistationary solutions. Neglecting the perturbing terms is equivalent to subtracting

$$\mathcal{H}_c{}'\equiv\begin{bmatrix}\mathcal{H}_2{}' & 0\\ 0 & 0\end{bmatrix} \quad\text{or}\quad \mathcal{H}_v{}'\equiv\begin{bmatrix}0 & 0\\ 0 & \mathcal{H}_1{}'\end{bmatrix} \tag{5}$$

from the Hamiltonian (1) when solving for the conduction or valence band, respectively.

It is shown in Appendix B that the tunneling current, using the golden rule, is proportional to $|\langle\psi^v|\mathcal{H}_c{}'|\psi^c\rangle|^2=|\langle\psi^c|\mathcal{H}_v{}'|\psi^v\rangle|^2$, where ψ^c and ψ^v are the quasistationary two-component functions obtained neglecting $\mathcal{H}_c{}'$ and $\mathcal{H}_v{}'$, respectively. Similarly, these approximate functions are used to calculate the optical absorption for the photon-assisted case, putting the vector potential of the radiation into the Hamiltonian (1) to obtain the perturbation due to the radiation. We calculate the tunneling current and optical absorption first for crossed fields, then for parallel fields.

III. WKB SOLUTIONS—CROSSED FIELDS

For crossed fields we have from Eq. (2), $\mathbf{A}=(0,Hx,0)$, so that the electric field is along the x axis, the magnetic field along the z axis. Putting $\mathbf{P}=\mathbf{p}+e\mathbf{A}/c$, neglecting one of $\mathcal{H}_c{}'$ and $\mathcal{H}_v{}'$, and separating each f into the form $f\sim\exp[i(k_yy+k_zz)]\varphi(x)$, we obtain from (3a) and (4a) the same equation for $\varphi_2{}^c(x)$ and $\varphi_1{}^v(x)$:

$$\left[\frac{p^2}{2m^*}-2e\left(\frac{\mathcal{E}}{\mathcal{E}_g}E-\frac{\hbar k_y}{m^*c}H\right)x-\frac{e^2E_{\text{eff}}{}^2}{\mathcal{E}_g}x^2\right]\varphi(x)=\left[\frac{1}{\mathcal{E}_g}(\mathcal{E}^2-\tfrac{1}{4}\mathcal{E}_g{}^2)-\frac{\hbar^2}{2m^*}(k_y{}^2+k_z{}^2)\right]\varphi(x)\,, \tag{6}$$

where $p\equiv(\hbar/i)d/dx$ and $E_{\text{eff}}{}^2\equiv E^2-H_{\text{eff}}{}^2$, $H_{\text{eff}}{}^2=H^2\mathcal{E}_g/2m^*c^2$. This equation is identical to that of ZL except that we have neglected the singular term, the electric field is along the x axis rather than the y axis, and we measure energies from the midpoint between the two band edges rather than from the conduction band edge.

Equation (6) has two types of solutions depending on

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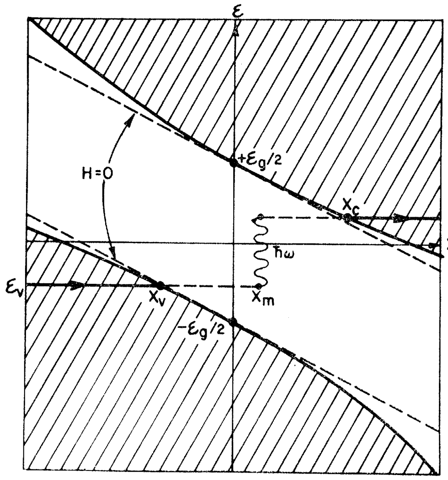

FIG. 1. Photon-assisted tunneling in crossed electric and magnetic fields for $E>H(\mathcal{E}_g/2m^*c^2)^{1/2}$. The sloping dashed lines represent the valence and conduction-band edges (or turning points) for $H=0$. The shaded regions represent the valence and conduction-band regions with both fields present. A valence electron of energy $\mathcal{E}_v$ tunnels from its turning point x_v to x_m, absorbs a photon, and tunnels to x_c, the turning point for a conduction electron of energy $\mathcal{E}_c=\mathcal{E}_v+\hbar\omega$. The magnetic field acts to reduce the absorption by curving the band edges away from each other, thus enlarging the tunneling region.

the sign of E_{eff}^2. In the magnetic-field region $E_{\text{eff}}^2<0$ or $Ec/H<(\mathcal{E}_g/2m^*)^{1/2}$, so that upon completing the square on x, the equation becomes a harmonic-oscillator equation with bound solutions and quantized Landau-like levels. In this region no Zener tunneling occurs. In the electric-field region $E_{\text{eff}}^2>0$ or $Ec/H>(\mathcal{E}_g/2m^*)^{1/2}$, with the analytic solutions to Eq. (6) being parabolic cylinder functions with nonbound states and a continuous-energy spectrum.

We calculate tunneling and photon-assisted tunneling in the electric-field region where $E_{\text{eff}}^2>0$. In order to find the matrix elements of the perturbations we obtain approximate solutions to Eq. (6) in the form of WKB functions as follows. We solve Eq. (6) for $p(x)$ which is now regarded as a number rather than an operator. We find the turning points of the motion where $p(x)=0$. Beyond these points, $p(x)$ is imaginary and the WKB solutions are decaying exponentials. In our case, this occurs within the forbidden energy gap.

The turning points of the motion are plotted in Fig. 1 for energies corresponding to the conduction and valence bands, for $k_y=k_z=0$. A valence electron of energy $\mathcal{E}_v$ tunnels from its turning point x_v to x_m, absorbs a photon if the tunneling is photon-assisted, then tunnels to x_c, the turning point for a conduction electron of energy $\mathcal{E}_c=\mathcal{E}_v+\hbar\omega$. We can easily see how the magnetic field acts to reduce the tunneling by curving the bands away from each other, thus enlarging the tunneling region. The effect of the magnetic field on photon-assisted tunneling becomes small as $\hbar\omega$ approaches $\mathcal{E}_g$ since then the optical transition takes place near $x=0$ where the bands are not moved apart by the magnetic field. This is observed experimentally.[4]

We can also see from this model how the magnetic field can become large enough to create bound states with no tunneling possible. In this case the valence-band edge is curved downward on the left, and the conduction-band edge is curved upward on the right. Within these bands the energy is quantized into Landau-like levels, where because of the electric field some of these levels are left closer together in energy than the zero-field gap. Between these levels oscillatory magnetoabsorption can take place for $\hbar\omega<\mathcal{E}_g$, as observed by Vrehen.[11] However, no Zener tunneling is possible between these levels because now there are no conduction and valence-band states with the same energy between which Zener tunneling can take place with energy conserved.

With this picture of the tunneling region we can now see how to treat the perturbations $\mathcal{H}_c'$ and $\mathcal{H}_v'$. The singularities of these perturbations [See Eqs. (3a), (4a), and (5)] are at $x_c^s=-(\mathcal{E}_c+\frac{1}{2}\mathcal{E}_g)/eE$ and $x_v^s=-(\mathcal{E}_v-\frac{1}{2}\mathcal{E}_g)/eE$. But x_v^s is to the right of $x_m=-(\mathcal{E}_c+\mathcal{E}_v)/2eE$ (See Appendix C), and x_c^s is to the left, by a distance $(\mathcal{E}_g+\mathcal{E}_c-\mathcal{E}_v)/2eE$. Since $\mathcal{E}_c-\mathcal{E}_v$ is either zero (Zener tunneling) or $\hbar\omega$ (FK effect), x_v^s is well to the right of the region where the valence-band solution ψ^v is used, and x_c^s is well to the left of the conduction-band region. Thus we can neglect $\mathcal{H}_v'$ when solving for ψ^v and $\mathcal{H}_c'$ when solving for ψ^c, as we did in obtaining Eq. (6). This analysis of the perturbations also holds for parallel fields.

For the valence band inside the forbidden region the WKB solution to Eq. (6) is

$$\varphi_1^v(x)=\frac{A}{|p_v(x)|^{1/2}}\exp\left(-\frac{1}{\hbar}\int_{x_v}^{x}|p_v(x)|dx\right). \quad (7a)$$

For the conduction band,

$$\varphi_2^c(x)=\frac{B}{|p_c(x)|^{1/2}}\exp\left(-\frac{1}{\hbar}\int_{x}^{x_c}|p_c(x)|dx\right), \quad (7b)$$

where at x_v, $|p_v(x)|=0$ and x_c, $|p_c(x)|=0$, and where $A\equiv(m^*/T_v)^{1/2}$, $B\equiv(m^*/T_c)^{1/2}$, with T_v and T_c the classical periods of the motion in the two bands.[8] A and B are normalization constants which cancel out when we multiply by the density of states.

To the left of x_v the valence-band solution is oscillatory, and similarly for the conduction-band solution to the right of x_c. The wave functions in these regions have no contribution to the transition matrix elements.

Using the WKB functions φ_2^c and φ_1^v, we form the large components f_2^c, f_1^v and from these, using Eqs. (3b) and (4b), calculate the small components f_1^c, f_2^v. Using the two-component solutions ψ^c, ψ^v thus obtained, we calculate the tunneling current and optical absorption following a procedure similar to that of Haering

[11] Q. H. F. Vrehen, Phys. Rev. Letters 14, 558 (1965).

of Adams.[8] We note that the solutions ψ^c and ψ^v are orthogonal even when $\mathcal{E}_c=\mathcal{E}_v$.

IV. TUNNELING CURRENT—CROSSED FIELDS

For tunneling in crossed fields the golden rule (See Appendix B) gives the tunneling rate from the valence band state ψ^v as

$$W_{vc}=\frac{2\pi}{\hbar}\sum_{\text{final states } c}|\langle\psi^c|\mathcal{H}_v'|\psi^v\rangle|^2\delta(\mathcal{E}_c-\mathcal{E}_v). \quad (8)$$

We calculate

$$\langle\psi^c|\mathcal{H}_v'|\psi^v\rangle=\delta_{k_yk_y'}\delta_{k_zk_z'}\int_{-\infty}^{\infty}[\varphi_1^c(x)]^*\mathcal{H}_1'\varphi_1^v(x)dx. \quad (9)$$

For $\mathcal{E}_c=\mathcal{E}_v=\mathcal{E}$ and therefore $|p_c(x)|=|p_v(x)|\equiv|p(x)|$, the integral becomes

$$I=\int_{x_v}^{x_c}dx\left\{\frac{[\pi_{21}{}^x]^*}{eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E}}\left(\frac{-\hbar}{i}\right)\frac{d}{dx}\left[\frac{B}{|p|^{1/2}}\exp\left(-\frac{1}{\hbar}\int_x^{x_c}|p|dx\right)\right]\frac{i\hbar\mathcal{E}_geE}{2m^*(eEx-\frac{1}{2}\mathcal{E}_g+\mathcal{E})^2}\times\left(\frac{\hbar}{i}\right)\frac{d}{dx}\left[\frac{A}{|p|^{1/2}}\exp\left(-\frac{1}{\hbar}\int_{x_v}^{x}|p|dx\right)\right]\right\}. \quad (10)$$

From Eq. (6) we have

$$|p|^2=(2m^*/\mathcal{E}_g)e^2E_{\text{eff}}{}^2(a^2-\xi^2), \quad (11)$$

where

$$a^2\equiv(2eE_{\text{eff}}{}^2)^{-2}[\mathcal{E}_g'^2E_{\text{eff}}{}^2+4\mathcal{E}'^2H_{\text{eff}}{}^2],$$
$$\xi\equiv x+\mathcal{E}/eE+(\mathcal{E}'/eE)(H_{\text{eff}}{}^2/E_{\text{eff}}{}^2), \quad (12)$$
$$\mathcal{E}_g'^2\equiv\mathcal{E}_g{}^2+2\mathcal{E}_g\hbar^2k_z{}^2/m^*, \quad \mathcal{E}'\equiv\mathcal{E}-\hbar k_yEc/H.$$

Then, using $\pi_{12}{}^x=(\frac{1}{3}\pi_{12}{}^2)^{1/2}=(\mathcal{E}_g/2m^*)^{1/2}/\sqrt{3}$, and assuming $eE\xi$, $eE_{\text{eff}}\xi\ll\frac{1}{2}\mathcal{E}_g$, we expand the integrand in I and find, to order ξ^2,

$$I\approx-\frac{8ABi\hbar e^2EE_{\text{eff}}}{\sqrt{3}m^*\mathcal{E}_g{}^2}\left(1-\frac{2\mathcal{E}'H_{\text{eff}}{}^2}{\mathcal{E}_gE_{\text{eff}}{}^2}\right)|a|^2[T(k_\perp)]^{1/2}, \quad (13)$$

where

$$T(k_\perp)\equiv\exp\left[-\frac{2}{\hbar}\left(\int_{x_v}^{x}|p|dx+\int_x^{x_c}|p|dx\right)\right]=\exp\left(-\frac{2}{\hbar}\int_{x_v}^{x_c}|p|dx\right). \quad (14)$$

Then including the density of initial and final states, with a factor of 2 for spin, the tunneling rate from all states $\mathcal{E}_v=\mathcal{E}$ to all states $\mathcal{E}_c=\mathcal{E}$ is

$$N_{vc}=\frac{2L_yL_z}{(2\pi)^2}\left(\frac{T_c}{2\pi\hbar}\right)\left(\frac{T_v}{2\pi\hbar}\right)\left(\frac{2\pi}{\hbar}\right)\times\int d\mathcal{E}\,f(\mathcal{E})\int dk_ydk_zI^2(k_\perp). \quad (15)$$

Since $I^2(k_\perp)$ includes an exponential factor $T(k_\perp)$ multiplied by slowly varying functions of k_y and k_z (through a, p, and $\mathcal{E}'$) we calculate the integrals on k_y and k_z by the method of steepest descent, evaluating the slowly varying factors at the saddle point of the exponent in $T(k_\perp)$. The calculation of the transmission coefficient $T\equiv\int dk_ydk_zT(k_\perp)$ is done in Eq. (C12). The saddle point is found to be $\mathcal{E}'=0$, $\mathcal{E}_g'=\mathcal{E}_g$. The resulting expression is independent of $\mathcal{E}$. Using

$$\int d\mathcal{E}f(\mathcal{E})=\int_{-L_x/2}^{+L_x/2}\frac{\partial\mathcal{E}}{\partial x}dx=eEL_x, \quad (16)$$

where $\mathcal{E}(x)$ is plotted in Fig. 1, we find the tunneling current $j=N_{vc}/L_xL_yL_z$ to be

$$j=\frac{e^2E^2}{3\pi^3\hbar^2}\left(\frac{2m^*}{\mathcal{E}_g}\right)^{1/2}\exp\left[-\frac{\pi}{2\hbar eE_{\text{eff}}}\left(\frac{m^*}{2}\right)^{1/2}\mathcal{E}_g{}^{3/2}\right]. \quad (17)$$

The above expression is nearly identical to that obtained by AP. The exponent is the same; this is the factor which is most sensitive to the effect of the magnetic field. In the prefactor, however, we have a factor $E^2/3\pi^3$, whereas AP obtain $E_{\text{eff}}{}^2/36\pi$. The effect of replacing E^2 by $E_{\text{eff}}{}^2$ is to make $j(H)/j(0)$ decay more sharply at low magnetic field and more slowly at high field. The effect is small except when the electric field becomes large. Even for low E the difference in the prefactor will affect the identification of the parameters involved when comparing the theory with experiment as is done in Fig. 4 of AP. The principal unknown parameter is the electric field E; however, the tunneling experiments are performed in diodes, so that the assumption of constant electric field is in question. However, the general $\exp(-H^2/E^3)$ behavior predicted by both theories for $j(H)/j(0)$ is confirmed by experiment.

V. PHOTON-ASSISTED TUNNELING—CROSSED FIELDS

We find the perturbation due to the incident radiation by replacing $\mathbf{P}\rightarrow\mathbf{P}+e\mathbf{A}_\omega/c$ in the Hamiltonian (1) and obtain

$$\mathcal{H}_\omega'=\begin{bmatrix}0 & \pi_{21}\cdot e\mathbf{A}_\omega/c\\ \pi_{12}\cdot e\mathbf{A}_\omega/c & 0\end{bmatrix}, \quad (18)$$

where in the dipole approximation $\mathbf{A}_\omega=(\mathbf{E}_\omega c/2i\omega)\times\exp i\omega t$, where $\mathbf{E}_\omega$ is the electric vector of the radiation. The matrix element of $\mathcal{H}_\omega'$ between the approximate wave functions ψ^c and ψ^v is given by

$$\langle\psi^c|\mathcal{H}_\omega'|\psi^v\rangle=[(e/c)\boldsymbol{\pi}\cdot\mathbf{A}_\omega](\langle f_1{}^c|f_2{}^v\rangle+\langle f_2{}^c|f_1{}^v\rangle), \quad (19)$$

but $\langle f_1{}^c|f_2{}^v\rangle\ll\langle f_2{}^c|f_1{}^v\rangle$ since the functions $f_1{}^c$ and $f_2{}^v$ are the small components of the respective states. [The ratio of the two terms is approximately $(\mathcal{E}_g-\hbar\omega)/2\mathcal{E}_g\ll1$.]

Therefore we have

$$\langle\psi^c|\mathcal{H}_\omega'|\psi^v\rangle\approx\frac{e}{c}\boldsymbol{\pi}_{12}\cdot\mathbf{A}\delta_{k_yk_y'}\delta_{k_zk_z'}\int dx(\varphi_c^2)^*\varphi_v^1. \quad (20)$$

The integral then becomes

$$I=\int dx(\varphi_2^c)^*\varphi_1^v=BA\int_{x_v}^{x_c}\frac{dx}{|p_c(x)p_v(x)|^{1/2}}\times\exp\left(-\frac{1}{\hbar}\int_{x_v}^{x}|p_v(x)|dx-\frac{1}{\hbar}\int_{x}^{x_c}|p_c(x)|dx\right). \quad (21)$$

We evaluate this integral by the method of steepest descent, using the expressions for $p_c(x)$, $p_v(x)$ in Appendix C, and find

$$I\approx BA(\pi\mathcal{E}_gT(k_\perp)/m^*\omega eE|p|)^{1/2}, \quad (22)$$

where

$$T(k_\perp)\equiv\exp\left(-\frac{2}{\hbar}\int_{x_v}^{x_m}|p_c(x)|dx-\frac{2}{\hbar}\int_{x_m}^{x_c}|p_c(x)|dx\right), \quad (23)$$

and at $x=x_m$, $|p_c(x)|=|p_v(x)|\equiv|p|$. The total transition rate N_{vc} is defined as for the tunneling case. N_{vc} is again proportional to a transmission coefficient $T=\int dk_ydk_zT(k_\perp)$ with other functions of k_y and k_z evaluated at the saddle point of the exponent of $T(k_\perp)$, which is found in Appendix C to be at $\mathcal{E}'=\frac{1}{2}\hbar\omega$, $\mathcal{E}_g'=\mathcal{E}_g$. The absorption coefficient is defined as

$$\alpha\equiv\frac{N_{vc}}{L_xL_yL_z}\frac{8\pi\hbar\omega}{ncE\omega^2}, \quad (24)$$

where n is the index of refraction. We use the expression (C13) for T in Appendix C and the fact that $(\boldsymbol{\pi}_{12}\cdot\mathbf{E}_\omega)^2=\mathcal{E}_gE_\omega^2/2m^*$ and make the approximation $(\mathcal{E}_g-\hbar\omega)\ll\mathcal{E}_g$ for absorption just below the gap. The absorption coefficient becomes

$$\alpha\approx e^3E_{\text{eff}}\mathcal{E}_g^2/8nc\hbar^3\omega^2(\mathcal{E}_g-\hbar\omega)\exp\left\{-\frac{4(\mathcal{E}_g-\hbar\omega)^{3/2}m^{*1/2}}{3\hbar eE}\times\left[1-\frac{(\mathcal{E}_g-\hbar\omega)}{20\mathcal{E}_g}\frac{(3E^2-8H_{\text{eff}}^2)}{E^2}\right]\right\} \quad (25)$$

or

$$\frac{\alpha(E,H)}{\alpha(E,0)}\approx\exp\left[-\frac{8(\mathcal{E}_g-\hbar\omega)^{5/2}}{15\hbar e\mathcal{E}_g}m^{*1/2}\frac{H_{\text{eff}}^2}{E^3}\right]. \quad (26)$$

Equations (25) and (26) are even closer to the corresponding expressions of AP than is the case for the tunneling current. Equation (26) and the exponent of (25) are exactly the same as the expressions of AP, and the experiments of the preceding paper[4] confirm the qualitative features of the behavior thereby predicted, especially the lack of magnetic effect on absorption of photons at $\hbar\omega\approx\mathcal{E}_g$. There are some differences in the full expression (25) for α: Where we have $E_{\text{eff}}\mathcal{E}_g^2/\hbar^3\omega^2$ in the prefactor, AP have $E\mathcal{E}_g/\hbar^2\omega$, a difference of a factor $(\hbar\omega E/\mathcal{E}_gE_{\text{eff}})\approx 1$. These differences are negligible when comparing the theoretical predictions with experiment.[4] The experiments approximately confirm the predictions of field and frequency behavior of the form $(\mathcal{E}_g-\hbar\omega)^{5/2}H^2/E^3$.

Also from Eq. (25) it can be shown[4] that the energy shift of the isoabsorption lines is approximately $2(\mathcal{E}_g-\hbar\omega)^2H^2/15m^*c^2E^3$, in agreement with the experimental results. This was predicted almost exactly by Lax[7] on the basis of the analogy to free relativistic electrons. Lax's result was also obtained by Reine, Vrehen, and Lax[12] from a generalization of the zero magnetic-field expressions consisting of the replacement of E by E_{eff}. However, this gives an expression for $\alpha(E,H)/\alpha(E,0)$ which incorrectly predicts exponential behavior of the form $(\mathcal{E}_g-\hbar\omega)^{3/2}H^2/E^3$. This was the subject of some discussion at the Kyoto conference.[13] We wish to emphasize that our result gives the $(\mathcal{E}_g-\hbar\omega)^{5/2}$ behavior in agreement with the result of AP.

VI. WKB SOLUTIONS—PARALLEL FIELDS

For parallel fields $\mathbf{A}=(0,0,Hy)$, with the electric and magnetic fields along the x axis. Putting $\mathbf{P}=\mathbf{p}+e\mathbf{A}/c$ and neglecting one of $\mathcal{H}_c'$, $\mathcal{H}_v'$, we obtain from Eqs. (3a) and (4a) the same equation for f_2^c, f_1^v:

$$\left\{[p_x^2/2m^*-(1/\mathcal{E}_g)(eEx+\mathcal{E})^2]+\left[\frac{p_y^2}{2m^*}+\frac{p_z^2}{2m^*}+\frac{eH}{m^*c}yp_z+\frac{e^2H^2y^2}{2m^*c^2}\right]+\frac{\mathcal{E}_g}{4}\right\}f=0. \quad (27)$$

We can separate this equation by substituting

$$f(x,y,z)=\varphi(x)\rho(y)\exp ik_zz, \quad (28)$$

giving for $\rho(y)$

$$[p_y^2/2m^*+\tfrac{1}{2}m^*\omega_c^2(y+\hbar k_z/m^*\omega_c)^2]\rho(y)=\lambda\rho(y), \quad (29)$$

where $\omega_c=eH/m^*c$. The solution to this equation is $\rho=\rho_n(y/L_m+L_mk_z)$, a harmonic-oscillator function with magnetic quantum number n and eigenvalue $\lambda=\lambda_n\equiv(n+\frac{1}{2})\hbar\omega_c$, with magnetic radius $L_m\equiv(\hbar c/eH)^{1/2}$ and orbit center $L_m^2k_z$. With this result for $\rho(y)$ the equation for $\varphi(x)$ becomes

$$[p^2/2m^*-(1/\mathcal{E}_g)(eEx+\mathcal{E})^2+\tfrac{1}{4}\mathcal{E}_g+(n+\tfrac{1}{2})\hbar\omega_c]\varphi(x)=0, \quad (30)$$

where $p\equiv(\hbar/i)d/dx$. The analytic solutions to Eq. (30), as to Eq. (6) for $E_{\text{eff}}^2>0$, are parabolic cylinder functions which reduce, for large energy gap, to the Airy

[12] M. Reine, Q. H. F. Vrehen, and B. Lax, Phys. Rev. Letters **17**, 582 (1966).

[13] B. Lax, J. Phys. Soc. Japan Suppl. **21**, 165 (1966).

functions obtained by RVL. We see this another way by noting that Eq. (30) reduces to Eqs. (3a) and (3b) of RVL when we solve for $(eEx+\mathcal{E})^2$ and take the square root of the operator equation, making the approximation $[p^2/2m^*+(n+\frac{1}{2})\hbar\omega_c]\ll\mathcal{E}_g$. We obtain

$$(\mathcal{E}+eEx)\varphi\approx\pm\{\tfrac{1}{2}\mathcal{E}_g+[p^2/2m^*+(n+\tfrac{1}{2})\hbar\omega_c]\}\varphi, \quad (31)$$

with the plus sign for the conduction band and the minus sign for the valence band. The solutions to Eq. (31) are the Airy functions obtained by RVL.

As for crossed fields, we obtain the perturbation matrix elements by using WKB solutions to (30). These solutions are given by Eqs. (7a) and (7b), where now $p(x)$ is obtained from Eq. (30). The turning points of the motion, where $p(x)=0$, are plotted in Fig. 2. In this case the bands are not curved, as they are for crossed fields as shown in Fig. 1, but are displaced by a constant amount from the $H=0$ lines. The energy separation of a pair of conduction and valence bands with the same quantum number n is

$$\mathcal{E}_n\equiv[\mathcal{E}_g^2+4\mathcal{E}_g(n+\tfrac{1}{2})\hbar\omega_c]^{1/2}.$$

Thus the effect of the magnetic field is to reduce the tunneling by increasing the effective energy gap to $\mathcal{E}_n>\mathcal{E}_g$. For large $\mathcal{E}_g$, $\mathcal{E}_n$ reduces to that obtained by RVL for parabolic bands, namely, $\mathcal{E}_n\rightarrow\mathcal{E}_g+2(n+\frac{1}{2})\hbar\omega_c$.

Using the WKB functions φ and the harmonic-oscillator functions ρ_n we form product functions f and from these, as for crossed fields, form the two-component solutions ψ to calculate the tunneling current and optical-absorption coefficient.

VII. TUNNELING CURRENT—PARALLEL FIELDS

We again use the golden rule (8) and calculate

$$\langle\psi^c|\mathcal{H}_v'|\psi^v\rangle=\delta_{k_zk_z'}\delta_{nn'}\int_{-\infty}^{\infty}[\varphi_{1,n'}{}^c(x)]^*\times\mathcal{H}_1'\varphi_{1,n}{}^v(x)dx. \quad (32)$$

Now $|p|^2$ is given by Eq. (11), where

$$a^2=(1/4e^2E^2)[\mathcal{E}_g^2+4\mathcal{E}_g(n+\tfrac{1}{2})\hbar\omega_c]=\mathcal{E}_n^2/4e^2E^2,$$
$$\xi=x+\mathcal{E}/eE. \quad (33)$$

Then the integral is found to be

$$I=-(8ABi\hbar e^2E^2/\sqrt{3}m^*\mathcal{E}_g^2)|a|^2T^{1/2}, \quad (34)$$

where T is defined by Eq. (16). The tunneling rate N_{vc} becomes

$$N_{vc}=\frac{2L_z}{(2\pi)}\left(\frac{T_c}{2\pi\hbar}\right)\left(\frac{T_v}{2\pi\hbar}\right)\left(\frac{2\pi}{\hbar}\right)\int d\mathcal{E}f(\mathcal{E})\sum_n\int dk_zI^2. \quad (35)$$

Since I^2 is independent and $\mathcal{E}$ and k_z, the sum over k_z (converted to an integral over values of k_z for which the center of the magnetic orbit $y=L_m^2k_z$ is contained in the crystal) contributes a factor $L_y/L_m^2=L_y(eH/\hbar c)$, and $\int d\mathcal{E}f(\mathcal{E})$ contributes a factor $eELx$. From this the tunneling current is calculated to be

$$j(H)=\frac{2e^2EH}{3\pi^2\hbar^2c}\sum_n\frac{\mathcal{E}_n^4}{\mathcal{E}_g^4}\exp\left[-\frac{\pi\mathcal{E}_n^2}{2\hbar eE}\left(\frac{m^*}{2\mathcal{E}_g}\right)^{1/2}\right]. \quad (36)$$

Because of the exponential factor the most important term is the $n=0$ term. We find $j(0)$ by converting the sum to an integral and taking the limit as $H\rightarrow 0$. The result is the same as the $H\rightarrow 0$ limit of the crossed-field result (17). The exponent of $j(H)/j(0)$ is proportional to H/E in comparison to H^2/E^3 for the crossed-field case.

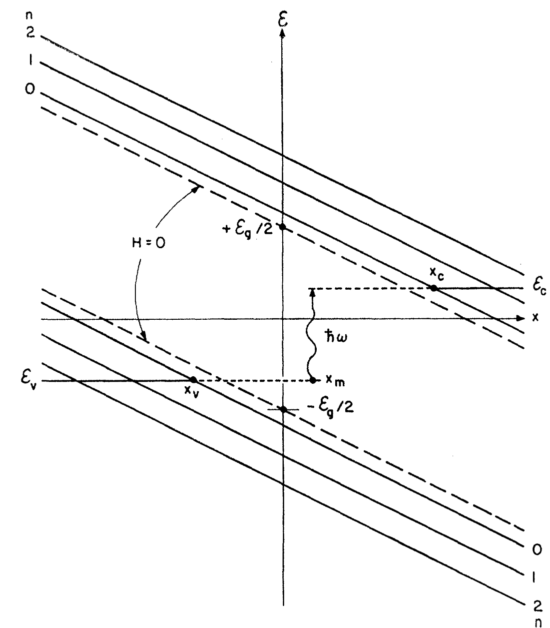

FIG. 2. Photon-assisted tunneling in parallel fields. The sloping solid lines represent the conduction and valence-band levels quantized by the magnetic field. A pair of levels with the same magnetic quantum number n are separated in energy by $\mathcal{E}_n=[\mathcal{E}_g^2+4\mathcal{E}_g(n+\frac{1}{2})\hbar\omega_c]^{1/2}$.

VIII. OPTICAL ABSORPTION—PARALLEL FIELDS

Using the perturbation $\mathcal{H}_\omega'$ defined in Eq. (18), we calculate for parallel fields

$$\langle\psi^c|\mathcal{H}_\omega'|\psi^v\rangle=\frac{e}{c}\pi_{12}\cdot\mathbf{A}_\omega\delta_{k_zk_z'}\delta_{n,n'}\int dx(\varphi_{2,n'}{}^c)^*\varphi_{1,n}{}^v. \quad (37)$$

As above, we find the integral by the method of steepest descent, to be given by Eq. (22), with T defined in Eq. (23). T is now independent of k_z and $\mathcal{E}$, and is calculated simply by evaluating the integrals in the exponent. As for crossed fields, $x_m=-(\mathcal{E}_c+\mathcal{E}_v)/2eE$. The absorption coefficient is found to be, for $\mathcal{E}_n-\hbar\omega\ll\mathcal{E}_n$,

$$\alpha(E,H)=\frac{e^3H\mathcal{E}_g^{3/2}}{2nm^{*1/2}c^2\hbar^2\omega}\times\sum_n\frac{\exp\left[-\frac{4m^{*1/2}}{3\hbar eE}\left(\frac{\mathcal{E}_n}{\mathcal{E}_g}\right)^{1/2}(\mathcal{E}_n-\hbar\omega)^{3/2}\right]}{\mathcal{E}_n^{1/2}(\mathcal{E}_n-\hbar\omega)^{1/2}}. \quad (38)$$

We calculate $\alpha(E,0)$ as we did $j(0)$ and obtain the same expression as the $H\to 0$ limit of the crossed-field result [Eq. (25)]. As for $j(H)$, the most important term of $\alpha(E,H)$ is the $n=0$ term. The result for $\alpha(E,H)$ is the same as that obtained by RVL for parabolic bands apart from the definition of $\mathcal{E}_n$, the factor $(\mathcal{E}_n/\mathcal{E}_g)^{1/2}\approx 1$ in the exponent, and a factor $(\mathcal{E}_g/\mathcal{E}_n)^{1/2}$ in the prefactor. For $\alpha(E,H)/\alpha(E,0)$ we obtain for small H the expression of RVL, and note that the exponent is proportional to $(\mathcal{E}_g-\hbar\omega)^{1/2}H/E$ rather than $(\mathcal{E}_g-\hbar\omega)^{5/2}H^2/E^3$ for crossed fields. Note that our result is the first term in an expansion in terms of $(\mathcal{E}_n-\hbar\omega)/\mathcal{E}_n\ll 1$, whereas RVL expand in terms of $(\mathcal{E}_n-\hbar\omega)/\hbar\theta\gg 1$, where $\theta^3=e^2E^2/\hbar m^*$. Thus there is only a limited range of ω for which both expressions hold, and neither calculation holds unless $\hbar\theta\ll\mathcal{E}_n$, or unless the electric field is relatively small.

IX. SUMMARY

We have calculated expressions for the Zener tunneling current and optical-absorption coefficient in the presence of crossed and parallel electric and magnetic fields, using WKB solutions to the two-band model. We have obtained results which are supported by the photon-assisted tunneling experiments in the preceding paper, and by other tunneling experiments. Apart from differences not resolved by these experiments, our results for crossed fields are the same as those obtained by Aronov and Pikus using a quite different treatment of a model identical to ours except for the inclusion of spin effects which do not appear to alter the results significantly. Our results for parallel fields are similar to those obtained for parabolic bands in the preceding paper. Thus we have shown that the main features of electron tunneling in crossed and parallel fields can be adequately explained using a semiclassical WKB treatment which in addition provides a good physical picture of the tunneling process.

ACKNOWLEDGMENT

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APPENDIX A: SOLUTIONS OF THE TWO-BAND EQUATION FOR s AND p BANDS IN CROSSED FIELDS

We calculate the $\mathbf{k}\cdot\mathbf{p}$ equation in the Kohn-Luttinger representation for two coupled bands, an s-like conduction band and a triply degenerate p-like valence band in the presence of crossed electric and magnetic fields. Neglecting spin and the free-electron mass term, we obtain

$$\begin{pmatrix} e\mathbf{E}\cdot\mathbf{r}+\frac{1}{2}\mathcal{E}_g-\mathcal{E} & \pi_{12}^-P_+ & \pi_{12}^+P_- & \pi_{12}^zP_z \\ \pi_{21}^-P_+ & e\mathbf{E}\cdot\mathbf{r}-\frac{1}{2}\mathcal{E}_g-\mathcal{E} & 0 & 0 \\ \pi_{21}^+P_- & 0 & e\mathbf{E}\cdot\mathbf{r}-\frac{1}{2}\mathcal{E}_g-\mathcal{E} & 0 \\ \pi_{21}^zP_z & 0 & 0 & e\mathbf{E}\cdot\mathbf{r}-\frac{1}{2}\mathcal{E}_g-\mathcal{E}\end{pmatrix}\begin{pmatrix} f_s \\ f_+ \\ f_- \\ f_z\end{pmatrix}=0, \quad (A1)$$

where f_i is the envelope function which multiplies the band-edge Bloch function for the ith band, and $P_\pm=P_x\pm iP_y$, with the other terms defined in Eq. (2). The valence band-edge functions corresponding to the envelope functions f_+, f_-, and f_z are the linear combinations of the three p-like states $|X\rangle$, $|Y\rangle$, and $|Z\rangle$ given by $|\pm\rangle=|X\rangle\pm i|Y\rangle$ and $|z\rangle=|Z\rangle$, which are eigenfunctions of the spin-orbit interaction. The functions $|X\rangle$, $|Y\rangle$, and $|Z\rangle$ transform as the coordinates along the cubic axes. Equation (A1) is an extension of an equation obtained by Yafet[14] for a magnetic field alone.

Assuming an isotropic mass with $(\pi_{12})^2=\mathcal{E}_g/2m^*$, Eq. (A1) can be solved for f_s (not for f_+, f_-, or f_z explicitly) since each of the last three equations couples f_s with only one of the f_+, f_-, and f_z. The complete solution can be written, for crossed-fields,

$$\psi=\begin{pmatrix} f_s \\ f_+ \\ f_- \\ f_z\end{pmatrix}=\begin{pmatrix} |eEx+\mathcal{E}_g/2+\mathcal{E}|^{1/2}\exp[i(k_yy+k_zz)]\varphi(x) \\ (\mathcal{E}_g/6m^*)^{1/2}(p_+f_s)/(eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E}) \\ (\mathcal{E}_g/6m^*)^{1/2}(p_-f_s)/(eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E}) \\ (\mathcal{E}_g/6m^*)^{1/2}(p_zf_s)/(eEx+\frac{1}{2}\mathcal{E}_g+\mathcal{E})\end{pmatrix}, \quad (A2)$$

where $\varphi(x)$ obeys Eq. (3a), which we rewrite in the

[14] Y. Yafet, Phys. Rev. **115**, 1172 (1959); R. Bowers and Y. Yafet, *ibid.* **115**, 1165 (1959).

form

$$\left[\frac{p^2}{2m^*}-2e\left(\frac{\mathcal{E}}{\mathcal{E}_g}E-\frac{\hbar k_y}{2m^*c}H\right)x-\frac{e^2E_{\text{eff}}^2}{\mathcal{E}_g}x^2\right]\varphi(x)+\frac{3\hbar^2}{8m^*}\frac{\varphi(x)}{[x+(\mathcal{E}+\frac{1}{2}\mathcal{E}_g)/eE]^2}=[(1/\mathcal{E}_g)(\mathcal{E}^2-\tfrac{1}{4}\mathcal{E}_g^2)-(h^2/2m^*)\times(k_y^2+k_z^2)]\varphi(x). \quad \text{(A3)}$$

This is the equation obtained by Zawadzki and Lax for two-spherical bands. The difficulty with this equation for our purposes is that it is only soluble when the singular term is small, that is, when solving for the conduction band to the right of the barrier. It is necessary to solve explicitly for the large component of the valence-band solution to the left of the barrier in order to obtain a perturbation which has no singularity in that region. Therefore, for the tunneling problem we have assumed no explicit character for the bands. This case of interacting simple bands occurs in real semiconductors such as Bi and PbTe.

APPENDIX B: PERTURBATIONS CAUSING THE TUNNELING CURRENT

The approximate equations we solve are

$$(\mathcal{H}-\mathcal{H}_c')\psi^c=\mathcal{E}_c\psi^c,\quad (\mathcal{H}-\mathcal{H}_v')\psi^v=\mathcal{E}_v\psi^v, \quad \text{(B1)}$$

where $\mathcal{H}$, $\mathcal{H}_c'$, and $\mathcal{H}_v'$ are defined in Eqs. (1), (3a), (4a), and (5). We seek the first-order corrections to the orthogonal functions ψ^c and ψ^v using the correct equation

$$\mathcal{H}\psi_1^{c,v}=\mathcal{E}_1^{c,v}\psi_1^{c,v}, \quad \text{(B2)}$$

where

$$\psi_1^{c,v}=\psi^{c,v}+A_{c,v}\psi^{v,c},\quad \mathcal{E}_1^{c,v}=\mathcal{E}^{c,v}+\Delta\mathcal{E}^{c,v}. \quad \text{(B3)}$$

Assuming the system is initially in a state ψ^v, we find, substituting Eq. (B3) into Eq. (B2) and using Eq. (B1) and the orthogonality of ψ^c and ψ^v, that the perturbed state ψ_1^v is given by

$$\psi_1^v=\psi^v+\sum_c\frac{\langle\psi^c|\mathcal{H}_v'|\psi^v\rangle}{\mathcal{E}_v-\mathcal{E}_c}\psi^c, \quad \text{(B4)}$$

so that the transition rate from the states ψ^v to the states ψ^c is given by the golden rule as

$$W_{vc}=\frac{2\pi}{\hbar}\sum_{v,c}|\langle\psi^c|\mathcal{H}_v'|\psi^v\rangle|^2\delta(\mathcal{E}_c-\mathcal{E}_v).$$

Similarly, the rate starting with the conduction-band solution is found to be

$$W_{cv}=\frac{2\pi}{\hbar}\sum_{v,c}|\langle\psi^v|\mathcal{H}_c'|\psi^c\rangle|^2\delta(\mathcal{E}_c-\mathcal{E}_v).$$

A calculation of the matrix elements shows that $W_{vc}=W_{cv}$. Thus we are justified in using different perturbations $\mathcal{H}_c'$ and $\mathcal{H}_v'$ for the conduction and valence bands since they leave the approximate solutions orthogonal and the transition rate is the same whether we start with the conduction-band or valence-band solution.

APPENDIX C: EVALUATION OF THE TRANSMISSION COEFFICIENT FOR CROSSED FIELDS

In order to calculate the tunneling current and optical-absorption coefficient in crossed fields we must calculate integrals $\int dk_ydk_zf(k_y,k_z)T(k_\perp)$, where $f(k_y,k_z)$ is a slowly varying function and $T(k_\perp)$ is the rapidly varying exponential defined in Eq. (23), and, for the special (Zener) case $\mathcal{E}_c=\mathcal{E}_v=\mathcal{E}$ or $\hbar\omega=0$, in Eq. (14). We calculate the integral by the method of steepest descent, and evaluate $f(k_\perp)$ at the saddle point k_{y0}, k_{z0} of the exponent of $T(k)$. The integral is then $f(k_{y0},k_{z0})$ times the transmission coefficient

$$T\equiv\int dk_ydk_zT(k_\perp), \quad \text{(C1)}$$

where

$$T(k_\perp)\equiv\exp[-\gamma],$$

and

$$\gamma\equiv\frac{2}{\hbar}\int_{x_v}^{x_m}|p_v(x)|dx+\frac{2}{\hbar}\int_{x_m}^{x_c}|p_c(x)|dx, \quad \text{(C2)}$$

where $p_c(x)$ and $p_v(x)$ are obtained from Eq. (6) with $\mathcal{E}=\mathcal{E}_c$, $\mathcal{E}_v$. Also, at $x=x_m$, $|p_c(x)|=|p_v(x)|\equiv|p(k_\perp)|$. We find, for i referring to the conduction or valence band,

$$|p_i(x)|=(2m^*/\mathcal{E}_g)^{1/2}eE_{\text{eff}}|y_i^2-a_i^2|^{1/2}=(2m^*/\mathcal{E}_g)^{1/2}eE_{\text{eff}}(a_i^2-y_i^2)^{1/2}, \quad \text{(C3)}$$

where, for $\mathcal{E}_c\equiv\mathcal{E}=\mathcal{E}_v+\hbar\omega$, and defining, for $H\neq0$,

$$\begin{aligned}\mathcal{E}'&\equiv\mathcal{E}-\hbar k_ycE/H,\\ \mathcal{E}_g'^2&\equiv\mathcal{E}_g^2+2\mathcal{E}_g\hbar^2k_z^2/m^*,\\ a_c^2&\equiv(2eE_{\text{eff}}^2)^{-2}[\mathcal{E}_g'^2E_{\text{eff}}^2+4\mathcal{E}'^2H_{\text{eff}}^2],\\ a_v^2&\equiv(2eE_{\text{eff}}^2)^{-2}[\mathcal{E}_g'^2E_{\text{eff}}^2+4(\hbar\omega-\mathcal{E}')^2H_{\text{eff}}^2],\end{aligned} \quad \text{(C4)}$$

and

$$y_{c,v}\equiv x+\mathcal{E}_{c,v}E/eE_{\text{eff}}^2-\mathcal{E}_g\hbar k_yH/2eE_{\text{eff}}^2m^*c. \quad \text{(C5)}$$

For the matching point x_m, we find

$$x_m=-(\mathcal{E}_c+\mathcal{E}_v)/2eE=-(2\mathcal{E}-\hbar\omega)/2eE. \quad \text{(C6)}$$

The integrals in γ can be evaluated exactly, to give

$$\gamma=\text{I}+\text{II}+\text{III},$$

where

$$\mathrm{I}=-\frac{\hbar\omega}{2\hbar eE_{\mathrm{eff}}^2}\left(\frac{2m^*}{\mathcal{E}_g}\right)^{1/2}[(\mathcal{E}_g'^2-\hbar^2\omega^2)E^2+(2\mathcal{E}'-\hbar\omega)^2H_{\mathrm{eff}}^2]^{1/2},$$

$$\mathrm{II}=\frac{1}{4\hbar eE_{\mathrm{eff}}^3}\left(\frac{2m^*}{\mathcal{E}_g}\right)^{1/2}[\mathcal{E}_g'^2E_{\mathrm{eff}}^2+4(\hbar\omega-\mathcal{E}')^2H_{\mathrm{eff}}^2]\times\sin^{-1}\left\{\frac{E_{\mathrm{eff}}[(\mathcal{E}_g'^2-\hbar^2\omega^2)E^2+(2\mathcal{E}'-\hbar\omega)^2H_{\mathrm{eff}}^2]^{1/2}}{E[E_g'^2E_{\mathrm{eff}}^2+4(\hbar\omega-\mathcal{E}')^2H_{\mathrm{eff}}^2]^{1/2}}\right\},$$

$$\mathrm{III}=\frac{1}{4\hbar eE_{\mathrm{eff}}^3}\left(\frac{2m^*}{\mathcal{E}_g}\right)^{1/2}[\mathcal{E}_g'^2E_{\mathrm{eff}}^2+4\mathcal{E}'^2H_{\mathrm{eff}}^2]\times\sin^{-1}\left\{\frac{E_{\mathrm{eff}}[(\mathcal{E}_g'^2-\hbar^2\omega^2)E^2+(2\mathcal{E}'-\hbar\omega)^2H_{\mathrm{eff}}^2]^{1/2}}{E[\mathcal{E}_g'^2E_{\mathrm{eff}}^2+4\mathcal{E}'^2H_{\mathrm{eff}}^2]^{1/2}}\right\}. \quad \text{(C7)}$$

We integrate $\exp\{-\gamma\}$ over k_y and k_z by the method of steepest descent, evaluating the k_y integral first. Since $\mathcal{E}'=\mathcal{E}-\hbar k_y cE/H$, we use $dk_y=-(H/\hbar cE)d\mathcal{E}'$. The saddle point $\mathcal{E}'$, where $\partial\gamma/\partial\mathcal{E}'=0$, is

$$\mathcal{E}'=\tfrac{1}{2}\hbar\omega. \quad \text{(C8)}$$

Then the integral is

$$T=\int dk_ydk_z\exp[-\gamma(\mathcal{E}',\mathcal{E}_g')]$$

$$=-(H/\hbar cE)\int\int d\mathcal{E}'dk_z\exp[-\gamma(\mathcal{E}',\mathcal{E}_g')]$$

$$=-(2\pi)^{1/2}(H/\hbar cE)\int dk_z\exp[-\gamma(\tfrac{1}{2}\hbar\omega,\mathcal{E}_g')]\times\left[\frac{\partial^2}{\partial\mathcal{E}'^2}\gamma(\tfrac{1}{2}\hbar\omega,\mathcal{E}_g')\right]^{-1/2}. \quad \text{(C9)}$$

The slowly varying function $\partial^2\gamma/\partial\mathcal{E}'^2$ is taken outside the integral, and it and the exponent are evaluated at the saddle point of the k_z integral which is the point where $\partial\gamma/\partial k_z=0$. This is at $k_z=0$ or

$$\mathcal{E}_g'=\mathcal{E}_g. \quad \text{(C10)}$$

The exponent of the final expression for T is

$$\gamma(\tfrac{1}{2}\hbar\omega,\mathcal{E}_g)=(1/\hbar eE_{\mathrm{eff}}^3)(m^*/2\mathcal{E}_g)^{1/2}\times\left\{(\mathcal{E}_g^2E_{\mathrm{eff}}^2+\hbar^2\omega^2H_{\mathrm{eff}}^2)\times\sin^{-1}\left[\frac{(\mathcal{E}_g^2-\hbar^2\omega^2)^{1/2}E_{\mathrm{eff}}}{(\mathcal{E}_g^2E_{\mathrm{eff}}^2+\hbar^2\omega^2H_{\mathrm{eff}}^2)^{1/2}}\right]-\hbar\omega EE_{\mathrm{eff}}(\mathcal{E}_g^2-\hbar^2\omega^2)^{1/2}\right\}. \quad \text{(C11)}$$

This exponent is exactly the same as that obtained by AP.

For the tunneling case $\mathcal{E}_c=\mathcal{E}_v$, $\hbar\omega=0$, we find the transmission coefficient to be

$$T_{\omega=0}=\frac{eE_{\mathrm{eff}}^2}{\hbar E}\left(\frac{2m^*}{\mathcal{E}_g}\right)^{1/2}\times\exp\left\{-\frac{\pi}{2\hbar eE_{\mathrm{eff}}}(\tfrac{1}{2}m^*)^{1/2}\mathcal{E}_g^{3/2}\right\}, \quad \text{(C12)}$$

and for the optical absorption, $\mathcal{E}_g-\hbar\omega\ll\mathcal{E}_G$,

$$T\approx(\pi eE_{\mathrm{eff}}/2\hbar)[m^*/(\mathcal{E}_g-\hbar\omega)]^{1/2}\times\exp\left\{-\frac{4(\mathcal{E}_g-\hbar\omega)^{3/2}m^{*1/2}}{3\hbar eE}\times\left[1-\frac{(\mathcal{E}_g-\hbar\omega)}{20\mathcal{E}_g}\frac{(3E^2-8H_{\mathrm{eff}}^2)}{E^2}\right]\right\}. \quad \text{(C13)}$$

The above development is not valid for $H=0$. For this case we define $\mathcal{E}'=\mathcal{E}$ and $\mathcal{E}_g'^2=\mathcal{E}_g^2+\hbar^2k_\perp^2/2m^*$. The saddle points of the integrals are at $k_y=k_z=0$, and the resulting expression for T is the same as that found by letting $H\to 0$ in the expressions for T calculated above.