Properties of Vortex Lines in Superconducting Barriers*

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There are two qualitatively different types of solution to the Josephson barrier equations. The first type corresponds to vortex lines in the barriers. For small damping, a rather complete picture of vortex line motion can be obtained including inertial and dissipative effects. The second type of solution corresponds to plasma oscillations. The stability of the solutions is investigated and results are obtained for the collective modes of the vortex lines and the propagation of plasma oscillations in a magnetic field.

1. INTRODUCTION

IN this paper we discuss some of the properties of a long Josephson[1] barrier between two superconductors. There are two qualitatively different types of solutions to the barrier equations. The first type corresponds to vortex lines in the barrier and has been discussed by a number of authors.[1,2] In the case where the damping is small and the barrier is long the transition to moving vortex lines can be accomplished by a Lorentz transformation. A rather complete picture of vortex motion can be easily obtained in this case including inertial and dissipative effects. When the damping is large the nonlinear equation of motion of the vortex lines is of the diffusion type and it does not appear easy to obtain analytic solutions. The case of large damping is closer to the situation in type-II superconductors.[3] In this paper we only consider the case of small damping (low temperatures).

The second type of solution is similar to a plasma oscillation or transverse electromagnetic wave propagating in the barrier. In the linear approximation these solutions have previously been discussed by Josephson.[4]

The stability of the above types of solutions is investigated. Results are obtained for the collective modes of oscillation of the vortex lines and also for the propagation of plasma oscillations in the presence of a magnetic field. In the case of a weakly nonlinear barrier these results, apart from the boundary condition, reduce to those of Eck *et al.*[5] Some considerations on the spectrum of small oscillations have also recently been made by Kulik.[6]

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[1] A recent review article containing previous references is B. D. Josephson, Advan. Phys. **14**, 419 (1965).
[2] R. A. Ferrell and R. E. Prange, Phys. Rev. Letters **10**, 479 (1963); P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966); Y. M. Ivanchenko, A. V. Svidzinskii, and V. A. Slyusarev, Zh. Eksperim. i Teor. Fiz. **51**, 194 (1966) [English transl.: Soviet Phys.—JETP **24**, 131 (1967)].
[3] J. Bardeen and M. J. Stephen, Phys. Rev. **140**, A1197 (1965).
[4] B. D. Josephson, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Company, Amsterdam, 1966).
[5] R. E. Eck, D. J. Scalapino, and B. N. Taylor, in *Proceedings of the Ninth International Conference on Low Temperature Physics* (Plenum Press, Inc., New York, 1965).
[6] I. O. Kulik, Zh. Eksperim. i Teor. Fiz. **51**, 1952 (1966) [English transl.: Soviet Phys.—JETP **24**, 1307 (1967)].

The nonlinear term in the equation of motion of the vortex lines is taken, in accordance with Josephson,[1] to be $\sin\phi$, where ϕ is the phase difference between the two superconductors. This simplifies the analysis considerably but the phenomena described here probably only depend quantitatively on the form of this term. Hobart[7] has indicated how to obtain solutions of some related nonlinear equations.

2. PROPERTIES OF VORTEX LINES

The phenomenological equations describing the macroscopic behavior of superconducting barriers have been obtained by Josephson[1] and we summarize his results here. We assume that the barrier lies in the x, y plane and only consider variations along x.

$$\partial\phi/\partial x=(2ed/\hbar c)H_y,\qquad \partial\phi/\partial t=(2e/\hbar)V, \tag{1}$$

$$J_z=j_1\sin\phi+\sigma V. \tag{2}$$

Here ϕ is the phase difference between the two superconductors, H is the magnetic field in the barrier, V is the voltage across the barrier, and $d=2\lambda+l$ where λ is the penetration depth and l is the barrier thickness. In the current density (2) the first term represents the supercurrents and the second term the Ohmic currents. From Maxwell's equations regarding the barrier as having a capacity C per unit area and (1) and (2), an equation for ϕ can be derived;

$$\left[\frac{\partial^2}{\partial x^2}-\frac{1}{\bar{c}^2}\frac{\partial^2}{\partial t^2}-\frac{\beta}{\bar{c}^2}\frac{\partial}{\partial t}\right]\phi=\frac{1}{\lambda_0^2}\sin\phi, \tag{3}$$

where $\bar{c}^2=c^2/4\pi dC$ is the phase velocity in the barrier, $\lambda_0^2=\hbar c^2/8\pi edj_1$ is the penetration depth, and $\beta=4\pi d\bar{c}^2\sigma/c^2$ is a damping constant.

Assuming the barrier to be of unit length in the y direction, the total free energy of the barrier, measured from the situation where $\phi=0$, is

$$F=\frac{\hbar j_1}{2e}\int dx\left[(1-\cos\phi)+\tfrac{1}{2}\lambda_0^2\left(\frac{\partial\phi}{\partial x}\right)^2+\tfrac{1}{2}\left(\frac{\lambda_0}{\bar{c}}\right)^2\left(\frac{\partial\phi}{\partial t}\right)^2\right]. \tag{4}$$

The first term represents the coupling energy of the

[7] R. H. Hobart (private communication).

two superconductors and the other terms the electromagnetic energy in the barrier. The rate of dissipation of energy from (3) and (4) is

$$dF/dt=-\sigma\int V^2dx \tag{5}$$

and is due to the Ohmic currents excited in the barrier.

It is interesting to note that (3) (with $\beta=0$) is the basic differential equation of the Frenkel-Kontorova lattice model of dislocations.[8] It has also been studied as a model one-dimensional field theory[9] and in differential geometry.[10]

Solutions of (3) representing vortex lines in the barrier are obtained as solutions of the time-independent equation[2]

$$\partial^2\phi/\partial x^2=(1/\lambda_0^2)\ \sin\phi, \tag{6}$$

which, except for a sign, is the equation of a pendulum.

In the case where the damping can be neglected ($\beta=0$), the transition to solutions representing vortex lines in uniform motion is accomplished by a Lorentz transformation.[8] When the damping is large the inertial term in (3) can be neglected and the equation resembles a diffusion equation rather than a wave equation. In this paper we will confine ourselves to the case of small damping which is appropriate at low temperatures. The case of large damping, however, is also of interest and is much closer to the situation in type-II superconductors.

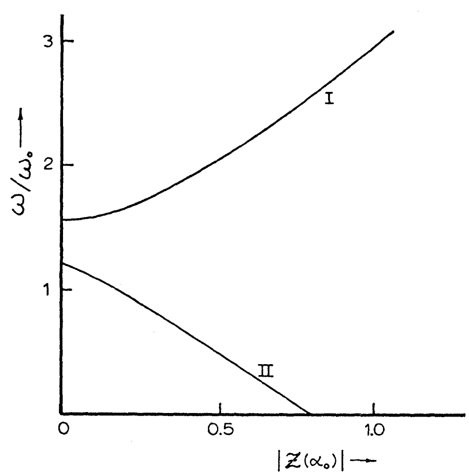

FIG. 1. Plot of the reduced frequency ω/ω_0 against $|Z(\alpha_0)|$ for $k^2=0.4$. Curve I corresponds to $0\leq\alpha_0\leq iK'$ and curve II to $K\leq\alpha_0\leq K+iK'$.

[8] J. Frenkel and T. Kontorova, Physik. Z. Sowjetuneon **13**, 1 (1938); A. Seeger and A. Kochendorfer, Z. Physik **130**, 321 (1951); A. Seeger, H. Dorth, and A. Kochendorfer, *ibid.* **134**, 173 (1953).

[9] J. K. Perring and T. H. R. Skyrme, Nucl. Phys. **31**, 550 (1962).

[10] L. P. Eisenhart, *A Treatise on Differential Geometry of Curves and Surfaces* (Ginn and Company, Boston, 1909).

Thus for a long barrier if $\phi(x/\lambda_0)$ is a solution of (6), a solution of (3) ($\beta=0$) is provided by

$$\phi[(x-vt)/\lambda], \tag{7}$$

where $v<\bar{c}$ is an arbitrary velocity and $\lambda=\lambda_0(1-v^2/\bar{c}^2)^{1/2}$ leads to a Lorentz contraction. In a similar manner if $\phi(\bar{c}t/\lambda_0)$ is a solution of

$$-\frac{1}{\bar{c}^2}\frac{\partial^2\phi}{\partial t^2}=\frac{1}{\lambda_0^2}\sin\phi, \tag{8}$$

then a solution of (3) ($\beta=0$) is provided by

$$\phi[(vt-x)/\lambda'], \tag{9}$$

where $v>\bar{c}$ and $\lambda'=\lambda_0(v^2/\bar{c}^2-1.)^{1/2}$ We refer to these solutions as M (magnetic) and E (electric) types.

Solutions of M type (7) are given in Ref. 8. They are

$$\tan(\tfrac{1}{4}\phi_0)=\exp[-(x-vt)/\lambda], \tag{10}$$

$$\sin[\tfrac{1}{2}(\phi_0-\pi)]=\mathrm{sn}[(x-vt)/k\lambda], \qquad v<\bar{c} \tag{11}$$

$$=k\ \mathrm{sn}[(x-vt)/\lambda], \tag{12}$$

where k is the modulus of the elliptic functions ($0\leq k\leq 1$). Solutions of this type have recently been discussed by Scott.[11]

The solution (10) represents a single-quantized vortex line in uniform motion and gives rise to a voltage pulse across the barrier. The energy per unit length of the moving line is $F=F_0/(1-v^2/\bar{c}^2)^{1/2}$, where F_0 is the rest energy ($F_0=4\hbar j_1\lambda_0/e$). This leads to a small effective mass per unit length of the vortex line $m=F_0/\bar{c}^2$. This of the order of the mass of an electron and is similar to that found by Suhl in a type-II superconductor.[12] The dissipation per unit length of the moving vortex line due to Ohmic currents is, from (5),

$$\partial F/\partial t=-2\sigma(\hbar^2v^2/e^2\lambda). \tag{13}$$

This result is only correct for small σ. By equating this to $-\eta_0v^2$ we arrive at a phenomenological viscosity coefficient.

$$\eta_0=2\sigma\hbar^2/e^2\lambda=\phi_0\sigma_nH_{c1}/c^2, \tag{14}$$

where ϕ_0 is the flux quantum, $\sigma_n=\sigma d$, and H_{c1} is the lower critical field.[1] This expression is of the same form as in a type-II superconductor except that H_{c1} relpaces H_{c2}.[13]

The solution (11) represents a moving vortex array with the distance separating adjacent vortices $L=2\lambda kK(k)$, where $K(k)$ is the complete elliptic integral of the first kind. The average field $\bar{H}$ and voltage $\bar{V}$ in

[11] A. C. Scott, Bull. Am. Phys. Soc. **12**, 308 (1967); and to be published.

[12] H. Suhl, Phys. Rev. Letters **14**, 226 (1965).

[13] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **139**, A1163 (1965).

the barrier are

$$\bar{H}_y = \hbar c\pi/2ed\lambda kK(k) = \tfrac{1}{4}\pi^2 H_{c1}/kK(k), \quad (15)$$

$$\bar{V} = -(v/c)\bar{H}_y d. \quad (16)$$

The voltage may be regarded as due to induction and owing to the structure of the vortex lines also contains an infinite series of harmonics of the frequency $(2e/h)\bar{V}$. Making use of the Fourier expansion of the elliptic function (11) the total voltage is

$$V(x,t) = \bar{V}\left[1 + \sum_{n=1}^{\infty} \frac{4q^n}{1+q^{2n}} \cos\left(\frac{n\pi}{\lambda kK}(x-vt)\right)\right], \quad (17)$$

where $q=\exp(-\pi K'/K)$ and $K'=K(k')$ when k' is the complementary modulus. The energy dissipation per unit length of a vortex line now leads to a viscosity coefficient of $\eta=\eta_0(E(k)/k)$, where $E(k)$ is the complete elliptic integral of the second kind. This is increased over that for a single line (14) and for strong field $\bar{H}>H_{c1}$ from (15) $\eta=\eta_0(H/H_{c1})$ and increases linearly. The energy per unit length of a vortex line also increases in this manner.

The solution (12) can be regarded as representing an array of vortex lines with alternating signs. The phase ϕ oscillates around the value π. We will not discuss this solution in detail as it is unstable owing to the attractive force between vortex lines of opposite sign.

Seeger *et al.*[8] have shown how to construct superpositions of the solutions (10), (11), and (12) using a Bäcklund transformation. One interesting solution[14,9] corresponds to the collision of two vortex lines of opposite sign moving in opposite directions

$$\tan\tfrac{1}{4}\phi = \frac{\bar{c}}{v}\frac{\sinh(vt/\lambda)}{\cosh(x/\lambda)}. \quad (18)$$

The collision occurs at $x=0$ at $t=0$ and as the vortex lines in one dimension do not have hard cores they pass through each other. It is also possible to get a stable bound state of the two vortex lines. Thus in (18) choose $v=i\nu\lambda_0$, where ν is a real frequency and we get a localized oscillatory solution

$$\tan\tfrac{1}{4}\phi = \frac{\omega_0}{\nu}\frac{\sin\omega t}{\cosh(x/\lambda'')}, \quad (19)$$

where

$$\omega = \nu/(1+\nu^2/\omega_0^2)^{1/2}, \qquad \lambda''=\lambda_0(1+\nu^2/\omega_0^2)^{1/2} \quad (20)$$

and $\omega_0=\bar{c}/\lambda_0$ is the cutoff frequency of the barrier. The frequency ω is less than ω_0 and the free energy is

$$F=2F_0/(1+\nu^2/\omega_0^2)^{1/2}=2F_0(1-\omega^2/\omega_0^2)^{1/2}, \quad (21)$$

where $2F_0$ is the free energy of two isolated vortex lines. This solution may be difficult to realize in practice, however.

Solutions of E type, Eq. (9), with $v>\bar{c}$, are immediately obtained from (10), (11), and (12) by making a phase change of π and replacing λ by $\lambda'=\lambda_0(v^2/\bar{c}^2-1)^{1/2}$. These solutions are

$$\tan\tfrac{1}{4}[\phi_0(x,t)+\pi]=\exp[-(x-vt)/\lambda'], \quad (22)$$

$$\sin\tfrac{1}{2}[\phi_0(x,t)]=\mathrm{sn}[(x-vt)/\lambda' k], \qquad v>\bar{c} \quad (23)$$

$$\sin\tfrac{1}{2}[\phi_0(xt)]=k\,\mathrm{sn}[(x-vt)/\lambda']. \quad (24)$$

These solutions represent, respectively, a single voltage pulse, a series of identical voltage pulses, and a series of alternating voltage pulses passing through the barrier. They correspond to the plasma oscillation discussed by Josephson.[4]

3. SMALL OSCILLATIONS

We now investigate the small oscillations of a static vortex array. If in Eq. (3) (with $\beta=0$) we let $\phi(x,t)=\phi_0(x)+u(x)e^{\pm i\omega t}$, where $\phi_0(x)$ is a solution of (6), and linearize with respect to u we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\omega^2}{\bar{c}^2}\right)u(x)=(1/\lambda_0^2)u(x)\cos\phi_0(x). \quad (25)$$

Here $\phi_0(x)$ can be any of the solutions (10), (11), or (12) with $v=0$. It turns out that no interesting localized modes of oscillation of a single vortex line (10) exist. Heliconlike modes, which occur in type-II superconductors, are absent in one dimension. Substituting (11) in (25) and introducing the new variable $\alpha=x/\lambda_0 k$, we find

$$\partial^2 u/\partial\alpha^2=[2k^2\,\mathrm{sn}^2\alpha-k^2(1+\omega^2/\omega_0^2)]u(\alpha), \quad (26)$$

where $\omega_0=\bar{c}/\lambda_0$. This is Lamé's equation[15] and three simple solutions are provided immediately by the ellipsoidal harmonics of degree one. These solutions and their corresponding frequencies are

(i) $u(\alpha)=\mathrm{sn}\alpha, \quad \omega^2=\omega_1^2=\omega_0^2/k^2,$ (27)

(ii) $u(\alpha)=\mathrm{cn}\alpha, \quad \omega^2=\omega_2^2=\omega_0^2(1-k^2)/k^2,$ (28)

(iii) $u(\alpha)=\mathrm{dn}\alpha, \quad \omega^2=0.$ (29)

The solution snα has nodes at the centers of the vortex lines and corresponds to a frequency $\omega_1>\omega_0$. This solution corresponds to the plasma oscillation described by Josephson.[4] The effect of the magnetic field is given by the factor k^{-1}. The solution cnα has a maximum amplitude at the centers of the vortex lines. It may be regarded as a collective mode of the vortex lines. The frequency vanishes exponentially for low

[14] A. Seeger, H. Dorth, and A. Kochendorfer, Z. Physik **134**, 173 (1952), Eq. (35). We are indebted to Dr. R. Hobart for bringing this to our attention.

[15] The relevant properties are given by E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1940), Chap. XXIII.

fields and increases linearly for high fields:

$$\omega_2 = 4\omega_0 \exp(-\pi^2 H_{c1}/4H), \qquad H \sim H_{c1}$$
$$= (2\omega_0/\pi)(H/H_{c1}), \qquad H > H_{c1}. \qquad (30)$$

The solution (29) corresponds to a translation of the vortex lines.

It is also possible to construct solutions of (26) for general values of the frequency. The problem is similar to that of an electron moving in a periodic potential and the solutions have the same form as the Bloch functions. The general solution[15] of (26) is

$$u(\alpha) = [H(\alpha-\alpha_0)/\Theta(\alpha)]e^{\rho\alpha}, \qquad (31)$$

where H is an eta function and Θ is a theta function. The quantity ρ plays the role of the wave vector and ρ and the frequency are related to the parameter α_0 by

$$\omega^2 = (\omega_0^2/k^2)\ \mathrm{dn}^2\alpha_0, \qquad \rho = -Z(\alpha_0), \qquad (32)$$

where Z is a zeta function. For small oscillations ρ must necessarily be imaginary and the spectrum of oscillations has two branches corresponding to α_0 in the intervals

$$0 \le \alpha_0 < iK'(k), \qquad K(k) \le \alpha_0 \le K(k) + iK'(k).$$

These two branches correspond to the two types of small oscillations (27) and (28), respectively. A typical dispersion curve is shown in Fig. 1.

The simplest manner in which to observe these oscillations would be to give the vortex lines a translational velocity v as in (11). It should be noted that (25) is separable in terms of the variables $x^* = x - vt$, $t^* = t - (v/\bar{c}^2)x$. When one of the frequencies associated with the translational motion of the vortex lines coincides with the frequency of a collective state of the vortex lines a dc current appears across the barrier. This dc current oscillates with respect to the spatial coordinate and whether or not there is a net dc current across the barrier will depend on the boundary conditions at the edges. Choosing the solution (28) as an example the condition for a dc current is

$$\omega_2 = (2n+1)\pi v/2K\lambda k, \qquad (33)$$

where n is an integer. From (15) and (16) this leads to a relation between the average field and voltage which takes the simple form when $H > H_{c1}$;

$$\bar{V}/\bar{H}d = \bar{c}/c((2n+1). \qquad (34)$$

Eck *et al.*[16] have observed such an effect corresponding to $n=0$. For long barriers other harmonics may also be observable. Their amplitude decreases like $(2n+1)^2 \exp(-\pi nK'/K)$. The effect of damping is easily included in (25). The solution is of the form $u(xt) = e^{\pm i\omega t - (\beta/2)t}u(x)$ and ω^2 is replaced by $\omega^2 + \beta^2/4$ in subsequent formulas.

Finally, we note that if (12) (with $v=0$) is substituted in (25) we again obtain Lamé's equation. The collective mode frequencies are now imaginary, indicating that the solution is unstable. Small oscillations around the solutions (23) and (24) may also be investigated. Solutions of type (24) are unstable, at least for long barriers, presumably against the formation of vortex lines.

ACKNOWLEDGMENT

The authors are indebted to Dr. R. Hobart for some valuable discussions and for bringing Refs. 8 and 9 to their attention.

[16] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Phys. Rev. Letters, **13**, 15 (1964). See also I. M. Dmitrenko, I. K. Yanson, and V. M. Svistunov, JETP Pis'ma V. Redaktsiyu **2**, 17 (1965) [English transl.: JETP Letters **2**, 10 (1965)].