Spatially Inhomogeneous Phonon Amplification in Solids

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A theory is presented for acoustic domains of both the stationary and propagating kinds. In the latter case, a mechanism of domain generation is also given. The basic assumption of the theory is that the spatially inhomogeneous amplification can be described by the stimulated emission of incoherent phonons within individual, macroscopic volume elements. By taking appropriate moments of the Boltzmann equations of the electron-phonon system, generalized to include drift terms, a set of three coupled integrodifferential equations is obtained. With suitable boundary conditions, these equations determine the three unknowns—drift velocity, phonon number, and macroscopic electric field—as a function of position and time. Some applications of the equations are given, and possible future applications are outlined.

I. INTRODUCTION

IN recent years, the internal generation of acoustic flux and its physical consequence have been studied by a number of workers. As is well known, this phenomenon was first observed by Smith,[1] who noted a marked saturation in the current-voltage characteristic of semiconducting CdS at a field at which the Ohmic drift velocity was very nearly equal to the velocity of sound. By placing a transducer at the downstream contact, McFee[2] later observed, among other things, that the current saturation was indeed accompanied by the buildup of acoustic noise. It is relevant to the present paper to note that the internally generated sound appeared incoherent. A physical argument for the saturation based on reversed acoustoelectric currents has been given by Hutson[3] in the classical regime. On the other hand, Yamashita and Nakamura[4] have presented a theory for the current saturation and incubation times based on the Boltzmann equations for the electron-phonon system. It is to be emphasized that both of these latter theories assume a spatially uniform situation.

However, recent experimental observation has shown that the assumption of spatial uniformity is unrealistic, and, in fact, that the spatial nonuniformity of the macroscopic electric field and flux distributions is an essential feature of the problem. Specifically, it has been shown that the large-scale current oscillations are associated with the generation and propagation of finite acoustic domains. This has been seen in CdS by Many and Balberg[5] and Hadyl and Quate,[6] among others. Domain motion has also been studied in a number of III-V compounds by Bray.[7] Moreover, the spatial nonuniformity is also characteristic of the steady-state situation; stationary domains have been observed under various conditions by Many and Balberg.[5] McFee and Tien,[8] and Maines and Paige.[9]

A second aspect of this problem is that the amplified flux is initially incoherent, being either the thermal phonon distribution or a shock-induced excitation. In the present paper, it is assumed that this incoherence remains up to the highest phonon levels of interest. Indeed, there is no definitive evidence that the internally generated flux becomes coherent (in the sense that one can define a classical strain field as a function of position and time). This is to be contrasted with the amplification of a coherently impressed sound wave.[10]

The approach of the present paper is to treat the effects of spatial dependence and incoherence in terms of the Boltzmann equations of the electron-phonon system, generalized to allow for the possibility that the distribution functions vary spatially over macroscopic volume elements.[11] This approach is perhaps better suited for the higher mobility III-V compounds studied by Bray[7] (p-GaSb, n-GaAs, n-InSb) for which[12] $ql>1$. However, in the absence of a classical incoherent theory of comparable simplicity,[13] it is assumed that the present theory is also applicable to CdS for which $ql<1$.

II. THEORY

As discussed in the Introduction, we take as the starting point of the present paper the coupled Boltzmann equations of the electron-phonon system, in which the distribution functions are allowed to be slowly varying

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[10] A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961); D. L. White, J. Appl. Phys. **33**, 2547 (1962).
[11] The conditions on the size of such elements will be discussed later in the text.
[12] Since the amplified frequencies have not been directly measured, this condition can only be inferred. Thus, at 77°K, for p-GaSb ($\mu=4300$ cm^2 V^{-1} sec^{-1}) and n-GaAs ($\mu=8150$ cm^2 V^{-1} sec^{-1}), we estimate $l\approx 800$ Å. The $ql=1$ condition corresponds to $\omega\approx 3\times 10^{10}$ sec^{-1}, which is to be compared with the classical frequency at maximum gain ($n=10^{16}$ cm^{-3}) $\omega_{\max}=(\omega_c\omega_D)^{1/2}\approx 10^{11}$ (see Ref. 10).
[13] An alternate approach would be the classical, incoherent theory of Gurevich and co-workers: e.g., V. L. Gurevich, Zh. Eksperim. i. Teor. Fiz. **46**, 598 (1964); **47**, 1291 (1964) [English transls.: Soviet Phys.—JETP **19**, 407 (1964); **20**, 873 (1965)].

functions of position. Specifically, the position is understood to be defined over volume elements large compared to the electronic mean free path and phonon wavelengths of interest, but small compared to the domain size. For representative values of the parameters, it is readily verified that these conditions are simultaneously realizable.[14]

Assuming that all spatial variations occur in the direction of the applied electric field, the coupled Boltzmann equations read

$$\frac{\partial f_k(\mathbf{r},t)}{\partial t}=\frac{eF}{\hbar}\frac{\partial f_k(\mathbf{r},t)}{\partial k_x}+\left(\frac{\partial f_k(\mathbf{r},t)}{\partial t}\right)_{ep}-(v_k)_x\frac{\partial f_k(\mathbf{r},t)}{\partial x}, \quad (2.1)$$

$$\frac{\partial N_q(\mathbf{r},t)}{\partial t}=\left(\frac{\partial N_q(\mathbf{r},t)}{\partial t}\right)_{ep}+\left(\frac{\partial N_q(\mathbf{r},t)}{\partial t}\right)_{pp}-s_x(q)\frac{\partial N_q(\mathbf{r},t)}{\partial x}. \quad (2.2)$$

Here f_k and N_q are the electron and phonon distribution functions, respectively. F is the external electric field applied in the x direction, $\mathbf{v}_k$ is the (group) velocity of an electron in Bloch state $\mathbf{k}$,and $s(\mathbf{q})$ is the sound velocity of a phonon of momentum $\mathbf{q}$. Further, the subscripts *ep* and *pp* refer to the rates of change due to collisions of electrons with phonons and phonons with phonons, respectively. Specifically,

$$\left(\frac{\partial f_k}{\partial t}\right)_{ep}=\frac{2\pi}{\hbar}\sum_q C_q^2\{[f_{k+q}(N_q+1)-f_kN_q]\times\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)+[f_{k-q}N_q-f_k(N_q+1)]\times\delta(\epsilon_{k-q}-\epsilon_k+\hbar\omega_q)\}, \quad (2.3)$$

and

$$\left(\frac{\partial N_q}{\partial t}\right)_{ep}=\frac{2\pi}{\hbar}\sum_{\mathbf{k}} C_q^2\{f_{k+q}(N_q+1)-f_kN_q\}\times\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q), \quad (2.4)$$

where ϵ_k is the energy of Bloch state $\mathbf{k}$; C_q^2, the electron-phonon coupling constant, will be specified later. For the phonon-phonon term of (2.2), we will make a relaxation-time approximation later.

As a matter of convenience, and in order to facilitate comparison with the spatially uniform theory, we generalize the work of Yamashita and Nakamura.[4] In particular, it is assumed that the distribution functions can be written

$$f_k=f_k^0(\epsilon_k)-\hbar v_d(x,t)k_x(\partial f_k^0/\partial\epsilon_k), \quad (2.5)$$

$$N_q=N_q^0+\xi_q(x,t), \quad (2.6)$$

where

$$f_k^0(\epsilon_k)=(2\pi)^3n(x)(h^2/2\pi mk_BT)^{3/2}e^{-\epsilon_k/k_BT} \quad (2.7)$$

is the local Maxwell distribution, $v_d(x,t)$ is the local drift velocity,[15] N_q^0 is the equilibrium Planck distribution, and $\xi_q(x,t)$ is the excess phonon population of mode q.

Of particular interest are the additional effects of electron drift, given by the last term on the right hand side of (2.1). Using (2.5), we obtain

$$-(v_k)_x\frac{\partial f_k}{\partial x}=-(v_k)_x\frac{\partial f_k^0}{\partial x}+(v_k)_x\frac{\partial f_k^0}{\partial\epsilon_k}k_x\times\left[\frac{\partial v_d(x,t)}{\partial x}+v_d(x,t)\frac{\partial\ln n(x)}{\partial x}\right]. \quad (2.8)$$

The second term arises from the spatial gradient of the anisotropic part of the distribution. Ordinarily, in the standard Boltzmann-equation treatments of electronic thermal conduction and particle diffusion,[16] this term is neglected as being of higher order for slow spatial variation. However, in the present case it must be included, since the spatial variation of v_d must, in principal, be taken into account.[17] For the first term, we have[18]

$$\frac{\partial f_k^0}{\partial x}=\left(\frac{\partial\ln n(\mathrm{x})}{\partial x}\right)f_k^0=-\left(\frac{\partial\ln n(x)}{\partial x}\right)(k_BT)\left(\frac{\partial f_k^0}{\partial\epsilon_k}\right), \quad (2.9)$$

the second equality following from the form of (2.7). The other terms of (21.) are treated just as in Ref. 4. Substituting (2.5) and (2.6) into (2.1) and incorporating (2.8) and (2.9), we obtain

$$\frac{\partial v_d(x,t)}{\partial t}k_x\frac{\partial f_k^0}{\partial\epsilon_k}=-\frac{v_d(x,t)-v_0(x,t)}{\tau(\epsilon_k)}k_x\frac{\partial f_k^0}{\partial\epsilon_k}-\frac{1}{\hbar}\left(\frac{\partial f_k}{\partial t}\right)_\xi-\left(\frac{k_BT}{m}\right)\left(\frac{\partial\ln n(x)}{\partial x}\right)k_x\frac{\partial f_k^0}{\partial\epsilon_k}-(v_k)_x\frac{\partial f_k^0}{\partial\epsilon_k}k_x\times\left[\frac{\partial v_d(x,t)}{\partial x}+v_d(x,t)\frac{\partial\ln n(x)}{\partial x}\right], \quad (2.10)$$

[14] Thus, for the III-V compounds, $\lambda_{ph}=(2\pi/q)\lesssim l\sim10^{-5}$ cm, while typical domain sizes are $30\,\mu\lesssim D\lesssim0.5$ mm.

[15] Although Yamashita and Nakamura initially assume a more general form for the distribution function, namely $v=v(\epsilon_k,t)$, they subsequently neglect the energy dependence, in which case it becomes just the common drift velocity of the distribution. In the present paper, this assumption is made from the outset.

[16] A. H. Wilson, *Theory of Metals* (Cambridge University Press, New York, 1958), 2nd ed., Chap. 1.

[17] This follows from the existence of a nonuniform electric-field distribution; such a variation in v_d would be required to ensure current continuity in the steady state, for example.

[18] In the present case, we allow for a spatial variation of the equilibrium distribution function only via its dependence on a spatially varying carrier concentration $n(x)$. In general, a temperature gradient might also be present, in which case the term $(\epsilon_k/k_BT^2)f_k^0(\partial T/\partial x)$ would add to the right-hand side of (2.9). However, we neglect the possibility of a temperature gradient in the present work.

where

$$v_0(x,t) = -eF(x,t)\tau(\epsilon_k)/m.$$

The quantity $(\partial f_k/\partial t)_\xi$ is the rate of change due to collisions with excess phonons, and is given by

$$\left(\frac{\partial f_k}{\partial t}\right)_\xi = \frac{2\pi}{\hbar}\sum_{\mathbf{q}} C_q{}^2\xi_q\{[f_{k+q}-f_k] \times\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)+[f_{k-q}-f_k] \times\delta(\epsilon_{k-q}-\epsilon_k+\hbar\omega_q)\}. \quad (2.11)$$

Following Ref. 4, the first moment of (2.10) is taken by multiplying through by $k_x\delta(\epsilon-\epsilon_k)$ and summing over **k**. For the left-hand side and first and third terms of the right-hand side of this equation, one must take sums of the form [for an arbitrary $g(\epsilon_k)$]

$$\sum_{\mathbf{k}} k_x\delta(\epsilon-\epsilon_k)k_x\frac{\partial f_k{}^0}{\partial\epsilon_k}g(\epsilon_k) = \frac{V}{(2\pi)^3} \times\int_0^\infty (2\pi)\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_k{}^{1/2}d\epsilon_k\frac{k^2}{3}\delta(\epsilon-\epsilon_k)g(\epsilon_k) = \frac{V}{12\pi^2}\left(\frac{2m}{\hbar^2}\right)^{5/2}\epsilon^{3/2}\frac{\partial f_k{}^0}{\partial\epsilon}g(\epsilon),$$

having replaced the sum by an integration, and having introduced the usual density of states for a parabolic band.

In taking the *first* moment, no contribution is obtained from the anisotropic part of the drift term, the last term in (2.10). This follows from the fact that the summand is odd in k_x:

$$-\sum_{\mathbf{k}} k_x\delta(\epsilon-\epsilon_k)(v_k)_x\frac{\partial f_k{}^0}{\partial\epsilon_k}k_x = -\frac{\hbar}{m}\sum_{\mathbf{k}} k_x{}^3\frac{\partial f_k{}^0}{\partial\epsilon_k}\delta(\epsilon-\epsilon_k)=0.$$

For the first moment of $(\partial f_k/\partial t)_\xi$, we use the result of Ref. 4, namely,

$$I_\xi{}^{(1)} = \sum_{\mathbf{k}}\frac{1}{\hbar}\left(\frac{\partial f_k}{\partial t}\right)_\xi k_x\delta(\epsilon-\epsilon_k) = -\frac{V}{2\pi\hbar}\left(\frac{m}{\hbar^2}\right)^2 \times\left(-\frac{\partial f_k{}^0}{\partial\epsilon_k}\right)\sum_q qC_q{}^2\xi_q\cos\theta_q[v_d\cos\theta_q-s], \quad (2.12)$$

where θ_q is the angle between the direction of the phonon of momentum **q** and the applied field.

Assembling results, the first moment of (2.10) becomes

$$\frac{\partial v_d(x,t)}{\partial t} = -\frac{v_d(x,t)-v_0}{\tau(\epsilon)} - \frac{3}{2(2m)^{1/2}\epsilon^{3/2}} \times\sum_q qC_q{}^2\cos\theta_q\{v_d(x,t)\cos\theta_q-s\}\xi_q - \left(\frac{k_BT}{m}\right)\left(\frac{\partial\ln n(x)}{\partial x}\right). \quad (2.13)$$

Averaging this equation over a Maxwell distribution, we obtain

$$\frac{\partial v_d(x,t)}{\partial t} = -\frac{v_d(x,t)-v_0}{\tau} - \frac{m}{2\pi\hbar^3 n(x)} \times\sum_q qC_q{}^2 f_0(\epsilon_q)\cos\theta_q\{v_d\cos\theta_q-s\}\xi_q - \frac{1}{n(x)\tau}D\frac{\partial n(x)}{\partial x}, \quad (2.14)$$

where

$$\epsilon_q = (\hbar^2/2m)(\tfrac{1}{2}q)^2,$$

and

$$D = \tfrac{1}{3}(v_{\rm th}{}^2\tau) \quad (2.15)$$

is the standard diffusion constant for a Maxwell-Boltzmann gas.

Hence, the sole modification of the first moment of the Boltzmann equation is to add a simple diffusion term to the right-hand side.

The treatment of the Boltzmann equation for the phonons, (2.2), is straightforward. Substituting (1.5) and (2.6), one obtains[19]

$$\left(\frac{\partial}{\partial t}+s\cos\theta_q\frac{\partial}{\partial x}\right)\xi_q(x,t) = \frac{VC_q{}^2}{2\pi}\left(\frac{m}{\hbar^2}\right)^2 f^0(\epsilon_q) \times(v_d(x,t)\cos\theta_q-s)\xi_q(x,t) - \frac{\xi_q(x,t)}{\tau_q}. \quad (2.16)$$

The modification here is to replace the time derivative by the total convective derivative for the particular mode in question.

Anticipating that $n(x)$ will later be related to $F(x)$ via Maxwell's equations, it is seen that (2.14) and (2.16) constitute only two equations in the three unknowns: v_d, ξ_q, and F. The third required relation is obtained by taking an even moment of (2.10), the simplest being the zeroth moment.[20]

Multiplying (2.10) to the left by $\delta(\epsilon-\epsilon_k)$ and summing over **k**, it is first noted that

$$\sum_{\mathbf{k}} k_x\left\{\frac{\partial v_d(x,t)}{\partial t}+\frac{v_d(x,t)-v_0}{\tau}-\frac{k_BT}{m}\frac{\partial\ln n(x)}{\partial x}\right\} \times\delta(\epsilon-\epsilon_k)=0, \quad (2.17)$$

and we are left with

$$0 = -I_\xi{}^{(0)} - \sum_{\mathbf{k}}\delta(\epsilon-\epsilon_k)(v_k)_x k_x\frac{\partial f_k{}^0}{\partial\epsilon_k} \times\left[\frac{\partial v_d(x,t)}{\partial x}+v_d(x,t)\frac{\partial\ln n(x)}{\partial x}\right], \quad (2.18)$$

[19] In this equation, only the linear phonon loss term is retained; the nonlinear phonon loss terms are neglected. Also, the so-called "spontaneous emission" source term [$\sim(N_q{}^0+1)$] is neglected for the reasons discussed in Ref. 4.

[20] Any even moment would suffice. In particular, the second moment, giving an energy balance, is equivalent. However, the zeroth moment is the simplest to consider.

where, in analogy with (2.12),

$$I_\xi^{(0)}=\frac{1}{\hbar}\sum_{\mathbf{k}}\delta(\epsilon-\epsilon_k)\left(\frac{\partial f_k^0}{\partial t}\right)_\xi. \tag{2.19}$$

This quantity is evaluated in Appendix A. It is shown there that one cannot make the usual high-temperature approximation for an acoustic mode ($\hbar\omega_q/k_BT=0$), but that the first nonvanishing contribution to $I_\xi^{(0)}$ occurs to first order in ($\hbar\omega_q/k_BT$). The result is

$$I_\xi^{(0)}=-\frac{V}{2\pi\hbar}\left(\frac{m}{\hbar^2}\right)^2\left(-\frac{\partial f_k^0}{\partial\epsilon_k}\right)\sum_q\left(\frac{\hbar qs}{k_BT}\right)\times C_q^2\{v_d(x,t)\cos\theta_q-s\}\xi_q(x,t). \tag{2.20}$$

The calculation of the second term of (2.18) is straightforward. Substituting this result and (2.20) into (2.18), one obtains, after some algebra,

$$0=-\frac{1}{\pi}\frac{ms}{k_BT}\left[\frac{3}{2(2m)^{1/2}\epsilon^{3/2}}\sum_q qC_q^2\xi_q\{v_d(x,t)\cos\theta_q-s\}\right]+\left[\frac{\partial v_d(x,t)}{\partial x}+v_d(x,t)\frac{\partial \ln n(x)}{\partial x}\right], \tag{2.21}$$

where the quantity in the square brackets of the first term is identical, except for the absence of a factor $\cos\theta_q$, with the corresponding term in the first-moment calculation [see (2.13)]. The thermal average of (2.21) is then taken with the result

$$0=\frac{1}{\pi}\left(\frac{ms}{k_BT}\right)\left(\frac{m}{2\pi\hbar^3n(x)}\right)\sum_q qC_q^2f^0(\epsilon_q)\times\{v_d(x,t)\cos\theta_q-s\}\xi_q(x,t)+\left[\frac{\partial v_d(x,t)}{\partial x}+v_d(x,t)\frac{\partial \ln n(x)}{\partial x}\right]. \tag{2.22}$$

The physical significance of (2.22) will be discussed in a later section of the paper in connection with the time-dependent solution.

Finally, it is desirable to express the diffusion term of (2.14) and the last term of (2.22) in terms of the electric field and its gradients. The current density is[21]

$$j=-nev_d(x,t).$$

By differentiation, one obtains

$$\left(\frac{\partial v_d}{\partial x}+v_d\frac{1}{n}\frac{\partial n}{\partial x}\right)=-\frac{1}{ne}\frac{\partial j}{\partial x}. \tag{2.23}$$

But from the equation of continuity

$$\partial j/\partial x+\partial\rho/\partial t=0,$$

and Poisson's equation

$$\partial F/\partial x=(4\pi/\epsilon)\rho,$$

where

$$\rho=-e[n(x)-n_0(x)],$$

with $n_0(x)$ the positive background charge, one obtains

$$\frac{\partial j}{\partial x}=-\frac{\epsilon}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right). \tag{2.24}$$

Also,[22]

$$\frac{\partial n}{\partial x}=-\frac{\epsilon}{4\pi e}\frac{\partial^2F}{\partial x^2}. \tag{2.25}$$

Substituting (2.23) and (2.24) into (2.22) and (2.25) into (2.14), we get our final set of coupled equations. They read

$$\frac{\partial v_d(x,t)}{\partial t}=-\frac{v_d(x,t)-v_0(x)}{\tau}-\frac{m}{2\pi\hbar^3n(x)}\times\sum_q qC_q^2f^0(\epsilon_q)\cos\theta_q\{v_d(x,t)\cos\theta_q-s\}\xi_q(x,t)+\left(\frac{\epsilon}{4\pi n(x)e}\right)\frac{1}{\tau}D\frac{\partial^2F(x,t)}{\partial x^2}, \tag{2.26}$$

$$\left(\frac{\partial}{\partial t}+s\cos\theta_q\frac{\partial}{\partial x}\right)\xi_q(x,t)=\frac{VC_q^2}{2\pi}\left(\frac{m}{\hbar^2}\right)^2f^0(\epsilon_q)\times(v_d(x,t)\cos\theta_q-s)\xi_q(x,t)-\frac{\xi_q(x,t)}{\tau_q}, \tag{2.27}$$

$$\left(\frac{3}{\pi}\frac{s}{v_{th}^2}\right)\left(\frac{m}{2\pi\hbar^3n(x)}\right)\sum_q qC_q^2f^0(\epsilon_q)\times\{v_d(x,t)\cos\theta_q-s\}\xi_q(x,t)+\left(\frac{\epsilon}{4\pi n(x)e}\right)\frac{\partial}{\partial t}\left(\frac{\partial F(x,t)}{\partial x}\right)=0. \tag{2.28}$$

III. APPLICATIONS

Before applying the basic equations (2.26), (2.27), and (2.28) of the previous section, it is convenient to rewrite them in terms of dimensionless variables. Following the notation of Ref. 4, we

(1) replace the space variable x by z,

(2) let x be the wave number in units of the thermal De Broglie wave number:

$$q=\frac{(2mk_BT)^{1/2}}{\hbar}x,$$

[21] This is the current density due to the combined effects of the applied field, the stimulated phonons, and diffusion. As is well known, this, in general, will result in $v_d\neq\mu F$.

[22] It is assumed that the internal fields required to offset the diffusion currents due to a possible inhomogeneous doping $\sim\partial n_0(x)/\partial x$, being of the order of the thermal voltage (k_BT/e), are negligible in comparison with the $\partial n/\partial x$ associated with the macroscopic field distribution. However, the explicit dependence of n on x is, in general, retained in the coefficients of (2.26), (2.27), and (2.28).

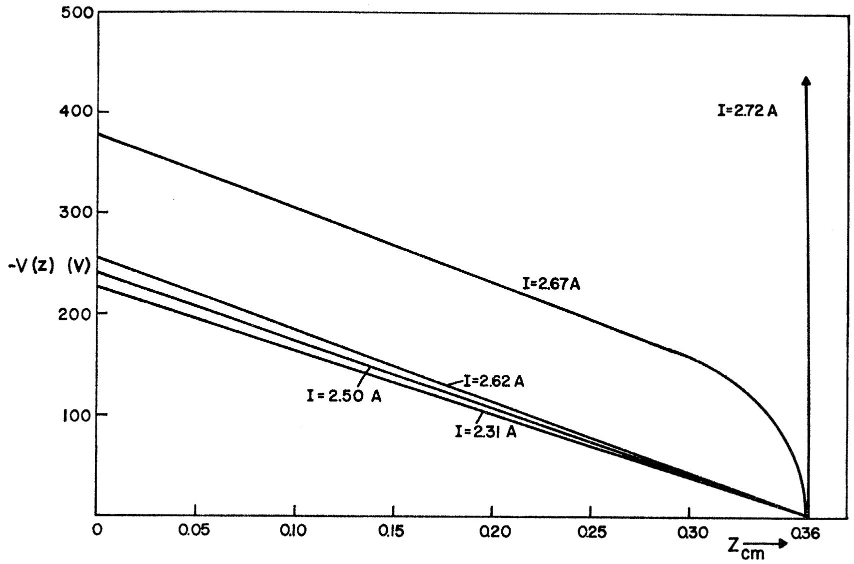

FIG. 1. Calculated potential distribution.

(3) let $y=\cos\theta_q$.

We then obtain the coupled equations:

$$\frac{\partial v_d(z,t)}{\partial t}=A[v_0(z,t)-v_d(z,t)]-B\int_0^\infty dx\,x^2 \times\left(\frac{x^2}{x^2+x_D^2}\right)^2 e^{-x^2/4}\int_{-1}^{1}dy\,y[v_d y-s]\xi(x,y,z,t) +\left(\frac{\epsilon}{4\pi n(z)e}\right)\mathfrak{D}A\frac{\partial^2 F}{\partial z^2}(z,t), \quad (3.1)$$

$$\left(\frac{\partial}{\partial t}+sy\frac{\partial}{\partial z}\right)\xi(x,y,z,t)=C_0\frac{e^{-x^2/4}}{x}\left(\frac{x^2}{x^2+x_D^2}\right)^2 \times\left(\frac{v_d(z,t)}{s}y-\frac{s_x^*}{s}\right)\xi(x,y,z,t), \quad (3.2)$$

$$0=-\frac{3}{\pi}\frac{s}{v_{th}^2}B\int_0^\infty dx\,x^2\left(\frac{x^2}{x^2+x_D^2}\right)^2 \times e^{-x^2/4}\int_{-1}^{1}dy[v_d(z,t)y-s]\xi(x,y,z,t) +\left(\frac{\epsilon}{4\pi n(z)e}\right)\frac{\partial}{\partial t}\left(\frac{\partial F(z,t)}{\partial z}\right), \quad (3.3)$$

where $v_0(z,t)=-\mu F(z,t)$ is the Ohmic drift velocity, $A=\tau^{-1}$ is the reciprocal electron collision time,

$$B=\frac{4\pi e^2\beta^2}{\epsilon_0^2}\frac{3}{\pi^{1/2}}\frac{m}{MNs\hbar^2},$$

where β is the piezoelectric constant, ϵ_0 is the static dielectric constant, N is the number of ions per unit volume, and M is the ionic mass; $\mathfrak{D}$ is the diffusion constant,

$$C_0=\left(\frac{4\pi e\beta}{\epsilon_0}\right)^2\left(\frac{\hbar}{2MN}\right)\frac{(2\pi)^{1/2}}{(k_BT)^{1/2}}n(z),$$

and x_D is the value of x corresponding to the Debye wave vector, $q_D=(4\pi ne^2/\epsilon_0 k_BT)^{1/2}$. In addition,

$$\frac{s_x^*}{s}=1+Dx^3\left(\frac{x^2+x_D^2}{x^2}\right)^2 e^{x^2/4}, \quad (3.4)$$

where

$$D=\frac{D_0Ts^2}{C_0}\frac{2mk_BT}{\hbar^2}; \quad [1/\tau_q=D_0T\omega_q^2]$$

represents the effective sound velocity for a given mode, taking into account linear phonon losses.

Stationary Domains

In this subsection, we calculate the stationary domains observed in CdS by Many and Balberg,[5] and the results are compared with their data. Under stationary conditions, all time derivatives are set equal to zero. Then, integrating (3.2), we obtain[23]

$$\xi(x,y,z)=\xi(x,y,0) \times\exp\left\{\frac{C_0}{s}\frac{e^{-x^2/4}}{x}\left(\frac{x^2}{x^2+x_D^2}\right)^2\frac{1}{y}\left(\frac{v_d}{s}y-\frac{s_x^*}{s}\right)z\right\}. \quad (3.5)$$

[23] Equation (3.2), being of first order, requires a single boundary condition; namely, the flux at the $z=0$ bounding plane $\xi(x,y,0)$, As in Ref. 4, we later take $\xi(x,y,0)=1$, on account of the insensitivity of the amplified flux to the choice of its initial value.

In obtaining this result, we assume that v_d is spatially constant. The validity of this assumption is important and will be examined later.

Equation (3.5) is next substituted into (3.1). In the latter equation, the diffusion term is neglected. This will be justified self-consistently later by estimating the diffusion term from the calculated $F(z)$ and verifying that it is indeed small with respect to the terms retained. With this assumption, the field distribution may be solved for explicitly. Finally, on account of the narrowness of the Čerenkov-like cone, it is assumed that the factor $(1/y)$ in the argument of the exponent of (3.5) may be replaced by unity.[24] Thus, one obtains

$$F(z)=-\frac{v_d}{\mu}-\frac{1}{\mu}\frac{B}{A}\int_0^\infty dx\, x^2\left(\frac{x^2}{x^2+x_D^2}\right)^2 \times e^{-x^2/4}\int_{-1}^{1} dy\, y(v_d y-s) \times\exp\left\{\frac{C_0}{s}\frac{e^{-x^2/4}}{x}\left(\frac{x^2}{x^2+x_D^2}\right)^2\left(\frac{v_d}{s}y-\frac{s_x^*}{s}\right)z\right\}, \quad (3.6)$$

where v_d is simply given by

$$v_d=\frac{I}{n_0eA'}, \quad (3.7)$$

where A' is the sample cross-sectional area.

The values of the parameters used in the calculation are[25]:

$$A=10^{13},$$
$$B=10''\times(0.356),$$
$$C_0=8\times10^6\times(0.356),$$
$$s=1.8\times10^5,$$
$$x_D=10^{-2},$$
$$D=10^2,$$
$$\mu=290,$$
$$\mathfrak{D}=3.3,$$
$$n_0=8\times10^{15}.$$

[24] The reason is entirely a technical consideration which facilitates the computer solution. A sample check for the case $v_d=1.91\times10^5$ cm sec^{-1} showed, for example, that the potential at $z=0.357$ cm was altered by 3 V out of 80 V by the inclusion of the $(1/y)$ factor. The reason for the small change is, of course, the narrowness of the Cerenkov-like cone. Moreover, the inclusion of this factor can clearly only increase the calculated $F(z)$ and $V(z)$. As will be seen later, the deviation from experiment, where it exists, is already in the direction of being too large.

[25] The numerical values are scaled to those of Ref. 4, except for B and C_0. On account of the extreme sensitivity of the exponent to the numerical value of its argument, it was found necessary to adjust C_0, in particular. The fit was made so as to get $V(0)=-375$ V for the case $I=2.67$ A. The required multiplication is by the factor (0.356) indicated. However, this corresponds to an adjustment in β of $(0.356)^{1/2}$, and is well within the uncertainties in the original estimate of Ref. 4.

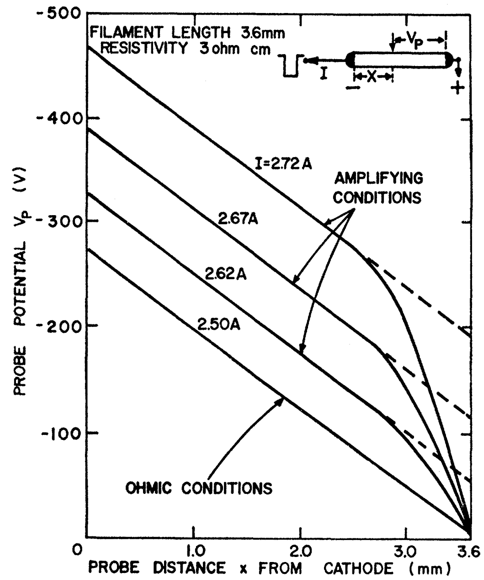

FIG. 2. Experimental potential distribution (Many and Balberg, Ref. 5).

All quantities are in cgs units except μ, which is in cm^2 V^{-1} sec^{-1} in order to obtain F in V cm^{-1}, and V is in volts.

With the above values of the parameters, (3.6) was numerically integrated on a digital computer. Results are presented on a mesh

$$0\leqslant z\leq L=0.36\text{ cm},$$

and for the values of total current,

$$I=2.31\text{ A}\quad(v_d=1.8\times10^5)$$
$$=2.50\text{ A}\quad(v_d=1.95\times10^5)$$
$$=2.67\text{ A}\quad(v_d=2.08\times10^5)$$
$$=2.72\text{ A}\quad(v_d=2.12\times10^5),$$

corresponding to the experimental conditions of Many and Balberg.[5]

The calculated potential distributions are shown in Fig. 1. These are to be compared with the experimental results shown in Fig. 2, obtained by potential probe measurements.[5] The following features are noted:

(1) The distributions for $I=2.31$ A, 2.50 A are quite linear, corresponding to Ohmic conditions. The steady phonon flux is negligible for these cases.

(2) The case $I=2.62$ A barely shows a departure from linearity near the anode. This is more clearly seen in Fig. 3.

(3) The case $I=2.67$ A, for which the fit was made, compares closely with the experimental results, except that (a) $V_{\text{calc}}(0)=375V</V_{\text{exp}}(0)/\lesssim400$ V; C_0 could not be so finely adjusted to obtain a precise fit.[25] (b) The calculated break point from linearity occurs nearer the anode by about 0.02 cm out of a sample length of

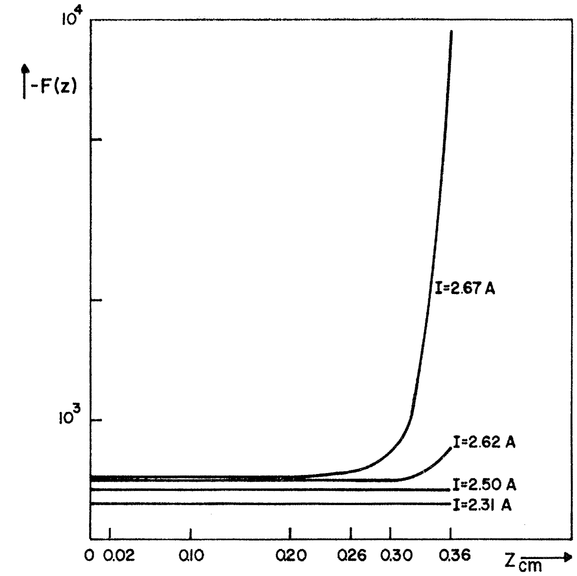

FIG. 3. Calculated field distribution.

0.36 cm. (c) Most important, the potential distribution for the case $I=2.72$ A rises precipitously from the anode as indicated, and lies far above the experimental points.

To understand the source of this latter difficulty, the calculated electric field distributions are shown in Fig. 3. These are quite Ohmic from the cathode to near the anode, where they rise rapidly because of the large concentration of phonon flux bordering the anode. The variation of field suggests that the concomitant space charge be estimated and compared with the equilibrium concentration n_0.

Estimating $F'(z)$ by its finite difference, it is found that

$$-\frac{\epsilon}{4\pi e}F'(z=0.35\text{ cm})\cong 2\times10^{12},\quad I=2.67\text{ A}$$
$$\cong 10^{14},\qquad I=2.72\text{ A}$$

to be compared with $n_0=8\times10^{15}$ cm^{-3}. In the latter case, the space charge amounts to 1/80 of n_0. In view of the sensitivity of the exponent to adjustment of C_0 by even this small amount, it is suggested that, instead of (3.6) and (3.7), the correct formulation of the problem is

$$F(z)=-\frac{v_d(z)}{\mu}-\frac{1}{\mu}\frac{B}{A}\int_0^\infty dx\,x^2\left(\frac{x^2}{x^2+x_D^2}\right)^2 \times e^{-x^2/4}\int_{-1}^{1}dy\,y[v_d(z)y-s]\exp\left\{\frac{C_0}{s}\frac{e^{-x^2/4}}{x} \times\left(\frac{x^2}{x^2+x_D^2}\right)\int_0^z dz'\left(\frac{v_d(z')}{s}y-\frac{s_x^*}{s}\right)\right\} \quad (3.8)$$

and

$$I=\text{const}=eA'\left(n_0-\frac{\epsilon}{4\pi e}\frac{\partial F(z)}{\partial z}\right)v_d(z). \quad (3.9)$$

Instead of an explicit solution, (3.8) and (3.9) correspond to two coupled equations for $v_d(z)$ and $F(z)$; the solution has not at present been carried out for this case. Both the occurrence of the break point too near the anode and the oversaturated current-voltage characteristic (easily inferred from Figs. 1 and 2) are manifestations of not allowing for such a space-varying v_d and amplification factor.

Finally, it is verified that the diffusion term is indeed negligible in all cases. Estimating the second derivative by finite differences, it is found that

$$\frac{\epsilon}{4\pi n_0 e}\mathfrak{D}\frac{\partial^2F(z=0.35\text{ cm})}{\partial z^2}\cong -1.2\times10^{-2},\quad I=2.67\text{ A}$$
$$\cong -0.7,\qquad \gamma=2.72\text{ A}.$$

Both estimates are negligible with respect to $v_d\sim s=1.8\times10^5$ cm sec^{-1}.

IV. FUTURE APPLICATIONS

In addition to the stationary domain solutions investigated in the preceding section, there are two other areas of application of the basic equations (2.26)–(2.28) which merit discussion.

Propagating Domains

The work of Bray,[7] Many and Balberg,[5] and others suggests the existence of fully formed domains which propagate without change of shape. One may seek solutions of this kind by transforming to a coordinate system moving with the domain velocity c, and requiring that there be no explicit time dependence in the moving system. Thus, making the transformation to moving coordinates (z',t'),

$$z'=z-ct,$$
$$t'=t,$$

we have

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}$$

and

$$\frac{\partial}{\partial t}=-c\frac{\partial}{\partial z'},$$

where variations with respect to t' are taken to be zero. Then (2.26)–(2.28) become

$$-c\frac{\partial v_d(z')}{\partial z'}=A[v_0(z')-v_d(z')]-B\int_0^\infty dx\,f(x) \times\int_{-1}^{1}dy\,y[v_d(z')y-s]\xi(x,y,z') +\left(\frac{\epsilon}{4\pi n(z')e}\right)\mathfrak{D}A\frac{\partial^2F}{\partial z'^2}, \quad (4.1)$$

$$(-c+sy)\frac{\partial\xi(x,y,z')}{\partial z'}=g(x)\left(\frac{v_d(z')}{s}y-\frac{s_x^*}{s}\right)\xi(x,y,z'),\quad(4.2)$$

$$0=\frac{3}{\pi}\frac{s}{v_{th}^2}\int_0^\infty dx\, f(x)\int_{-1}^{1}dy[v_d(z')y-s]\times\xi(x,y,z')-c\left(\frac{\epsilon}{4\pi n(z')e}\right)\frac{\partial^2F(z')}{\partial z'^2},\quad(4.3)$$

where

$$f(x)=x^2\left(\frac{x^2}{x^2+x_D^2}\right)^2e^{-x^2/4},$$

and

$$g(x)=\frac{e^{-x^2/4}}{x}\left(\frac{x^2}{x^2+x_D^2}\right)^2C_0.$$

Evidently, using (4.3), one can eliminate the field derivative in (4.1) in favor of the phonon number. Then integrating (4.1), v_d can be found in terms of ξ. This result may then be substituted into (4.2), giving an integrodifferential equation in ξ alone. To obtain a localized domain, it remains to demonstrate that a localized phonon packet is a solution of this equation. This problem is under investigation.

Transient Behavior: Domain Incubation

In this subsection, a discussion of the physical significance of the basic equations is presented for the time-dependent case. On the basis of these equations, we suggest a mechanism of domain generation which does not depend on shock excitation or some unspecified fluctuation in the local phonon number.

The interpretation of (2.26) and (2.27) is self-evident. On the other hand, the physical significance of (2.28) [or, equivalently, (2.22)] merits some discussion. It will be recalled that this equation is obtained by taking the zeroth moment of (2.10). This operation gives a relation for the time rate of change of the total number of particles. The change in particle number due to the randomizing collisions, electric field, diffusion, and explicit time dependence are all zero. This follows from the fact that all of these quantities are proportional either to the deviation from equilibrium of the displaced Maxwellian (2.5), or to $\mathbf{v}_k$ itself; hence, they are anisotropic in $\mathbf{k}$ and cancel in the sum over $\mathbf{k}$.

The zeroth moment of the last term of (2.10) is essentially

$$\frac{\partial}{\partial x}\sum_{\mathbf{k}}(v_k)_x f_k^{(1)}=\frac{\partial j_x}{\partial x}.$$

In the absence of stimulated phonons, the particle balance would require that the above divergence in the current vanish, insuring current continuity. However, in the presence of amplification, the time rate of change of particle number due to interaction of the electrons with the stimulated phonons is nonzero. Though small by a factor $(\hbar\omega_q/k_BT)$, it is nevertheless finite, as shown in Appendix A. The physical explanation of this is that, since the stimulated phonons are emitted within the forward Cerenkov-like cone, the electron recoil is preferentially in the backward direction. Hence, the sum on $\mathbf{k}$ does not give cancelling contributions as is the case with the Ohmic collisions. Hence, there is a small but finite divergence in the current density, or, equivalently a local time rate of change of space charge or electric field gradient.

The implications of this result are twofold: First, it correlates the presence of amplification in a localized region of the sample with a time rate of change of electric-field gradient. This is consistent with the concept of a propagating domain (although it does not, in itself, necessarily imply the existence of such domains). Secondly, it implies that during the rise time of the applied voltage pulse, a stimulated emission of phonons will occur at a point at which there is a finite field gradient: notably, at a contact or inhomogeneity. It is postulated that the localized phonons so generated will then be amplified by the Ohmic current, as will thermal phonons throughout the sample. The phonon level in this packet, however, because it is initially larger, should first reach the level at which strong scattering of the electrons by the stimulated phonons occurs. The electron drift velocity will then decrease locally, resulting in an electron accumulation on the upstream side of the domain and a deficiency on the downstream side, resulting in a dipole domain. We suggest this as a mechanism of domain generation; however, the verification of this mechanism will require detailed numerical solution of the time-dependent equations during the initial transient.

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APPENDIX

In this Appendix, the quantity $I_\xi^{(0)}$, defined by Eq. (2.19), is evaluated. We have

$$I_\xi^{(0)}=\frac{2\pi}{\hbar^2}\sum_{\mathbf{k},\mathbf{q}}C_q^2\xi_q[(f_{k+q}-f_k)\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)+(f_{k-q}-f_k)\delta(\epsilon_{k-q}-\epsilon_k+\hbar\omega_q)]\delta(\epsilon-\epsilon_k).$$

As for $I_\xi^{(1)}$, we set $\mathbf{k}+\mathbf{k}+\mathbf{q}$ for fixed $\mathbf{q}$ in the second sum, giving

$$I_\xi^{(0)}=\frac{2\pi}{\hbar^2}\sum_{\mathbf{k},\mathbf{q}}C_q^2\xi_q(f_{k+q}-f_k)\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)\times[\delta(\epsilon-\epsilon_k)-\delta(\epsilon-\epsilon_k-\hbar\omega_q)].\quad(\text{A1})$$

Substituting (2.5) and expanding, for an acoustic mode,

$$f^0(\epsilon+\hbar\omega_q)=f^0(\epsilon_q)+\hbar sq\frac{\partial f^0(\epsilon_k)}{\partial\epsilon_k},$$

we obtain

$$I_\xi^{(0)}=\frac{2\pi}{\hbar}\sum_{\mathbf{q}} qC_q^2\xi_q\{v_d(x,t)\cos\theta_q-s\}$$

$$\times\sum_{\mathbf{k}}\left(-\frac{\partial f_k^0}{\partial\epsilon_k}\right)\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)$$

$$\times[\delta(\epsilon-\epsilon_k)-\delta(\epsilon-\epsilon_k-\hbar\omega_q)].$$

The sum over **k** is first taken, with **q** as the polar axis. Thus,

$$S\equiv\sum_{\mathbf{k}}\left(-\frac{\partial f_k^0}{\partial\epsilon_k}\right)\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)$$

$$\times[\delta(\epsilon-\epsilon_k)-\delta(\epsilon-\epsilon_k-\hbar\omega_q)]$$

$$=\frac{V}{(2\pi)^3}\int_{q/2}^{\infty}dk\,k^2\int_0^{\pi}d\Theta\,\sin\Theta\int_0^{2\pi}d\Phi\left(-\frac{\partial f_k^0}{\partial\epsilon_k}\right)$$

$$\times\delta(\epsilon_{k+q}-\epsilon_k-\hbar\omega_q)[\delta(\epsilon-\epsilon_k)-\delta(\epsilon-\epsilon_k-\hbar\omega_q)].$$

We first have that

$$\int_0^{\pi}d\Theta\,\sin\Theta\,\delta\left(\frac{\hbar^2}{m}kq\left[\cos\Theta+\frac{q}{2k}-\frac{ms}{\hbar k}\right]\right)=\frac{m}{\hbar^2kq},$$

and

$$S=\frac{V}{(2\pi)^3}(2\pi)\left(\frac{m}{\hbar^2}\right)^2\frac{1}{q}\int_{\epsilon_q}^{\infty}d\epsilon_k\left(-\frac{\partial f_k^0}{\partial\epsilon_k}\right)$$

$$\times[\delta(\epsilon-\epsilon_k)-\delta(\epsilon-\epsilon_k-\hbar\omega_q)]$$

$$=\frac{V}{(2\pi)^2}\left(\frac{m}{\hbar^2}\right)^2(-)\frac{1}{q}\left\{\left(\frac{\partial f_k^0}{\partial\epsilon_k}\right)_{\epsilon_k=\epsilon}-\left(\frac{\partial f_k^0}{\partial\epsilon_k}\right)_{\epsilon_k=\epsilon-\hbar\omega_q}\right\}$$

$$=\frac{V}{(2\pi)^2}\left(\frac{m}{\hbar^2}\right)^2(-)\frac{1}{q}\hbar\omega_q\left(\frac{\partial^2 f_k^0}{\partial\epsilon_k^2}\right)_{\epsilon_k=\epsilon}$$

$$=\frac{V}{(2\pi)^2}\left(\frac{m}{\hbar^2}\right)^2\left(\frac{\hbar s}{k_BT}\right)\left(\frac{\partial f_k^0}{\partial\epsilon}\right),$$

the last equality following from (2.7).

The result (2.20) and the preceding remarks then follow directly.