Excitons in Metals: Infinite Hole Mass

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The optical conductivity is evaluated for interband transitions between a flat valence band and a parabolic conduction band. The conduction band is filled with electrons to a Fermi energy μ_F. The conductivity is calculated assuming that the electron-hole interaction is attractive, static, and short range. The final-state interactions between the electron and hole cause a divergence in the conductivity at the interband threshold. This divergence appears to go as a power law. For this case of an infinite hole mass, the exciton binding energy vanishes, since the singularity in the scattering amplitude occurs just at threshold.

I. INTRODUCTION

PREVIOUS calculations[1-4] have discussed the important contribution of final-state interactions to the interband optical transition in metals or heavily doped semiconductors. The electron-hole interactions can, in some cases, significantly alter the shape and strength of the interband absorption threshold. Virtual plasmon excitations, introduced by the dynamic screening, are also an important source of final-state interactions.[4]

The present calculation is concerned with the final-state interactions which contribute to a specific and simple type of interband transition. This has the valence band flat and infinitely narrow—i.e., the hole mass is infinite. This band is separated from the conduction band by a gap E_G. The conduction band is parabolic, and is filled up to a Fermi energy μ_F. The electron-hole interaction is assumed to be attractive, static, and short-range. Our neglect of plasmon effects arising from dynamic screening means that the model does not describe a real metal. Yet the calculations may still have applications in interpreting the soft x-ray spectra of metals. These considerations are discussed in Sec. IV.

The present model is interesting because the exciton problem is directly related to the donor problem for an ionized impurity. In the conventional calculation of the electron's self-energy from ionized impurity scattering,[5] the exclusion-principle factors all cancel out. Any bound state, if one exists, must occur beneath the bottom of the conduction band. Yet in the exciton case, exclusion-principle effects do limit the final-state scattering.[3] This causes logarithmic singularities in the electron-hole scattering amplitude, which suggests the existence of a bound exciton state just below the interband threshold. The present calculation investigates in detail this disparity between the exciton- and donor-state calculations.

We have found that the exciton binding energy is zero when the hole mass is infinite. That is, the singularity in scattering amplitude occurs right at the interband threshold. The interband oscillator strength diverges as an inverse power law at threshold. The strength of this power-law divergence is determined by the strength of the coupling constant for electron-hole interactions. These results appear to be the exact, infinite order, solution which is deduced from our results, which extend to third order.

Since we are primarily concerned with the analytical properties of the various logarithmic divergences, it was decided to use the contact model for the electron-hole interaction. This enormously simplifies the investigation of the singular characteristics of the final-state interactions. The attractive electron-hole interaction is characterized by a constant Δ [$\Delta \sim N_F|V_0|$] up to a cutoff energy ξ_0. In terms of these constants, the solution to the optical conductivity has the form

$$\sigma \sim \frac{1}{\omega}\left(\frac{\xi_0}{\omega-\omega_c}\right)^{2\Delta}\theta(\omega-\omega_c), \tag{1.1}$$

where $\omega_c = E_G + \mu_F$ is the threshold frequency. This result has the power-law divergence discussed above. Our solution is just valid to "logarithmic accuracy." This means that only the important powers of the parameter

$$L(\omega) = \Delta \ln|(\omega-\omega_c)/\xi_0| \tag{1.2}$$

are retained when investigating the logarithmic divergences of the electron-hole scattering function.

In the solutions to the Kondo problem, the phrase "logarithmic accuracy" also implies that one does not keep track of the phases of the logarithmic singularity. This has led to a controversy[6] over the position and nature of the Kondo bound-state pole. This does not happen in the present calculation. By using a linear vertex equation, it is quite easy to keep track of the phases. A bound state, if any existed, would have the pole falling on the real ω axis at $\omega < \omega_c$. In fact, it appears that one can similarly find the phase of the scattering integral in the Kondo problem by using a linear vertex equation. This possibility is discussed in Sec. IV.

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[5] R. H. Parmenter, Phys. Rev. **104**, 22 (1956).
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II. THE INTERBAND OPTICAL CONDUCTIVITY

The optical properties of a metal are determined by the dielectric function[7,8]

$$\epsilon(\omega)=1+4\pi i\sigma(\omega)/\omega. \tag{2.1}$$

The Kubo formula is used to evaluate the conductivity $\sigma(\omega)$. For an isotropic system at zero temperature,[9,10]

$$\sigma(\omega)=\frac{2}{3\omega}\int_0^\infty dt\, e^{i\omega t}\langle[j_\nu(0),j_\nu(t)]\rangle. \tag{2.2}$$

For interband transitions, we assume that the $\mathbf{p}$ operator in $\mathbf{j}$ operates between Bloch functions. This matrix element is taken to be independent of energy, as is done in most calculations. The correlation function (2.2) is most easily evaluated in the Matsubara formalism.[10] This will be discussed first for an arbitrary set of bands, and then for the particular example of interest.

Denote $C_{j\mathbf{k}}^\dagger$ as the creation operator for an electron in band j with momentum $\mathbf{k}$. We define the correlation function

$$\Pi(i\omega)=\frac{1}{V}\sum_{\mathbf{k},\mathbf{k}',ijnm}\int_0^\beta d\tau\, e^{i\omega(\tau-\tau')} \times\langle T_\tau C_{j\mathbf{k}}(\tau)C_{i\mathbf{k}}^\dagger(\tau)C_{m\mathbf{k}'}(\tau')C_{n\mathbf{k}'}^\dagger(\tau')\rangle. \tag{2.3}$$

The retarded form of this correlation function $\Pi_{\rm ret}(\omega)$ is obtained by setting $i\omega\to\omega+i\delta$. In terms of this retarded function, the interband conductivity is

$$\mathrm{Re}\sigma(\omega)=\frac{e^2\langle p\rangle^2}{3\omega m^2}A(\omega),$$
$$A(\omega)=-2\,\mathrm{Im}\Pi_{\rm ret}(\omega).$$

The function $A(\omega)$ has a simple physical interpretation. The correlation function (2.3) can be evaluated by a diagrammatic expansion as in Fig. 1, where dotted lines are the electron-hole interactions. If one ignores these interactions, then the correlation function is just given by the first term [Fig. 1(a)]. Call this function $\Pi^{(0)}(i\omega)$, and

$$A^{(0)}(\omega)=-2\,\mathrm{Im}\Pi_{\rm ret}{}^{(0)}(\omega),$$
$$A^{(0)}(\omega)=2\pi\sum_{ij}\int_{E_i<\mu_f,E_j>\mu_f}\frac{d^3k}{(2\pi)^3} \times\delta(E_i(k)+\omega-E_j(k)). \tag{2.4}$$

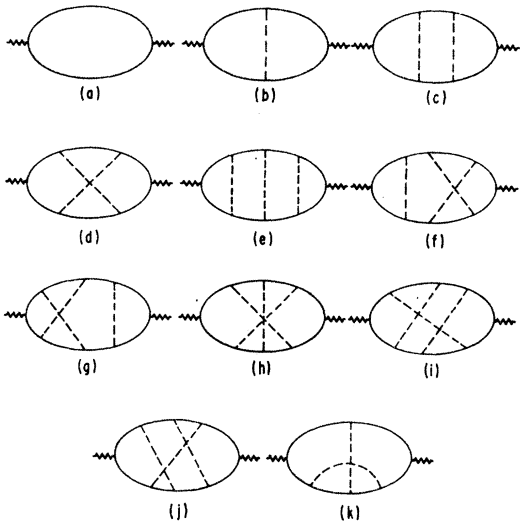

FIG. 1. Some types of correlation functions which contribute to the final-state interactions. The dashed lines represent the static, attractive, electron-hole interactions.

The function $A^{(0)}(\omega)$ is proportional to the joint density of states. Specifically,

$$A^{(0)}(\omega)=\pi\rho(\omega).$$

Previous calculations[7,8] of the optical properties of metals have assumed that the transition probability is proportional to the joint density of states. This is no longer adequate when final-state interactions are important,[1] since then $A(\omega)$ is quite different than $A^{(0)}(\omega)$.

These equations simplify when applied to the present problem. For a contact interaction, the retarded correlation function is

$$\Pi_{\rm ret}(\omega)=\int_0^{\xi_0}\frac{d\xi}{\omega-\omega_c+\xi+i\delta}\Gamma(\xi,\omega), \tag{2.5}$$

where ξ is the conduction electron's energy as measured from its Fermi surface. The vertex function $\Gamma(\xi,\omega)$ is generated by the diagrammatic expansion of Fig. 1. This procedure is discussed in the next section. The poles and structure of $\Gamma(\xi,\omega)$ determine the nature of the interband absorption. This is the interesting aspect of optical experiments, since one directly measures the scattering function.

III. HIGHER-ORDER INTERACTIONS

A. Perturbation Expansion

We begin the discussion of higher-order interactions by examining their contribution to a perturbation expansion for the correlation function. A solution to the correlation function is given in the following sections. The perturbation-series result is useful for providing a qualitative introduction to the effects of these higher

[7] H. Ehrenreich and H. R. Philipp, Phys. Rev. 128, 1622 (1962); B. R. Cooper, H. Ehrenreich, and H. R. Philipp, *ibid.* 138, A494 (1965).

[8] H. Ehrenreich, H. R. Philipp, and B. Segall, Phys. Rev. 132, 1918 (1963).

[9] W. Kohn, in *Optical Properties and Electronic Structure of Metals and Alloys*, edited by F. Abeles (John Wiley & Sons, Inc., New York, 1966).

[10] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

interaction terms. It also provides an expansion against which to check the exact result after it is obtained.

As discussed above, the perturbation-series results will only be evaluated to logarithmic accuracy. The result is a power series in $L(\omega)$ given in (1.2). When the electron-hole potential is an attractive Coulomb potential with Fermi Thomas screening, then the coupling parameter is

$$\Delta \cong (r_s/12)\ln(1+6/r_s),$$

where r_s is the standard density parameter. It was shown in Ref. 1 that the contribution of the ladder diagrams to the interband absorption may be expressed as

$$A_L(\omega)=A^{(0)}(\omega)\left|1-\frac{\Lambda_2(\omega)}{1+\Lambda_1(\omega)}\right|^2. \tag{3.1}$$

To logarithmic accuracy, the terms appearing in (3.1) are

$$\begin{aligned}&\Lambda_1(\omega)=\Lambda_2(\omega)=L(\omega)-i\pi\Delta\theta(\omega-\omega_c),\\ &A^{(0)}(\omega)=2\pi\theta(\omega-\omega_c).\end{aligned} \tag{3.2}$$

Expanding (3.1), and ignoring the terms ImΛ, gives

$$A_L(\omega)=A^{(0)}(\omega)\{1-2L(\omega)+3L(\omega)^2-4L(\omega)^3\cdots\}.$$

The terms in the expansion originate from the ladder diagrams in Figs. 1(a), 1(b), 1(c), and 1(e). The term $-2L(\omega)$ corresponds to Fig. 1(b). Since it is the only vertex term of order Δ, this logarithmic singularity cannot be canceled by any other diagram. The first important higher-order interaction is the cross diagram Fig. 1(d). When the hole mass is infinite, this contributes to the absorption an amount

$$A_{1d}=-A^{(0)}(\omega)L(\omega)^2.$$

Note that this only partially cancels the $3L^2(\omega)$ contribution from Fig. 1(c). The third-order diagrams (e) to (j) in Fig. 1 contribute in units of L^3: -4, $\frac{4}{3}$, $\frac{4}{3}$, $-\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$. A consideration of all terms in Fig. 1 from (a) to (j) gives

$$A(\omega)=A^{(0)}(\omega)\{1-2L+2L^2-\tfrac{4}{3}L^3\cdots\}. \tag{3.3}$$

It is evident that these logarithmic singularities remain in the conductivity even when higher-order interactions are included. When the hole mass is finite, the nonladder diagrams do not contribute logarithmic singularities of the same order as the ladder terms. This is what makes the infinite hole mass an interesting special case of study, since here all of the diagrams of the kind in Fig. 1 from (a) to (j) contribute significant logarithmic singularities. In the impurity scattering calculation, the logarithmic singularities from the nonladder diagrams exactly cancel those from the ladder terms. This obviously does not occur in the present exciton calculation.

Vertex corrections such as Fig. 1(k) have not been included in this analysis. In this exciton problem, they do not contribute logarithmic singularities of the type in (3.3). This diagram was important in the recent analysis by Luttinger and Kohn[11] of a superconductive instability in low-temperature electron gases. The present calculation differs from theirs in several respects. Since both of their particle propagators are electrons in the Fermi sea, they have a symmetry which is lacking in the electron-hole problem. Also, they only found instabilities in states of odd angular momentum, while we are explicitly concerned with relative s states.

The series in (3.3) appears to be

$$e^{-2L}=1-2L+2L^2-\tfrac{4}{3}L^3+\cdots. \tag{3.4}$$

Of course, one cannot guarantee that the series (3.3) is an exponential without evaluating the series to all orders. However, the solution of the scattering integral, which is given below, also suggests that this is the correct series limit. Thus we proceed on the assumption that the series (3.3) is just an exponential. Since the correlation function is an analytic function of ω, the real and imaginary parts are related. Using the definition of $L(\omega)$, this further suggests that our result is

$$\begin{aligned}A(\omega)&=\Delta^{-1}\,\mathrm{Im}\\ &\times\exp\left[-2\Delta\ln\left|\frac{\omega-\omega_c}{\xi_0}\right|+2\Delta\pi i\theta(\omega-\omega_c)\right],\end{aligned} \tag{3.5}$$

$$A(\omega)=\frac{\theta(\omega-\omega_c)}{\Delta}\left|\frac{\xi_0}{\omega-\omega_c}\right|^{2\Delta}\sin(2\Delta\pi).$$

The distressing observation that this is negative for $\Delta>\frac{1}{2}$ appears to be an artifact of our approximation, as is discussed below. Note that the result does become $A^{(0)}(\omega)$ as $\Delta\to 0$. The divergence at the critical frequency is a power law, rather than logarithmic. This result could only be obtained from an infinite-order solution, which in this case has only been guessed. It also shows that there is no bound state in the infinite hole mass case. Essentially, the binding energy has become zero since the singularity occurs precisely at $\omega=\omega_c$.

B. Solution to the Vertex Equation

We now proceed to solve for the correlation function (2.5). For the irreducible interactions of interest, it is easy to show that one can write a linear vertex equation

$$\Gamma(\xi,\omega)=1+\int_0^{\xi_0}\frac{d\xi'}{\omega-\omega_c+\xi'+i\delta}\Gamma(\xi',\omega)V(\xi,\xi',\omega). \tag{3.6}$$

The potential V is the sum of all irreducible interactions. For the present calculation, we have retained contribu-

[11] J. M. Luttinger, Phys. Rev. **150**, 202 (1966); W. Kohn and J. M. Luttinger, Phys. Rev. Letters **15**, 524 (1966).

tions to V up to third order in Δ

$$V(\xi,\xi',\omega)=-\left\{\Delta+\Delta^2\ln\left(\frac{\xi+\xi'-\omega+\omega_c}{\xi_0}\right)\right.$$
$$\left.-\Delta^3L_2\left(\frac{\xi}{\xi+\xi'-\omega+\omega_c}\right)-\Delta^3L_2\left(\frac{\xi'}{\xi+\xi'-\omega+\omega_c}\right)\right\},\quad(3.7)$$

where $L_2(z)$ is the Euler dilogarithmic function

$$L_2(z)=-\int_0^z\frac{dt}{t}\ln(1-t).$$

The imaginary parts of V are found by the condition, determined by causality, that the imaginary part of ω be infinitesimally real in the region of branch cuts. The first term Δ is the direct interaction [Fig. 1(b)]. The second term is the exchange interaction [Fig. 1(d)]. The terms of order Δ^3 are the sum of the three interactions h, i, and j in Fig. 1. The result (3.7) is derived in the Appendix.

The vertex equation (3.6) is solved by a slight modification of Noyes's[12] method. Define two functions $t(\omega)$ and $f(\xi,\omega)$ by

$$\Gamma(\xi,\omega)=t(\omega)f(\xi,\omega),\quad(3.8)$$

with the additional normalization requirement that

$$f(\xi_1,\omega)=1,\quad(3.9)$$

for some value of $\xi=\xi_1$. Then these equations can be manipulated to give

$$f(\xi,\omega)=1+\int_0^{z_0}\frac{d\xi'}{\omega-\omega_c+\xi'+i\delta}f(\xi',\omega)$$
$$\times[V(\xi',\xi_1,\omega)-V(\xi',\xi,\omega)],\quad(3.10)$$

$$t(\omega)=1/[1+\Lambda(\omega)],$$

$$\Lambda(\omega)=\int_0^{\xi_0}\frac{d\xi'}{\omega-\omega_c+\xi'+i\delta}f(\xi',\omega)V(\xi',\xi_1,\omega).\quad(3.11)$$

Equation (3.10) provides a method of systematically evaluating $f(\xi,\omega)$ as a power series in Δ. Then one evaluates $\Lambda(\omega)$ and $\Pi(\omega)$. The poles of $\Pi(\omega)$ specify the nature of the bound state.

The potentials in (3.7) have the property

$$\mathrm{Im}V(\xi,\xi',\omega)=0,\quad\xi,\xi'>0,\quad\omega<\omega_c.\quad(3.12)$$

Although this has been explicitly verified only up to and including third-order interactions, it seems reasonable to conclude that it is generally true. This makes $f(\xi,\omega)$ entirely real for $\omega<\omega_c$ and $\xi>0$. This means that a bound-state pole, if one exists for $\omega<\omega_c$, falls on the real axis.

Equations (3.8) to (3.10) are first solved for $\omega<\omega_c$. The normalizing condition (2.9) is chosen at $\xi_1=\omega_c-\omega$.

[12] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965); K. L. Kowalski, *ibid.* **15**, 798 (1965).

Evaluating (3.10) and (3.11) up to order Δ^3 gives

$$f(\xi,\omega)=1-\Delta^2\left[L_2\left(\frac{\xi}{\omega-\omega_c}\right)-L_2(-1)\right],\quad(3.13)$$

$$\Lambda(\omega)=L(\omega)+\frac{1}{2!}L(\omega)^2+\frac{1}{3!}L(\omega)^3+O(\Delta^4).\quad(3.14)$$

With the knowledge of $f(\xi,\omega)$, the full correlation function in (2.5) can be evaluated for $\omega<\omega_c$,

$$\Pi(\omega)=\frac{1}{\Delta}\frac{L+L^3/3!+O(\Delta^5)}{1+L+L^2/2!+L^3/3!+O(\Delta^4)}.\quad(3.15)$$

The denominator series appears to be $\exp(L)$, while the numerator we take to be $\sinh(L)$. Assuming that we have correctly guessed the result, we get

$$\Pi(\omega)=\frac{1}{2\Delta}[1-\exp(-2L)],\quad(3.16a)$$

$$\Pi(\omega)=\frac{1}{2\Delta}\left[1-\left(\frac{\xi_0}{\omega-\omega_c}\right)^{2\Delta}\right].\quad(3.16b)$$

The singularity occurs at $\omega=\omega_c$. There is no bound-state pole, but just a cut which starts at $\omega=\omega_c$.

Equations (3.8) to (3.10) have also been solved for $\omega>\omega_c$. Here one must keep track of both the real and imaginary parts of the various functions. It is convenient to set $\xi_1=\omega-\omega_c+\delta$ in (3.9). The calculation is tedious, but the result, correct to logarithmic accuracy, is for $\omega>\omega_c$

$$\Pi(\omega)=\frac{1}{\Delta}\frac{L+L^3/3!-i\pi\Delta(1+L^2/2!)}{1+L+L^2/2!+L^3/3!-i\pi\Delta(1+L+L^2/2!)}.$$

This gives for $A(\omega)$

$$A(\omega)=2\pi\theta(\omega-\omega_c)\{(1+L+L^2/2!+L^3/3!)^2$$
$$+\Delta^2\pi^2(1+L+L^2/2!)^2\}^{-1}.$$

If we assume that the series is exponential, then

$$A(\omega)=\frac{2\pi\theta(\omega-\omega_c)}{1+\Delta^2\pi^2}\left(\frac{\xi_0}{\omega-\omega_c}\right)^{2\Delta}.\quad(3.17)$$

This result appears slightly different from (3.5). Both equations contain the same $(\omega-\omega_c)^{-2\Delta}$ divergence. The other constant terms differ because we have not properly kept track of them in deriving either (3.5) or (3.17). For instance, consider the integral

$$\mathrm{Re}\int_0^{\xi_0}\frac{d\xi}{\omega-\omega_c+\xi}\ln\left|\frac{\xi}{\xi_0}\right|=\tfrac{1}{2}\ln^2\left|\frac{\omega-\omega_c}{\xi_0}\right|$$
$$-\tfrac{1}{6}\pi^2-\tfrac{1}{4}\pi^2\theta(\omega-\omega_c)+O\left(\frac{\omega-\omega_c}{\xi_0}\right).$$

In solving the problem to logarithmic accuracy, we have been ignoring the constant terms such as $\pi^2/6$ and $\pi^2/4$. The existence of such terms explains the differences between (3.17) and (3.5). The difficulty with including such constant terms is that the problem has two singular regions, one at $\xi\sim 0$ and one at $\xi\sim\xi_0$. The unphysical singularity at $\xi\sim\xi_0$ is an artifact of the contact model. Yet, the constant terms cannot be calculated accurately without treating the singularities at $\xi\sim\xi_0$ in some arbitrary fashion. The fact that these constant terms may be appreciable when Δ is not small may explain the disturbing observation that (3.5) is negative for $\Delta>\frac{1}{2}$.

IV. DISCUSSION

A solution has been obtained to the correlation function specifying interband optical absorption for the case of an infinite hole mass. The analytic structure of the divergences near the absorption threshold has been investigated using a contact model for the attractive Coulomb interaction. We found that both the perturbation-series result, and the solution to the scattering integral, suggest that the divergence is a power law. If ω is the photon frequency, then

$$\sigma(\omega)\sim\frac{1}{\omega}\left(\frac{\xi_0}{\omega-\omega_c}\right)^{2\Delta}\theta(\omega-\omega_c),$$

where ω_c is the threshold frequency, ξ_0 is a cutoff energy, and Δ is the strength of the attractive electron-hole interaction.

The absorption and emission spectra of soft x rays constitute a physical system which approximates the present model. The emission spectra should have a similar divergence since, as discussed by Suna,[13] the emission and absorption differ only by factors relating to the occupation factors. Some x-rays data[14,15] do show structure at threshold. Some of these cases, such as magnesium, are satisfactorily explained in terms of density-of-states effects. But the L and M absorption edges of[14,15] Ni, Cu, and Zn do show peaks at threshold which apparently are not caused by the density of states. Although this feature could be caused by exciton processes, it is sobering to note that the spectra also have much other unexplained structure, such as the high-energy tail in K emission.

For interband transition in real metals, final-state interactions involving the virtual excitation of plasmons are also important. These plasmons arise from the dynamic nature of the screening of the electron-hole Coulomb interaction. These dynamic interactions appear to be more important than the statically screened interaction in interband optical transitions in real metals. In these cases, however, the statically screened Coulomb interaction does not lead to logarithmic singularities. Nevertheless, it might be necessary to include the plasmon processes before the present calculations can be applied to describe real x-ray transitions.

One can also envision processes where the hole is in a d state, and where the important final-state interactions are caused by an s-d interaction. The nature of exciton states in this process is related to the Kondo problem.[16,17] For this interaction, the linear vertex equation (3.6) becomes a matrix equation because of spin.

$$\Gamma=\Gamma_s+\boldsymbol{\sigma}\cdot\mathbf{s}\Gamma_v,$$
$$V=V_s+\boldsymbol{\sigma}\cdot\mathbf{s}V_v.$$

Each of the higher-order irreducible interactions, such as Fig. 1(d), contributes both to V_s and V_v. Since the analytic properties of the irreducible interactions are the same for the exciton and Kondo problems, then (3.12) applies. If one solves $\Gamma=N/D$, then N and D are both real for $\omega<\omega_c$. Hence, the zeros of D for $\omega<\omega_c$ give poles in Γ which correspond to a donor state. Abrikosov's[17] solution to the Kondo problem indicates that such poles do exist.

Note added in manuscript. J. J. Hopfield[18] has suggested that the ground state of an electronic system with and without an impurity are orthogonal. P. W. Anderson[19] has recently suggested a proof of this theorem, and notes that this causes some transitions to have divergences. His functions diverge as a power law with an exponent

$$\epsilon=\frac{1}{3\pi^2}\sum_l(2l+1)\sin^2\delta_l=\Delta^2/3.$$

This exponent of $\Delta^2/3$ is quite different, and much weaker, than the present result 2Δ. Thus, Anderson's calculation predicts divergences which are much smaller than those discussed here.

APPENDIX: THE IRREDUCIBLE INTERACTIONS

Equation (3.6) indicates that the correlation function can be evaluated using a linear integral equation. This is only true because we have assumed that the basic electron-hole interaction is static. That one can use a linear integral equation of this form is deduced by writing out explicitly the lowest six or eight terms. Here we will elaborate on this by deriving (3.7) for the irreducible interactions.

The two equations (3.6) and (3.7) will first be written in terms of momentum variables. This is conventionally the way these results are expressed, and it is instructive

[13] A. Suna, Phys. Rev. 135, A111 (1964).
[14] L. G. Parratt, Rev. Mod. Phys. 31, 616 (1959).
[15] D. H. Tomboulian, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 30, p. 246.
[16] K. Yoshida, Phys. Rev. 147, 223 (1966).
[17] A. A. Abrikosov, Physics 2, 5 (1965); 2, 61 (1965).
[18] J. J. Hopfield (private communication).
[19] P. W. Anderson, Phys. Rev. Letters 18, 1049 (1967).

to see them before one invokes the contact model.

$$\Gamma(\mathbf{p},\omega)=1+\int\frac{d^3p'}{(2\pi)^3}\times\frac{[1-n_F(p')]\Gamma(\mathbf{p}',\omega)V(\mathbf{p},\mathbf{p}',\omega)}{\omega-\omega_c+\xi_{p'}+i\delta}, \tag{A1}$$

$$V(\mathbf{p},\mathbf{p}',\omega)=\sum_m V_m(\mathbf{p},\mathbf{p}',\omega).$$

The potential is the sum of all irreducible interactions. The m subscript corresponds to Fig. 1(m); thus V_b is the irreducible interaction first introduced in Fig. 1(b).

$$V_b(\mathbf{p}_1,\mathbf{p}_2,\omega)=\mathfrak{U}(\mathbf{p}_1-\mathbf{p}_2),$$

$$V_d(\mathbf{p}_1,\mathbf{p}_2,\omega)=-\int\frac{d^3p_3}{(2\pi)^3}\frac{n_F(p_3)\mathfrak{U}(\mathbf{p}_1-\mathbf{p}_3)\mathfrak{U}(\mathbf{p}_3-\mathbf{p}_2)}{\omega-\omega_c+\xi_3-\xi_1-\xi_2+i\delta},$$

$$V_h(\mathbf{p}_1,\mathbf{p}_2,\omega)=\int\frac{d^3p_3d^3p_4}{(2\pi)^6}\frac{\mathfrak{U}(\mathbf{p}_1-\mathbf{p}_3)\mathfrak{U}(\mathbf{p}_3-\mathbf{p}_4)\mathfrak{U}(\mathbf{p}_4-\mathbf{p}_2)n_F(p_3)n_F(p_4)}{(\omega-\omega_c-\xi_1-\xi_2+\xi_3+i\delta)(\omega-\omega_c-\xi_1-\xi_2+\xi_4+i\delta)},$$

$$V_i(\mathbf{p}_1,\mathbf{p}_2,\omega)=-\int\frac{d^3p_3d^3p_4}{(2\pi)^6}\frac{\mathfrak{U}(\mathbf{p}_1-\mathbf{p}_3)\mathfrak{U}(\mathbf{p}_3-\mathbf{p}_4)\mathfrak{U}(\mathbf{p}_4-\mathbf{p}_2)n_F(p_3)(1-n_F(p_4))}{(\omega-\omega_c-\xi_1-\xi_2+\xi_3+i\delta)(\omega-\omega_c-\xi_1-\xi_4+\xi_3+i\delta)},$$

$$V_j(\mathbf{p}_1,\mathbf{p}_2,\omega)=-\int\frac{d^3p_3d^3p_4}{(2\pi)^6}\frac{\mathfrak{U}(\mathbf{p}_1-\mathbf{p}_3)\mathfrak{U}(\mathbf{p}_3-\mathbf{p}_4)\mathfrak{U}(\mathbf{p}_4-\mathbf{p}_2)(1-n_F(p_3))n_F(p_4)}{(\omega-\omega_c-\xi_2-\xi_3+\xi_4+i\delta)(\omega-\omega_c-\xi_1-\xi_2+\xi_4+i\delta)},$$

where

$$\xi_n=(p_n^2-p_F^2)/2m,\quad n=1,2,3,4.$$

In the contact model, one replaces

$$\int\frac{d^3p}{(2\pi)^3}\mathfrak{U}\to-\Delta\int_{-\xi_0}^{\xi_0}d\xi,$$

where $\mathfrak{U}$ is attractive and $\Delta>0$. This immediately takes (A1) into (3.6). Similarly, the irreducible interactions become

$$V_b(\xi,\xi',\omega)=-\Delta,$$

$$V_d(\xi,\xi',\omega)=-\Delta^2\int_{-\xi_0}^{0}\frac{d\xi_3}{\omega-\omega_c+\xi_3-\xi-\xi'+i\delta}=-\Delta^2\ln\left(\frac{\xi+\xi'-\omega+\omega_c-i\delta}{\xi_0}\right),$$

$$V_h(\xi,\xi',\omega)=-\Delta^3\left\{\int_{-\xi_0}^{0}\frac{d\xi_3}{\omega-\omega_c+\xi_3-\xi-\xi'+i\delta}\right\}^2=-\Delta^3\ln^2\left(\frac{\xi+\xi'-\omega+\omega_c-i\delta}{\xi_0}\right),$$

$$V_i+V_j=\Delta^3\int_{-\xi_0}^{0}d\xi_3\int_{0}^{\xi_0}d\xi_4\frac{1}{\omega-\omega_c+\xi_3-\xi-\xi'+i\delta}\left[\frac{1}{\omega-\omega_c-\xi_4+\xi_3-\xi+i\delta}+\frac{1}{\omega-\omega_c-\xi_4+\xi_3-\xi'+i\delta}\right]$$

$$=\Delta^3\left\{\ln^2\left(\frac{\xi+\xi'-\omega+\omega_c-i\delta}{\xi_0}\right)+L_2\left(\frac{\xi}{\xi+\xi'-\omega+\omega_c-i\delta}\right)+L_2\left(\frac{\xi'}{\xi+\xi'-\omega+\omega_c-i\delta}\right)\right\}.$$

The sum of these terms gives (3.7).