$U(6)\otimes U(6)\otimes O(3)$ with collinear $U(6)\otimes O(2)$ invariance for the vertices. These sets then *separately* saturate the sum rule for $t=0$ and the results for the lowest multiplet remain unchanged.[7] Similarly for the baryons it may be possible to introduce higher states in sets corresponding to higher representations of some suitable group containing $U(6)\otimes U(6)$ so that each set separately satisfies the superconvergence relation for $t=0$.

In general, the saturation of our sum rules with particles of definite mass and spin corresponds to a power-series expansion of the integrals over absorptive parts around $t=0$.[8] The exact saturation for a finite interval in $t$ requires, of course, an infinite set of particles with unlimited spin. In order to saturate our sum rules for small finite values of $t$, we may try to use the sequence of states which saturate the forward superconvergence relations. The nonforward superconvergence relation will certainly imply stringent additional restrictions on the mass spectrum and on the vertices.

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[7] Recent calculations by Oehme, as well as by Freund and Rotelli, for the case of mesons, indicate that this is indeed what happens if higher representations of $U(6)\otimes U(6)\otimes O(3)$ are included (private communication). See also P. G. O. Freund, R. Oehme, and P. Rotelli, Phys. Rev. (to be published).

[8] For a detailed discussion of these points, see Ref. 2.

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# Infinite Multiplets and Crossing Symmetry.* I. Three-Point Amplitudes

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This paper investigates what remains of crossing symmetry in theories that are conventional local field theories in all but one respect: that infinite irreducible representations of the homogeneous Lorentz group are used. Only vertex functions are studied here; results for scattering amplitudes will be reported in a sequel. It is found that: (i) Form factors for scattering ($t<0$) and form factors for annihilation ($t>4m^2$) are strongly related to each other by the requirement that the interaction Lagrangian density be local, but they are *not* connected by analytic continuation. (ii) In the case of half-integral-spin fields, the empirical fact that the parities of particles and antiparticles are opposite makes it necessary to use a pair of conjugate irreducible representations, rather than a single unitary irreducible representation. An analog of the Dirac equation allows one to avoid parity doubling and to ensure a proper physical interpretation, provided that quantization is carried out with anticommutators.

## I. INTRODUCTION

LOCAL field theory possesses a number of "good" properties of a general sort, such as microcausality and crossing symmetry; and some "bad" specific properties, for example, the fact that the first Born approximation is an extremely poor representation of experimental form factors. Infinite-component field theories were first introduced because of the ease with which they can accommodate internal symmetries of the type of $SU(6)$, but even apart from $SU(6)$ they turned out to have considerable intrinsic interest. In these theories the first Born approximation to the form factors is remarkably similar to the best parametric fits to experimental data.[1] On the other hand, it is not clear that their general properties are satisfactory.[2] In a previous paper[3] it has been shown that locality, in the dual sense of a local Lagrangian density and local commutation relations, can be satisfied, and that the conventional relation between spin and statistics is at least favored. The purpose of the present paper is to show precisely what are the crossing properties of a sample infinite-component "local" field theory.

The conclusions that have been reached here, with regard to vertex functions, are as follows. The requirement that the Born approximation be given by a local interaction Lagrangian density implies that scattering and annihilation form factors are strongly related to each other. However, the form factors for the two channels are *not* related to each other by analytic continuation in the invariant momentumt ransfer. This does not mean that analyticity is lacking, but only that the analytic continuation of a vertex function from negative to positive values of the invariant momentum transfer has no direct physical significance. In the case of half-integral-spin theories, it is found (as first pointed out to

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[1] See, e.g., G. Cocho, C. Fronsdal, H. Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966).

[2] See, e.g., E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967).

[3] C. Fronsdal, Phys. Rev. **156**, 1653 (1967).

us by Leutwyler[4]) that the use of unitary representations conflicts with the empirical fact that particles and antiparticles have opposite parities. A satisfactory theory can be constructed with the aid of two conjugate "almost unitary" representations that are related by an analog of the Dirac equation. In such a theory, crossing symmetry is found to be similar to the integral-spin case, with some additional features that are familiar from the Dirac theory. Although a simple model without internal degrees of freedom forms the main basis for the exposition, it is shown that all the results are quite general. In particular, it is shown that the Dirac-like equations for the half-integral-spin case can be extended to $SL(6,C)$, and the Dirac-Majorana matrices are calculated for this case.

## II. INTEGRAL-SPIN THEORIES

Let $\psi(x)$ be an infinite-component field that transforms according to a unitary irreducible representation of $SL(2,C)$. For definiteness, consider the particular representation $D(N,0)$ whose basis is the traceless symmetric tensor

$$\psi_{\mu_1\cdots\mu_N}. \tag{1}$$

This representation is equivalent to a unitary representation if

$$(N+1)^2<1. \tag{2}$$

Reduction according to the rotation subgroup reveals that the spin $j$ has a simple spectrum consisting of the values 0, 1, 2, $\cdots$. The basis for the contragredient representation is a tensor $\bar{\psi}$, with indices as follows:

$$\bar{\psi}^{\mu_1\cdots\mu_N}. \tag{3}$$

The fact that $D(N,0)$ is equivalent to a unitary representation means that there exists a positive definite matrix $\beta$ such that the relation

$$\bar{\psi}=\psi^*\beta \tag{4}$$

is consistent with the transformation properties of $\psi^*$ and of $\bar{\psi}$. If the representation is made unitary, by the introduction of properly normalized basis vectors in Hilbert space, then $\beta$ is the unit matrix. Consequently, the matrix $\beta$ may be regarded as the metric in Hilbert space.

The $x$-space properties of the field $\psi(x)$, in the absence of interactions, may be characterized by complete mass degeneracy,

$$(p^2-m^2)\psi(x)=0, \tag{5}$$

or more generally, by a mass spectrum determined by a wave equation of the form

$$(\Gamma^{\mu\nu}p_\mu p_\nu-m^2)\psi(x)=0, \tag{6}$$

where $\Gamma^{\mu\nu}$ is a set of constant matrices with the requisite transformation properties. It turns out that, except in the case when (6) reduces to the special form (5), there are always solutions for spacelike momenta. Therefore, in order that our considerations be strictly relevant to physics, it is necessary to generalize the Majorana-like theories considered here.[5]

Suppose that the positive frequency components of $\psi(x)$ create a set of particle states, then antiparticles, if they exist, must be described by a field that transforms contragrediently to $\psi(x)$. Let this field be denoted by $\psi_{(c)}(x)$, or

$$\psi_{(c)}{}^{\mu_1\cdots\mu_N}, \tag{7}$$

where the subscript means "charge conjugate field." The negative-frequency components of $\psi(x)$ may now be given a physical interpretation by means of the identification

$$\psi_{(c)}(x)=\bar{\psi}(x). \tag{8}$$

On this background one may set up a local theory of free fields based on the Lagrangian density

$$\mathcal{L}_0=\bar{\psi}(\Gamma^{\mu\nu}p_\mu p_\nu-m^2)\psi, \tag{9}$$

with local, canonical, Bose-Einstein commutation relations. The logical coherence of such theories cannot be guaranteed at the present time. The limited aim of this work is to find out what type of crossing symmetry they exemplify.

In order to study crossing symmetry with the least possible technical complications, we introduce a conventional scalar neutral field $A(x)$, and a local, nonderivative interaction Lagrangian density

$$\mathcal{L}_I=g\bar{\psi}(x)\psi(x)A(x). \tag{10}$$

Here $g$ is a coupling constant and $\bar{\psi}\psi$ is short for the invariant

$$\bar{\psi}^{\mu_1\cdots\mu_N}\psi_{\mu_1\cdots\mu_N}. \tag{11}$$

In the lowest order in the coupling constant there are two different scattering amplitudes. First, there is the scattering of a particle with spin $j$ by the external source $A(x)$, with or without change of $j$; this is characterized by the amplitude

$$g\langle j'|\bar{\psi}(p)\psi(q)|j\rangle A(p-q), \tag{12}$$

where the four-vectors $p$ and $q$ both have positive energy components. This process may be compared with an annihilation process in which a particle and an antiparticle are annihilated by the external source. In this case the amplitude is

$$g\langle 0|\psi_{(c)}(p)\psi(q)|j,j'\rangle A(p+q), \tag{13}$$

where again $p_0$ and $q_0$ are positive.

[4] H. Leutwyler (private communication).

[5] Equations that have no unphysical solutions have been found by Y. Nambu and by the author. See Y. Nambu, Phys. Rev. **160**, 1171 (1967); C. Fronsdal, *ibid.* **156**, 1665 (1967).

Crossing symmetry, to lowest order in the coupling constant, is simply the statement that the same constant $g$ appears in (12) and in (13). To see what this means we must evaluate the "kinematical factors"

$$K_{jj'}(t)=\langle j'|\bar{\psi}(p)\psi(q)|j\rangle,$$
$$K_{jj'}{}^x(s)=\langle 0|\psi_{(c)}(p)\psi(q)|j,j'\rangle,$$

where

$$t=(p-q)^2,\quad s=(p+q)^2,$$

in the relevant regions of $t$ or $s$; that is, for negative $t$ and positive $s$, respectively. Taking $j=j'=0$ for simplicity, one finds[6]

$$K_{00}(t)=\frac{\sinh[(N+1)\sinh^{-1}[t(t-4m^2)(2m^2)^{-2}]^{1/2}]}{(N+1)[t(t-4m^2)(2m^2)^{-2}]^{1/2}}$$
$$K_{00}{}^x(s)=\frac{\sinh[(N+1)\sinh^{-1}[s(s-4m^2)(2m^2)^{-2}]^{1/2}]}{(N+1)[s(s-4m^2)(2m^2)^{-2}]^{1/2}}. \tag{14}$$

Both functions have the same functional dependence on $pq$, and the correct branch of the inverse sinh is, in both cases, that which is analytic in the neighborhood of $pq=m^2$. Thus $K_{00}(t)$ is regular at $t=0$ and has a square-root-type singularity at $t=4m^2$, while $K_{00}{}^x(s)$ is regular at $s=4m^2$ and has a square-root type singularity at $s=0$. In the special case of non-negative integer values of $N$, the functions are entire functions, analytic in the whole complex plane of their respective variables.

In conventional local field theory, crossing symmetry means that the function $K_{00}(t)$, continued analytically from the physical region of $t$, which is the lower side of the negative real axis, through a path that passes between $t=0$ and $t=4m^2$, to a point $s$ on the upper side of the positive real axis to the right of the point $4m^2$, coincides with the function $K_{00}{}^x(s)$ evaluated at that point. We see that our functions satisfy this type of crossing symmetry only if $N$ is a non-negative integer; that is, only if $D(N,0)$ is finite-dimensional. When $N$ is in the range (2), which is the range in which $D(N,0)$ is unitary, there is no finite path of analytic continuation that connects $K_{00}(t)$ to $K_{00}{}^x(s)$ in their respective physical regions.

Although the difference between crossing symmetry in conventional theories and unitary theories appears to be a subtle one in general, it turns out to be quite dramatic in special cases. To illustrate, let us take $N=-\frac{1}{2}$ to obtain

$$K_{00}(t)=(1-t/4m^2)^{-1/2},$$
$$K_{00}{}^x(s)=(s/4m^2)^{-1/2}. \tag{15}$$

Here crossing symmetry, in the conventional sense, has been completely obfuscated.

[6] This result was reported in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1966). However, at that time the analytic structure was not well understood.

## III. UNITARY THEORIES WITH HALF-INTEGRAL SPINS

Consider now the representation $D(N,K)$, whose basis is the symmetric spinor tensor

$$\psi_{A_1\cdots A_{N+k}}{}^{\bar{B}_1\cdots\bar{B}_N}. \tag{16}$$

This representation is equivalent to a unitary representation if $N=-\frac{1}{2}(k+2)+i\rho$, $\rho=$real; reduction according to the rotation subgroup reveals the spin content $j=\frac{1}{2}k$, $\frac{1}{2}(k+2)$, $\cdots$. The basis for the contragredient representation is

$$\bar{\psi}_{\bar{B}_1\cdots\bar{B}_N}{}^{A_1\cdots A_{N+k}}, \tag{17}$$

and unitarity means that there exists a positive-definite operator $\beta$, the metric operator in Hilbert space, such that the relation

$$\bar{\psi}=\psi^*\beta \tag{18}$$

is consistent with the transformation properties of $\bar{\psi}$ and of $\psi^*$.

Let us postpone questions of canonical commutation or anticommutation relations and postulate, for the present, only the wave equation (5). As before, let the charge-conjugate field $\psi_{(c)}(x)$ be identified with $\bar{\psi}(x)$, and let us introduce the interaction Lagrangian (10). Then we may calculate the kinematical factors $K_{jj'}(t)$ and $K_{jj'}{}^x(s)$ and investigate the behavior of these functions under conventional crossing.

Physical application of this type of model is not possible unless a parity operator exists. If $P$ is the operator of space reflection, and if $\psi(x)$ transforms according to $D(-\frac{1}{2}(k+2)+i\rho,\ k)$, then $P\psi(x)$ transforms according to $D(-\frac{1}{2}(k+2)-i\rho,\ k)$. Consequently, the operator of intrinsic parity exists within the space of a representation $D(N,k)$ if $N=-\frac{1}{2}(k+2)$ only. Taking $N=-\frac{1}{2}\times(k+2)$, one has, in momentum space,

$$P\psi(\mathbf{p},p_0)=\pm(-1)^{j-1/2}\psi(-\mathbf{p},p_0). \tag{19}$$

In the case of half-integral spins, particles and antiparticles have opposite parities, and since the charge-conjugate field is related to $\psi$ by Eq. (8), it follows that the double sign in (19) must be coupled to the sign of the energy. For positive frequency components,

$$P\psi(\mathbf{p},p_0)=(-1)^{j-1/2}\psi(-\mathbf{p},p_0),$$
$$P\psi_{(c)}(\mathbf{p},p_0)=(-1)^{j+1/2}\psi_{(c)}(-\mathbf{p},p_0). \tag{20}$$

Now we shall see that this gives trouble for crossing symmetry.

Explicit evaluation[7] of $K_{\frac{1}{2}\frac{1}{2}}(t)$ gives, for $N=-\frac{3}{2}$, $k=1$,

$$K_{\frac{1}{2}\frac{1}{2}}(t)=(1-t/4m^2)^{-3/2}\times\bar{u}^A(p)[\tfrac{1}{2}\delta_A{}^B+\tfrac{1}{2}p_A{}^{\bar{E}}q_{\bar{E}}{}^B]u_B(q), \tag{21}$$

where $\bar{u}^A(p)$ and $u_B(q)$ are the Pauli 2-spinors associated with the two spin-$\frac{1}{2}$ states. The two terms in brackets are parity conjugate, and the whole expression is invariant

[7] The calculations are carried out in Appendix A.

under space reflection. Similarly,

$$K_{\frac{1}{2}\frac{1}{2}}{}^{x}(s)=(s/4m^2)^{-3/2}\times u_{(c)}{}^{A}(p)[\tfrac{1}{2}\delta_A{}^B+\tfrac{1}{2}p_A{}^{\bar{E}}q_{\bar{E}}{}^B]u_B(q). \quad (22)$$

But this is odd under space reflection. In other words, if particles and antiparticles have opposite parities, and if they are assigned to a pair of mutually contragredient, irreducible representations, then parity conservation forbids the single-quantum annihilation process. In such circumstances there can be no meaningful concept of crossing symmetry. This was first pointed out by Leutwyler,[4] who also suggested the following remedy.

Suppose that, instead of one irreducible representation, we have two, $D(N,k)$ and $D(N',k)$, with bases $\psi$ and $\psi'$, that are parity conjugates in the sense that, instead of (20),

$$\begin{aligned} P\psi(\mathbf{p},p_0)&=(-1)^{j-1/2}\psi'(-\mathbf{p},p_0)\,,\\ P\psi_{(c)}(\mathbf{p},p_0)&=(-1)^{j+1/2}\psi_{(c)}'(-\mathbf{p},p_0)\,, \end{aligned} \quad (23)$$

for the positive energy components. Then, to the interaction Lagrangian (10) must be added the parity conjugate, and the kinematical form factors become

$$K_{jj'}(t)=\langle j'|\bar{\psi}(p)\psi(q)+\bar{\psi}'(p)\psi'(q)|j\rangle\,, \quad (24)$$

$$K_{jj'}{}^{x}(s)=\langle 0|\psi_{(c)}(p)\psi(q)-\psi_{(c)}'(p)\psi'(q)|jj'\rangle\,, \quad (25)$$

both of which are reflection invariant and different from zero.

Two representations $D(N,k)$ and $D(N',k)$ are parity conjugates if $N+N'=-k-2$. Thus, if both are unitary, then $N=-\frac{1}{2}(k+2)+i\rho$, $N'=-\frac{1}{2}(k+2)-i\rho$, $\rho=$real. However, we do not believe that this choice of representations is satisfactory.[8] It amounts, in practical terms, to parity doubling, a concept that has never found support in nature. It is, in fact, necessary to double the representation space when one deals with the half-integral-spin fields because annihilation must occur, but at the same time it is important to avoid parity doubling. One way to accomplish this is to impose a wave equation that effectively identifies the particle contents of the two representations. This leads, as we shall see, to the use of nonunitary representations, although physical unitarity, in the sense of conservation and positiveness of probability, will be preserved.

## IV. UNITARY-CONJUGATE REPRESENTATIONS AND FIRST-ORDER WAVE EQUATIONS

The foregoing arguments for the relevance of pairs of parity-conjugate irreducible representations linked by a wave equation receive support from different considerations. Efforts to quantize with anticommutators have shown[3] that this requires the existence of a set of Majorana matrices $\Sigma_\mu$, that transform among themselves like a four-vector, and a relation of the type

$$m\bar{\psi}=\psi^*\Sigma_\mu p_\mu \quad (26)$$

between the complex conjugate and the contragredient fields. In order to achieve the correct normalization of the particle states, it was found necessary to impose the condition that all the eigenvalues of $\Sigma p$ equal $+1$, which implies the inverse relation

$$m\psi^*=\bar{\psi}\Sigma_\mu p_\mu. \quad (27)$$

In addition, it is possible, but not necessary, that the relation (18) still holds. In general there are two fields, $\psi$ and $\bar{\psi}^*$, and their contragredients, $\bar{\psi}$ and $\psi^*$, linked by a pair of wave equations that are very similar to the two halves of the Dirac equation.

The simplest possibility is to select a unitary, irreducible representation that allows the existence of parity operator and Majorana matrices acting within the irreducible Hilbert space. This approach was followed by Majorana,[9] who found the only representation with these properties: $D(-\frac{3}{2},1)$. In such a theory, the field has only positive- (or only negative-) energy Fourier components. An alternative, which allows for both signs of the energy, is to use a pair of these representations. Neither possibility will be investigated here, for two reasons. Firstly, both theories forbid annihilation. Secondly, it is necessary to develop a theory that is capable of generalization to more interesting models, employing groups that are larger than $SL(2,C)$; familiar examples of such groups do not have representations with all the special properties that characterize $D(-\frac{3}{2},1)$. In contrast, the following procedure will be shown to be typical.[10]

A variety of methods may be employed to determine all pairs of irreducible representations that can be linked by a first-order wave equation. However, this is not the place to exhibit these techniques, since the result for $SL(2,C)$ has been known for a long time.[11] The result is that two irreducible representations $D(N,k)$ and $D(N',k)$ can be related to each other by a first-order wave equation if and only if $N-N'=0$ or $\pm 1$.[12] The requirement that the representations be parity conjugates gives the additional condition $N+N'=-k-2$.

[8] It is true that the procedure adopted below can be applied to this pair of representations if $k=1$, since a set of Majorana matrices that links them exists. This alternative was developed by A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 754 (1967). We have rejected this possibility because it is too special; it does not generalize to representations $D(N,k)$ with $k>1$, and it has no analogs with larger groups.

[9] This is the only irreducible representation with half-integral spins that admits Majorana matrices. See E. Majorana, Nuovo Cimento **9**, 335 (1932). The Majorana theory was later rediscovered and generalized; see I. M. Gelfand and A. M. Yaglom, as quoted by M. A. Naimark, in *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., London, 1964).

[10] See Appendix B.

[11] See Naimark's book, Ref. 9.

[12] The exceptional case of the pair $D(-\frac{3}{2}\pm i\rho,1)$ was discussed above.

Since the solution $N=N'=-\frac{3}{2}$ has already been rejected, a unique solution remains,

$$N=-\tfrac{1}{2}(k+3),\quad N'=-\tfrac{1}{2}(k+1). \tag{28}$$

The two representations $D(N,k)$ and $D(N',k)$ are not unitary, but unitary-conjugate to each other, which means that $\psi^*$ transforms equivalently to the contragredient of $\psi'$. This situation is strictly analogous to what is familiar in the Dirac theory.

It is convenient to normalize the basis vectors so that $\psi^*$ transforms precisely like $\tilde{\psi}'$. Then, the usual expansions of the generalized tensors take the form (from now on we shall take $k=1$)

$$\psi_{A_1\cdots A_{N+1}}{}^{\bar{B}_1\cdots\bar{B}_N}=\sum_{t=0}^{\infty}\frac{(-)^t[(2t+2)!]^{1/2}}{t!}\times S\tilde{\psi}_{A_1\cdots A_{t+1}}{}^{B_1\cdots B_t}\delta_{A_{t+2}}{}^{B_{t+1}}\cdots\delta_{A_{N+1}}{}^{B_N}, \tag{29}$$

and similarly for $\psi'$, the coefficient being

$$(-)^t[(2t+2)!]^{1/2}/(t+1)!.$$

With this normalization the noncompact generators of $SL(2,C)$ take the form

$$\begin{aligned}s_{k0}\tilde{\psi}_{A_1\cdots A_{t+1}}{}^{B_1\cdots B_t}={}&(\sigma_k)_B{}^A\Bigg\{(t+1)\left[\frac{t+2}{2(2t+3)}\right]^{1/2}\hat{\psi}_{AA_1\cdots A_{t+1}}{}^{BB_1\cdots B_t}-i(N+\tfrac{3}{2})(2t+3)^{-1}\\&\times S[t\delta_A{}^{B_1}\tilde{\psi}_{A_1\cdots A_{t+1}}{}^{BB_2\cdots B_t}+(t+1)\delta_{A_1}{}^{B}\tilde{\psi}_{AA_2\cdots A_{t+1}}{}^{B_1\cdots B_t}-2t(t+1)(2t+1)^{-1}\delta_{A_1}{}^{B_1}\tilde{\psi}_{AA_2\cdots A_{t+1}}{}^{BB_2\cdots B_t}]\\&+t[(t+1)/2(2t+1)]^{1/2}S[\delta_A{}^{B_1}\delta_{A_1}{}^{B}\tilde{\psi}_{A_2\cdots A_{t+1}}{}^{B_2\cdots B_t}-(t-1)(2t+1)^{-1}\delta_A{}^{B_1}\delta_{A_1}{}^{B_2}\tilde{\psi}_{A_2\cdots A_{t+1}}{}^{BB_3\cdots B_t}\\&-t(2t+1)^{-1}\delta_{A_1}{}^{B}\delta_{A_2}{}^{B_1}\tilde{\psi}_{AA_3\cdots A_{t+1}}{}^{B_2\cdots B_t}+(t-1)[2(2t+1)]^{-1}\delta_{A_1}{}^{B_1}\delta_{A_2}{}^{B_2}\tilde{\psi}_{AA_3\cdots A_{t+1}}{}^{BB_3\cdots B_t}]\Bigg\},\end{aligned} \tag{30}$$

and similarly for the action of $s_{k0}'$ on $\psi'$, with $N$ replaced by $N'$.

The fact that our two representations may be related to each other by a first-order wave equation is obvious;

$$\begin{aligned}m\psi_{A_1\cdots A_{N+1}}{}^{\bar{B}_1\cdots\bar{B}_N}&=p_{\bar{B}_{N+1}}{}^{A_{N+2}}\psi'_{A_1\cdots A_{N+2}}{}^{\bar{B}_1\cdots N+1},\\m\psi'_{A_1\cdots A_{N+2}}{}^{\bar{B}_1\cdots\bar{B}_{N+1}}&=Sp_{A_{N+2}}{}^{\bar{B}_{N+1}}\psi_{A_1\cdots A_{N+1}}{}^{\bar{B}_1\cdots\bar{B}_N}.\end{aligned} \tag{31}$$

These equations are of the form

$$\begin{aligned}m\psi&=\Sigma_\mu p_\mu\psi',\\m\psi'&=\hat{\Sigma}_\mu p_\mu\psi.\end{aligned} \tag{32}$$

In order to determine the matrix elements of $\Sigma_\mu$ and $\hat{\Sigma}_\mu$ we transform to the rest system and expand $\psi$ and $\psi'$ as in (29). The result is

$$(\Sigma_0)_{jj'}=(\hat{\Sigma}_0)_{jj'}=(j+\tfrac{1}{2})\delta_{jj'}. \tag{33}$$

The remaining $\Sigma$ matrices may be calculated by means of their transformation laws:

$$\begin{aligned}i\Sigma_k&=s_{k0}\Sigma_0-\hat{\Sigma}_0 s_{k0}',\\i\hat{\Sigma}_k&=s_{k0}'\hat{\Sigma}_0-\Sigma_0 s_{k0}.\end{aligned} \tag{34}$$

The result is that $\Sigma_k$ and $\hat{\Sigma}_k$ have the same matrix elements as $is_{k0}$, except that (i) the sign of the first term is changed, and (ii) the factor $N+\frac{3}{2}$ is replaced by $-(t+1)$ in the formula for $\Sigma_k$ and by $+(t+1)$ in the formula for $\hat{\Sigma}_k$. This difference in sign between $\Sigma_k$ and $\hat{\Sigma}_k$ corresponds to the fact that three of the Dirac $\gamma$ matrices are non-Hermitian.

In order to simplify the notation, and to stress the similarity to the Dirac theory, let us introduce

$$\phi=\begin{pmatrix}\psi\\\psi'\end{pmatrix},\quad\bar{\phi}=(\tilde{\psi},\tilde{\psi}'). \tag{35}$$

Here $\tilde{\psi}$ and $\tilde{\psi}'$ are fields that transform contragrediently to $\psi$ and $\psi'$, respectively. We have noted that $\tilde{\psi}$ transforms like $\psi'^*$, and $\tilde{\psi}'$ transforms like $\psi^*$; consequently, we may identify

$$\bar{\phi}=\phi^\dagger\beta,\quad\beta=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{36}$$

The wave equation (32) becomes

$$(\Gamma_\mu p_\mu-m)\phi=0,\quad\Gamma_\mu=\begin{pmatrix}0&\Sigma_\mu\\\hat{\Sigma}_\mu&0\end{pmatrix}. \tag{37}$$

This may be derived from the Lagrangian density

$$\mathcal{L}_0=\bar{\phi}(x)(\Gamma_\mu p_\mu-m)\phi(x), \tag{38}$$

which is Hermitian since

$$\Gamma_\mu\beta=\beta\Gamma_\mu{}^\dagger, \tag{39}$$

exactly as in the Dirac theory. Note, however, that $\beta\neq\Gamma_0$. The operator of intrinsic parity is

$$P=(-1)^{j-1/2}\beta. \tag{40}$$

In the rest system, the wave equation reduces to

$$[p_0(j+\tfrac{1}{2})\beta-m]\phi=0. \tag{41}$$

This shows that the eigenvalues of $p_0$ are $\pm m(j+\frac{1}{2})^{-1}$, and that the sign of the energy is related to the sign of the eigenvalue of $\beta$. Consequently, the parity of the antiparticle is opposite to that of the particle.

Finally, we note that this theory, if it can be quantized at all, can only be quantized with anticommutators. For the invariant inner product,

$$\begin{aligned}\bar{\phi}\phi&=(1/m)\bar{\phi}\Gamma_\mu p_\mu\phi\\&=(1/m)[\phi'^*\Sigma_\mu p_\mu\phi'+\phi^*\hat{\Sigma}_\mu p_\mu\phi]\end{aligned} \tag{42}$$

has the same sign as the energy; hence it is positive for particle states and negative for antiparticle states.

## V. CROSSING SYMMETRY

We are finally prepared to investigate what crossing symmetry means in a theory that (i) contains only half-integral spins, (ii) contains antiparticles with parity opposite to that of the particles, (iii) is unitary, in the sense that the probability is positive definite when quantized with anticommutators. Our model does not have a reasonable mass spectrum, and for this reason it is unlikely that it can be made into a completely consistent field theory, but it comes closer to that goal than theories considered previously.

To the free Lagrangian (38) we add a local interaction with an ordinary scalar field $A(x)$:

$$\begin{aligned}\mathfrak{L}_I&=g\bar{\phi}\phi A\\&=g[\bar{\psi}(x)\psi(x)+\bar{\psi}'(x)\psi'(x)]A(x).\end{aligned}\tag{43}$$

Projecting out the spin-$\frac{1}{2}$ components, we obtain[7]

$$\begin{aligned}\langle\tfrac{1}{2}|\bar{\psi}\psi|\tfrac{1}{2}\rangle=&\tfrac{1}{2}(z^2-1)^{-1}\{[z-(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\\&\times\bar{u}^A(p)u_A(q)+[-1+z(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\\&\times\bar{u}^A(p)p_A{}^{\bar{E}}q_{\bar{E}}{}^Bu_B(q)\},\end{aligned}\tag{44}$$

$$\begin{aligned}\langle\tfrac{1}{2}|\bar{\psi}'\psi'|\tfrac{1}{2}\rangle=&\tfrac{1}{2}(z^2-1)^{-1}\{[-1+z(z^2-1)^{-1/2}\\&\times\sinh^{-1}(z^2-1)^{1/2}]\bar{u}'^A(p)u_A'(q)+[z-(z^2-1)^{-1/2}\\&\times\sinh^{-1}(z^2-1)^{1/2}]\bar{u}'^A(p)p_A{}^{\bar{E}}q_{\bar{E}}{}^Bu_B'(q)\}.\end{aligned}\tag{45}$$

$$z=pq/m^2.$$

The kinematical form factor is the sum of these two expressions. In the case of scattering, the wave equation reduces to $u'=u$ and $\bar{u}'=\bar{u}$; hence

$$\begin{aligned}K_{\frac{1}{2}\frac{1}{2}}(t)=&(z+1)^{-1}[1+(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\\&\times\bar{u}^A[\tfrac{1}{2}\delta_A{}^B+\tfrac{1}{2}p_A{}^{\bar{E}}q_{\bar{E}}{}^B]u_B,\end{aligned}\tag{46}$$

which has positive parity. In the case of pair annihilation, the field $\bar{\psi}(x)$ must be represented by its negative-frequency component, so that $u'=u$ but $\bar{u}'=-\bar{u}$, and thus

$$\begin{aligned}K_{\frac{1}{2}\frac{1}{2}}{}^x(s)=&(z-1)^{-1}[1-(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\\&\times\bar{u}^A[\tfrac{1}{2}\delta_A{}^B-p_A{}^{\bar{E}}q_{\bar{E}}{}^B]u_B,\end{aligned}\tag{47}$$

which also has positive parity, since in this case

$$\bar{u}(p)=u_{(c)}(p)C^{-1},\tag{48}$$

and the intrinsic parity of $u_{(c)}(p)$ is opposite that of $u(q)$. (Note: In the above formulas $p$ and $q$ are the physical four-momenta, with positive energies.)

In order better to appreciate the meaning of these results it is useful to introduce Dirac four-spinors to describe the particle and antiparticle wave functions for spin $\frac{1}{2}$. Thus, we write

$$\chi(q)=\begin{pmatrix}u_A(q)\\u_{\bar{A}}(q)\end{pmatrix},\qquad \bar{\chi}(p)=\begin{pmatrix}\bar{u}^A(p)\\\bar{u}^{\bar{A}}(p)\end{pmatrix},\tag{49}$$

where the dotted spinors are defined by the Dirac equations

$$\begin{aligned}mu_{\bar{A}}(q)&=q_{\bar{A}}{}^Bu_B(q),\\m\bar{u}^{\bar{B}}(p)&=\bar{u}^A(p)p_A{}^{\bar{B}},\quad\text{scattering case}\\&=-\bar{u}^A(p)p_A{}^{\bar{B}},\quad\text{annihilation case.}\end{aligned}\tag{50}$$

In this notation,

$$\begin{aligned}K_{\frac{1}{2}\frac{1}{2}}(t)=&\tfrac{1}{4}(1-t/4m^2)^{-1}\\&\times\left\{1+\frac{\sinh^{-1}[t(t-4m^2)(2m^2)^{-2}]^{1/2}}{[t(t-4m^2)(2m^2)^{-2}]^{1/2}}\right\}\bar{\chi}\chi,\\K_{\frac{1}{2}\frac{1}{2}}{}^x(s)=&\tfrac{1}{4}(s/4m^2)^{-1}\\&\times\left\{1-\frac{\sinh^{-1}[s(s-4m^2)(2m^2)^{-2}]^{1/2}}{[s(s-4m^2)(2m^2)^{-2}]^{1/2}}\right\}\bar{\chi}\chi.\end{aligned}\tag{51}$$

Does this theory possess crossing symmetry? Certainly it violates the rules of crossing symmetry in the conventional sense. In a more general sense, "crossing symmetry" is a set of rules that relate $K(t)$ to $K^x(s)$; the details of the rules are a property of the particular field theory under consideration. Thus, in the Dirac theory, one writes

$$K_{\frac{1}{2}\frac{1}{2}}(t)=\bar{\chi}(p)\chi(q)D(t)$$

and postulates that the same expression gives $K_{\frac{1}{2}\frac{1}{2}}{}^x(s)$ when $D(t)$ is continued analytically, along a selected path, from negative values of $t$ to the point $s>4m^2$. A similar postulate could be made in the integer-spin theory with equal masses studied in Sec. II. In that theory, application of Feynman rules to the calculation of corrections to the bare vertex $\bar{\psi}\psi A$ gives a corrected scattering vertex of the form $\bar{\psi}\psi AD(t)$ and an annihilation vertex of the form $\bar{\psi}\psi AD^x(s)$, where $D^x(s)$ is the analytic continuation of $D(t)$. Such a simple result does not remain valid when the mass dengeneracy is lifted, because the spin dependence of the propagators introduces, in each order of perturbation, a more complicated spin-dependent correction. In the half-integral-spin case there is no avoiding the spin dependence of the propagators; therefore, there is probably no simple way to express the crossing symmetry of the theory in terms of the physical amplitudes. The fact remains that the Lagrangian density $\mathfrak{L}_0+\mathfrak{L}_I$ given by Eqs. (38) and (43) implies an intimate relationship between "crossed" amplitudes.

In Appendix B we show that the preceding treatment of the half-integral-spin case can be applied with no additional difficulty to a theory with internal degrees of freedom.

## VI. DISCUSSION

Two lines of future research are indicated. Firstly, the present discussion must be extended to include crossing properties of scattering amplitudes. This is more involved than the case of vertex functions, because of the appearance, in the first Born approximation, of the one-particle propagator. Secondly, once the nature of crossing in infinite-component theories has been established, it is important to determine whether or not those experiments that are usually quoted in support of conventional crossing are in conflict with the predictions of such theories. In particular, the consistency of the various determinations of the pion-nucleon coupling constant and the verification of certain dispersion relations are areas that should be investigated.

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## APPENDIX A. EVALUATION OF THE FORM FACTORS

It is required to calculate the contribution to

$$\tilde{\psi}_{\bar{B}_1\cdots\bar{B}_N}{}^{A_1\cdots A_{N+k}}(p)\psi_{A_1\cdots A_{N+k}}{}^{\bar{B}_1\cdots\bar{B}_N}(q) \qquad \text{(A1)}$$

from the first terms in both expansions; that is, the quantity

$$\tilde{\psi}^{A_1\cdots A_k}(p)p_{\bar{B}_1}{}^{A_{k+1}}\cdots p_{\bar{B}_N}{}^{A_{N+k}}S\tilde{\psi}_{A_1\cdots A_k}(q) \times q_{A_{k+1}}{}^{\bar{B}_1}\cdots q_{A_{N+k}}{}^{\bar{B}_N}. \qquad \text{(A2)}$$

This is easily rewritten in the form[1]

$$\sum_{j=0}^{k} F_j(pq)\tilde{\psi}^{A_1\cdots A_k}(p)(q_{A_1}{}^{\bar{E}_1}p_{\bar{E}_1}{}^{B_1})\cdots \times(q_{A_j}{}^{\bar{E}_j}p_{\bar{E}_j}{}^{B_j})\tilde{\psi}_{B_1\cdots B_jA_{j+1}\cdots A_k}, \qquad \text{(A3)}$$

$$F_j(pq)=\sum_{i=j}^{k}\binom{k}{i}\binom{i}{j}(-1)^{i+j}Q_{N-i+j}{}^{(i)}(pq). \qquad \text{(A4)}$$

The $Q$ functions have been evaluated in the following cases: $n=2$, 4, 6, that is, for degenerate representations of $SL(2,C)$, $SL(4,C)$, and $SL(6,C)$; $k=1$, 2, 3. The following formulas hold in all those cases, but a derivation for general values of $n$ and $k$ has not yet been carried out:

$$Q_N{}^{(i)}=\sum_{r=0}^{i+\frac{1}{2}n-1}\binom{r+i+\frac{1}{2}n-1}{r}\binom{N+\frac{1}{2}n-1-r}{N-i} \times\left(y-\frac{1}{y}\right)^{-i-r-\frac{1}{2}n} \times[(-1)^r y^{N+\frac{1}{2}n-r}+(-1)^{i+\frac{1}{2}n}y^{-N-\frac{1}{2}n+r}] \qquad \text{(A5)}$$

$$=\binom{i+\frac{1}{2}n+N-1}{N}(pq)^{N-i} \times {}_2F_1\left(\frac{i-N}{2},\frac{i+1-N}{2};-N-\tfrac{1}{2}n+1;(pq)^{-2}\right), \qquad \text{(A6)}$$

where

$$y^{\pm1}=m^{-2}\{pq\pm[(pq)^2-m^4]^{1/2}\}. \qquad \text{(A7)}$$

For practical calculations the following simple formulas are to be preferred:

$$Q_N{}^{(1-\frac{1}{2}n)}=(y-1/y)^{-1}(y^{N+\frac{1}{2}n}-y^{-N-\frac{1}{2}n}), \qquad \text{(A8)}$$

$$Q_N{}^{(i+1)}=(2i+n)^{-1}\frac{\partial}{\partial z}Q_N{}^{(i)}, \qquad \text{(A9)}$$

$$z=pq/m^2. \qquad \text{(A10)}$$

When $n=2$ and $k=0$, the quantity (A3) reduces to $F_0=Q_N{}^{(0)}$, which according to (A8) is the function

$$(y-1/y)^{-1}(y^{N+1}-y^{-N-1}) \sim(z^2-1)^{-1/2}\sinh[(N+1)\ln y],$$

which, properly normalized, gives us the functions (13), (14).

When $k\neq0$ it is convenient to rearrange the expression (A3) by writing

$$(1/m^2)q_A{}^{\bar{E}}p_{\bar{E}}{}^B=2z-(1/m^2)p_A{}^{\bar{E}}q_{\bar{E}}{}^B.$$

The result is

$$\sum_{l=0}^{k} G_l(z)\tilde{\psi}^{A_1\cdots A_k}(p_{A_1}{}^{\bar{E}_1}q_{\bar{E}_1}{}^{B_1})\cdots \times(p_{A_l}{}^{\bar{E}_l}q_{\bar{E}_l}{}^{B_l})\tilde{\psi}_{B_1\cdots B_lA_{l+1}\cdots A_k}, \qquad \text{(A11)}$$

where

$$G_l(z)=\sum_{k\geq i\geq j\geq l}\sum\binom{k}{i}\binom{i}{j}\binom{j}{l} \times(-1)^{i+j+l}Q_{N-i+j}{}^{(i)}(z)(2z)^{j-l}. \qquad \text{(A12)}$$

When $n=2$ and $k=1$, (A11) reduces to

$$G_0(z)\tilde{\psi}^A(p)\psi_A(q)+(1/m^2)G_1(z)\tilde{\psi}^A(p)p_A{}^{\bar{E}}q_{\bar{E}}{}^B\psi_B(q),$$

where

$$G_0(z)=Q_N{}^{(0)}-Q_{N-1}{}^{(1)}+2zQ_N{}^{(1)},\quad G_1(z)=-Q_N{}^{(1)}.$$

If $N=-\frac{3}{2}$, (A8) and (A9) give

$$Q_N{}^{(0)}=-(y^{1/2}+y^{-1/2})^{-1}=-(2z+2)^{-1/2},$$
$$Q_{N-1}{}^{(0)}=-(2z+1)(2z+2)^{-1/2},$$
$$Q_N{}^{(1)}=\tfrac{1}{2}(2z+2)^{-3/2},$$
$$Q_{N-1}{}^{(1)}=-(z+\tfrac{3}{2})(2z+2)^{-3/2},$$

or

$$G_0(z)=G_1(z)=-\tfrac{1}{2}(2z+2)^{-3/2},$$

which is the result used to calculate $K_{\frac{1}{2}\frac{1}{2}}(t)$, Eq. (21).

If $n=2$, $k=1$, and $N=-1$, then (A8) and (A9) give

$$Q_N{}^{(0)}=\lim_{N\to-1}[(N+1)(N+2)]^{-1}(y-y^{-1})^{-1}\times(y^{N+1}-y^{-N-1})=(z^2-1)^{-1/2}\cosh^{-1}z,$$
$$Q_N{}^{(1)}=-\tfrac{1}{2}z(z^2-1)^{-3/2}\cosh^{-1}z+\tfrac{1}{2}(z^2-1)^{-1},$$
$$Q_{N-1}{}^{(0)}=(1-z^{-2})^{-1/2}\cosh^{-1}z-1/(N+1),$$
$$Q_{N-1}{}^{(1)}=-\tfrac{1}{2}(z^2-1)^{-3/2}\cosh^{-1}z+\tfrac{1}{2}z(z^2-1)^{-1},$$

or

$$G_0(z)=\tfrac{1}{2}(z^2-1)^{-1}\times[z-(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}],$$
$$G_1(z)=-\tfrac{1}{2}(z^2-1)^{-1}\times[1-z(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\text{ for }N=-1.$$

If $n=2$, $k=1$, and $N=-2$, then

$$Q_N{}^{(0)}=-(1-z^{-2})^{-1/2}\cosh^{-1}z=1/(N+2),$$
$$Q_{N-1}{}^{(0)}=-(2z^2-1)(z^2-1)^{-1/2}\cosh^{-1}z+2z/(N+2),$$
$$Q_N{}^{(1)}=\tfrac{1}{2}(z^2-1)^{-3/2}\cosh^{-1}z-\tfrac{1}{2}z(z^2-1)^{-1},$$
$$Q_{N-1}{}^{(1)}=\tfrac{1}{2}(3-2z^2)z(z^2-1)^{-3/2}\cosh^{-1}z-\tfrac{1}{2}(2z^2-1)(z^2-1)^{-1}+1/(N+2),$$

or

$$G_0(z)=-\tfrac{1}{2}(z^2-1)^{-1}\times[1-z(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}],$$
$$G_1(z)=\tfrac{1}{2}(z^2-1)^{-1}\times[z-(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}]\text{ for }N=-2.$$

These are the formulas that were used in Eqs. (44) and (45).

The complete calculation for the case $n=6$ and $k=3$ has been reported previously.[1]

### APPENDIX B. CROSSING IN $SL(n,c)$

We introduce a pair of representations

$$\psi_{NA_1\cdots A_{N+k}}{}^{\bar{B}_1\cdots\bar{B}_N},\ \psi_{N'A_1\cdots A_{N'+k}}{}^{\bar{B}_1\cdots\bar{B}_{N'}}$$

which are not necessarily unitary, their contragredients

$$\bar{\psi}_{N\bar{B}_1\cdots\bar{B}_N}{}^{A_1\cdots A_{N+k}},\ \bar{\psi}_{N'\bar{B}_1\cdots\bar{B}_{N'}}{}^{A_1\cdots A_{N'+k}},\qquad\text{(B1)}$$

and complex conjugates

$$\psi_N{}^*{}_{B_1\cdots B_N}{}^{A_1\cdots A_{N+k}},\ \psi_{N'}{}^*{}_{B_1\cdots B_{N'}}{}^{A_1\cdots A_{N'+k}}.$$

For decomposition of these representations with respect to the compact subgroup $SU(n)$, action of operators, etc., see Refs. 13 and 14. For convenience we take Re $N'>$ Re$N$. We know that a representation is unitary if $N=-\frac{1}{2}(k+n)+i\rho$. We will determine $N$, $N'$ such that the pair of representations $\psi_N$, $\psi_{N'}$ admit the definition of a four-vector $\Sigma_\mu$ analogous to the case of $SL(2,c)$. Define the operators

$$\Sigma_A{}^B=(\bar{\psi}_N\psi_{N'})_A{}^B,\qquad\hat{\Sigma}_A{}^B=(\bar{\psi}_{N'}\psi_N)_A{}^B.\qquad\text{(B2)}$$

In order that these exist we must have $N'=N+1$. $\Sigma_0$ is then given by the trace of $\Sigma_A{}^B$.

Introduce an invariant inner form in the doubled space exactly as in $SL(2,c)$:

$$I=I_N+I_{N'}=\bar{\psi}_N\psi_N+\bar{\psi}_{N'}\psi_{N'}.\qquad\text{(B3)}$$

Then for $I$ to be Hermitian we must require that $\bar{\psi}_N$ be equivalent to $\psi_{N'}{}^*$, and $\bar{\psi}_{N'}$ to $\psi_N{}^*$. Therefore, there must exist a matrix $\beta^N$ connecting $\bar{\psi}_N$ to $\psi_{N'}{}^*$. Determine $\beta^N$ as follows. The expression

$$I_N=\sum_t a_t{}^N\bar{\psi}_t{}^N\hat{\psi}_t{}^N\qquad\text{(B4)}$$

is an invariant.[15] Using

$$\bar{\psi}_t{}^N=\beta_t{}^N\hat{\psi}^*{}_t{}^{N'},\quad(\bar{\psi}_t{}^{N'}=\beta_t{}^{N'}\hat{\psi}^*{}_t{}^N)$$
$$I_N=\sum a_t{}^N\beta_t{}^N\hat{\psi}^*{}_t{}^{N'}\hat{\psi}_t{}^N,\qquad\text{(B5)}$$

and this can be true only if

$$\lambda'{}_A{}^B\bar{\psi}_t{}^N=\beta_t{}^N\lambda'{}_A{}^B\hat{\psi}^*{}_t{}^{N'}.\qquad\text{(B6)}$$

Using formula (VII-15) of Ref. 14,

$$-(N-t)\bar{\psi}_{t+1}{}^N=\beta_t{}^N(N'^*-1)\hat{\psi}^*{}_{t+1}{}^{N'},\qquad\text{(B7)}$$

which gives

$$\beta_t{}^N=(-)^t\frac{(t-N'^*-1)!}{(t-N-1)!}f(N,N'),\qquad\text{(B8)}$$

where $f(N,N')$ is arbitrary and will henceforth be ignored.

Renormalizing according to

$$\hat{\psi}_t{}^N\to\hat{\psi}_t{}^N/\alpha_t{}^N,\quad\hat{\psi}_t{}^{N'}\to\hat{\psi}_t{}^{N'}/\alpha_t{}^{N'},\qquad\text{(B9)}$$

we find that

$$\beta_t{}^N=(-)^t\frac{(t-N'^*-1)!}{(t-N-1)!}\frac{\alpha_t{}^N}{(\alpha_t{}^{N'})^*}.\qquad\text{(B10)}$$

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[13] A Salam and J. Strathdee, Proc. Roy. Soc. (London) **292**, 314 (1966).

[14] C. Fronsdal, Trieste Report No. IC/66/51 (unpublished); R. White, Lecture Notes on the $SL(n,c)$ Symmetry, Summer School in Theoretical Physics, Udaipur, India 1966 (unpublished).

[15] $\hat{\psi}_t{}^N$ refers to the $t$th $SU(n)$ irreducible tensor in the decomposition of $\psi_N$. We use the basis of Ref. 14:

$$a_t{}^N=(-)^t\frac{(2t+k+n-1)!(t-N-1)!}{t!(t+k)!(t+k+n+N-1)!}.$$

Similarly, $\beta_t^{N'}=1/(\beta_t^N)^*$. Hermiticity of the invariant (B3) then requires that $(\beta_t^N)^2=1$, or

$$\alpha_t^N=\pm(\alpha_t^{N'})^*\frac{(t-N-1)!}{(t-N'^*-1)!}. \quad \text{(B11)}$$

The operators $\Sigma_0$ and $\hat{\Sigma}_0$ can be simply calculated:

$$\Sigma_0=\operatorname{tr}(\bar{\psi}_N\psi_{N'A}{}^B)=\sum_t \bar{\psi}_t{}^N a_t{}^N \hat{\psi}_t{}^{N'}. \quad \text{(B12)}$$

Then, using the relation

$$\bar{\psi}_t{}^N{}_{D_1\cdots D_t}{}^{C_1\cdots C_{t+k}}\hat{\psi}_t{}^N{}_{S_1\cdots S_{t+k}}{}^{R_1\cdots R_t}=(1/a_t{}^N)S(\delta_S{}^C)^{t+k}S(\delta_D{}^R)^t, \quad \text{(B13)}$$

we find that in a general basis

$$(\Sigma_0)_t=\alpha_t{}^N/\alpha_t{}^{N'}, \quad \text{(B14)}$$

and by a similar calculation

$$(\hat{\Sigma}_0)_t=\frac{\alpha_t{}^{N'}}{\alpha_t{}^N}\frac{a_t{}^N}{a_t{}^{N'}}.$$

Note that (B13) is true only for those tensors not involved in the trace condition, i.e., all $C_i\neq$ all $D_k$. For simplicity we restrict ourselves to such tensors.

Then the operator

$$\Gamma_0=\begin{pmatrix}0 & \Sigma_0\\ \hat{\Sigma}_0 & 0\end{pmatrix}$$

can be made Hermitian by requiring that $\hat{\Sigma}_0=\Sigma_0^*$, or

$$\frac{(t-N^*-1)!(t-N-1)!}{(t-N'-1)!(t-N'^*-1)!}=|t-N-1|^2=\frac{a_t{}^N}{a_t{}^{N'}}, \quad \text{(B15)}$$

which has the solution $\mathrm{Re}N'=-\tfrac{1}{2}(k+n)+\tfrac{1}{2}$.

Now find the operators $\Sigma_A{}^B$, $\hat{\Sigma}_A{}^B$:

$$\Sigma_A{}^B=(\bar{\psi}_N\psi_{N'})_A{}^B. \quad \text{(B16)}$$

Expand

$$\Sigma_A{}^B=\sum_t a_t{}^N\bar{\psi}_t{}^N{}_{(D)}{}^{(C)}\psi_{t+1}{}^{N'}{}_{(C)}{}^{(D)}{}_A{}^B, \quad \text{(B17)}$$

where $\psi_{t+1}{}^{N'}$ is *not* irreducible.

But we have that

$$\psi_{t+1}{}^{N'}{}_{(D)}{}^{(C)}{}_A{}^B=\hat{\psi}_{t+1}{}^{N'}{}_{(D)}{}^{(C)}{}_A{}^B+\frac{1}{2t+k+n}S\begin{bmatrix}\delta_A{}^B\hat{\psi}_{t(D)}{}^{(C)}+(t+k)\delta_D{}^B\hat{\psi}_{t(D)A}{}^{(C)}\\ +t\delta_A{}^C\hat{\psi}_{t(D)}{}^{(C)B}+t(t+k)\delta_D{}^C\hat{\psi}_{t(D)A}{}^{(C)B}\end{bmatrix}$$
$$+\frac{t(t+k)}{(2t+k+n-1)(2t+k+n-2)}S[\delta_D{}^B\delta_A{}^C\hat{\psi}_{t-1}{}^{N'}{}_{(D)}{}^{(C)}]+\text{terms proportional to }\delta_D{}^C. \quad \text{(B18)}$$

Then in the general basis given by (B9),

$$\Sigma_A{}^B\hat{\psi}_t{}^N{}_{(D)}{}^{(C)}=\frac{\alpha_t{}^N}{\alpha_{t+1}{}^{N'}}\hat{\psi}_{t+1}{}^{N'}{}_{(D)}{}^{(C)}{}_A{}^B+\frac{\alpha_t{}^N}{\alpha_t{}^{N'}}\frac{S}{2t+k+n}$$
$$\times[\delta_A{}^B\hat{\psi}_t{}^{N'}+(t+k)\delta^B\hat{\psi}_{tA}+t\delta_A\hat{\psi}_t{}^B]+\frac{\alpha_t{}^N}{\alpha_{t-1}{}^{N'}}\frac{t(t+k)}{(2t+k+n-1)(2t+k+n-2)}S[\delta^B\delta_A\hat{\psi}_{t-1}{}^{N'}], \quad \text{(B19)}$$

where for simplicity we have considered a tensor with simple orthonormality properties (i.e., all $C_i\neq$all $D_i$). Similarly, we find in a general basis

$$\hat{\Sigma}_A{}^B\hat{\psi}_t{}^{N'}=\frac{\alpha_t{}^{N'}}{\alpha_{t-1}{}^N}\frac{a_{t-1}{}^N}{a_t{}^{N'}}S\hat{\psi}_{t-1}{}^N\delta^B\delta_A+\frac{\alpha_t{}^{N'}}{\alpha_t{}^N}\frac{a_t{}^N}{a_t{}^{N'}}\frac{S}{2t+k+n}$$
$$\times[\delta_A{}^B\hat{\psi}_t{}^N+t\delta_A\hat{\psi}_t{}^{NB}+(t+k)\delta^B\hat{\psi}_t{}^N{}_A]+\frac{\alpha_t{}^{N'}}{\alpha_{t+1}{}^N}\frac{a_{t+1}{}^N}{a_t{}^{N'}}\frac{(t+1)(t+k+1)}{(2t+k+n+1)(2t+k+n)}\hat{\psi}_{t+1}{}^N{}_A{}^B. \quad \text{(B20)}$$

The matrix operator

$$\Gamma_A{}^B=\begin{pmatrix}0 & \Sigma_A{}^B\\ \hat{\Sigma}_A{}^B & 0\end{pmatrix}$$

may be made skew Hermitian. Requiring that

$$\langle t|\Sigma_A{}^B|t-1\rangle^*=-\langle t-1|\hat{\Sigma}_B{}^A|t\rangle,$$

we find the condition

$$|\alpha_{t-1}{}^N/\alpha_t{}^{N'}|=(-a_{t-1}{}^N/a_t{}^{N'})^{1/2}.$$

But the condition that $(-a_{t-1}{}^N/a_t{}^{N'})^{1/2}$ be real, along with the fact that $N'=N+1$ fixes $\mathrm{Im}N'=0$, and thus

$$\left|\frac{\alpha_{t-1}{}^N}{\alpha_t{}^{N'}}\right|=\left[\frac{t(t+k)(t+\frac{1}{2}k+\frac{1}{2}n-\frac{3}{2})(t+\frac{1}{2}k+\frac{1}{2}n-\frac{1}{2})}{(2t+k+n-1)(2t+k+n-2)}\right]^{1/2}. \quad \text{(B21)}$$

The skew-Hermiticity condition further fixes the phases [along with (B11)]. Finally, then, in the "natural" basis,

$$\langle t|\Sigma_A{}^B|t-1\rangle=\langle t-1|\hat{\Sigma}_B{}^A|t\rangle$$
$$=-\langle t-1|\Sigma_A{}^B|t\rangle=-\langle t|\hat{\Sigma}_B{}^A|t-1\rangle$$
$$=i\left[\frac{t(t+k)(t+\frac{1}{2}k+\frac{1}{2}n-\frac{3}{2})(t+\frac{1}{2}k+\frac{1}{2}n-\frac{1}{2})}{(2t+k+n-1)(2t+k+n-2)}\right]^{1/2}. \quad \text{(B22)}$$