# Reggeized Bootstrap of the $K^*$ Meson

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A bootstrap of the $K^*$ meson is attempted in the $K\pi$ channel via a representation due to Abbe, Kaus, Nath, and Srivastava. Here a total amplitude is constructed purely in terms of Regge trajectories, and crossing is used to determine the parameters. An unknown coupling constant $G_{\rho K\bar{K}}$ enters the caluclation. When we take its $SU(3)$ value, our bootstrapped values are $m_{K^*}=960$ MeV (experimental value is 890 MeV) and $\Gamma_{K^*}=28$ MeV (experimental value is 50 MeV). When we choose for $G_{\rho K\bar{K}}$ two times its $SU(3)$ value, a *simultaneous* bootstrap of $m_{K^*}$ and $\Gamma_{K^*}$ occurs at the *experimental* value. We have also plotted the $S$-, $P$-, and $D$-wave phase shifts for the $K\pi$ system in the low-energy region. The scattering lengths for $I=\frac{1}{2}$ $S$ and $P$ waves are found to be: $a_0\approx -0.078$ and $a_1\approx +0.017$.

## 1. INTRODUCTION

It is by now well-known that single-particle exchange forces coupled with the $N/D$ partial-wave dispersion relations fail to produce any reasonable quantitative agreement for mesonic amplitudes. In particular, such models appear oversimplified in their handling of the exchange forces to generate narrow widths for the mesonic resonances. Among other things, these models leave much to be desired in the way of a consistent treatment of particles as "dynamic" in the direct as well as the cross channels. A truly dynamic approach was first attempted by Chew[1] based on bootstrapping entire Regge trajectories. Various refinements of Chew's proposal (the strip approximation) have recently appeared in the literature.[2] The strip approximation, however, introduces arbitrary parameters (e.g., the strip width) into the theory and also mutilates the Mandelstam analyticity for the scattering amplitudes. As a consequence, for instance, the imaginary part of the scattering amplitude fails to develop the correct threshold behavior.

Recently, a theory was proposed for bootstrapping the entire Regge trajectories.[3] In this scheme the total scattering amplitudes are constructed such that they are unitary, respect Mandelstam analyticity, and are free of any arbitrary, undetermined parameters. Furthermore, this bootstrap program involves the Regge trajectories alone, in contrast to the earlier proposals which also involve the Regge residues as extra input. A prerequisite for any practical theory of Reggeized bootstraps is that it must converge very fast in terms of the top few trajectories. The encouraging feature of I is that in situations where comparisons with the exact results were possible, the representation converged very rapidly in terms of the number of Regge trajectories included.

The program in I involves evaluation of the background integral in the Regge continuation, since we need the amplitudes $A(s,t)$ for $s$, $t>4m_\pi^2$ for the $\pi\pi$ case; and in this region the partial-wave expansion fails to converge. The computation of the background integral, on the other hand, is rather hard because it converges on the cut in $t$ ($t<4m_\pi^2$) only in a limiting sense as one approaches the real axis from the complex $t$ plane. Thus, a simplified version of the full program proposed in I was applied to calculate the $\rho$-meson parameters self-consistently.[4] The simplification basically consists of replacing the $t$ cut (due to the $2\pi$ continuum) beginning at $t=4m_\pi^2$ by that due to $\rho$ and $f^0$ mesons. This calculation produced a width for the $\rho$ meson which was 125 MeV in contrast to about 600-MeV width produced by the usual $N/D$ calculations.[5]

In view of this significant reduction in the width of the $\rho$ meson, we attempt here a similar calculation for the $K^*$ bootstrap. In brief, the procedure is as follows:

We construct the partial-wave amplitudes for the $K\pi$ system via the modified Cheng representation [see Eqs. (2.7), (2.9), and (2.10)] derived in I. As can easily be checked, irrespective of the number of trajectories included, the representation is unitary and possesses the correct threshold and asymptotic behaviors. The total amplitudes [Eq. (2.3)] constructed through these partial-wave amplitudes converges up to $\cos\theta_s=1+2m_\pi^2/k^2$, where $m_\pi$ and $k$ denote the pion mass and the c.m. momentum in the $s$ ($\pi K$) channel. However, to impose crossing symmetry, a wider region

* Work supported in part by the Connecticut Research Foundation.
† Supported in part by a grant from the National Science Foundation.
[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[2] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964); D. C. Teplitz and V. L. Teplitz, Phys. Rev. **137**, B142 (1965).
[3] W. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **141**, 1513 (1966); hereafter referred to as I.
[4] W. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **154**, 1515 (1967).
[5] A survey of various $N/D$ attempts to bootstrap the $\rho$ meson is made by F. Zachariasen in Lectures on Bootstraps, given at the Pacific International Summer School in Physics, 1965, Honolulu, Hawaii (unpublished).

in $\cos\theta_s$ is needed. In order to implement it without recourse to the background integral evaluation, which is hard to do numerically,[6] we approximate the beginning of the cuts in $\cos\theta_s$ given by the $2\pi$ and $\pi K$ continua by $\rho$ and $K^*$ exchange singularities. The Regge parameters are then determined by the maximum satisfication of crossing ($s\leftrightarrow u$) via a figure-of-merit method discussed in the text.

We have investigated the self-consistency of both the mass and the width of the $K^*$. The calculation requires a knowledge of $G_{\rho K\bar{K}}$, which is unknown. Initially, choosing the $SU_3$ value, we find the bootstrapped $K^*$ width to be smaller than the experimental value while the corresponding mass is larger. However, if $G_{\rho K\bar{K}}$ is taken to be about 2 times its $SU_3$ value, then *both the mass and width are found to be remarkably close to the experimental value*. This seems to us to be an interesting result.

## 2. FORMALISM

We consider the reaction

$$\pi(p_1)+K(k_1)\rightarrow\pi(p_2)+K(k_2) \tag{2.1}$$

and define the usual Mandelstam variables

$$\begin{aligned} s&=(p_1+k_1)^2=(p_2+k_2)^2=[(k^2+m_K^2)^{1/2}+(k^2+m_\pi^2)^{1/2}]^2\\ t&=(p_1-p_2)^2=(k_1-k_2)^2=-2k^2(1-\cos\theta_s)\\ u&=(p_1-k_2)^2=(k_1-p_2)^2=+2m_K^2+2m_\pi^2-s-t, \end{aligned} \tag{2.2}$$

where $k$ is the $s$-channel c.m. momentum, $\theta_s$ the scattering angle, and $m_K$, $m\pi$ are the masses of $K$ and $\pi$ mesons, respectively.

$$A_s{}^I(s,t,u)=\sum_{l=0}^{\infty}(2l+1)A_l{}^I(s)P_l(\cos\theta_s), \tag{2.3}$$

where

$$A_l{}^I(s)=\frac{S_l{}^I(s)-1}{2i\rho(s)},$$

$$\rho(s)=k/\sqrt{s}. \tag{2.4}$$

The crossing matrix ($s\leftrightarrow u$) reads

$$A^I(s,u)=\sum_{I'}\chi^{II'}A^{I'}(u,s), \tag{2.5}$$

where

$$\chi^{II'}=\begin{pmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}; \tag{2.6}$$

here $I$ stands for the isotopic spin.

Now the Regge continuation of the partial-wave amplitudes has to be made for odd and even $l$'s separately. We represent the $S$ matrix in terms of the direct-channel Regge poles $[\alpha_n(s)]$ and the Born terms via the new representation obtained in I. Thus we can write

$$\ln S_l{}^{\pm I}=\sum_n\int_{\alpha_{n\pm I}(s)}^{\alpha_{n\pm I}{}^*(s)}dl'\frac{e^{(l'-l)\xi_2(s)}}{l'-l}+\ln B_l{}^{\pm I}(s), \tag{2.7}$$

where the Born terms, $B_l{}^{\pm I}(s)$, are obtained via particle exchanges in the crossed $t$ and $u$ channels. Here $+(-)$ stands for even (odd) signatures. $\xi_2(s)$ is defined as

$$\cosh\xi_2(s)=\frac{s+(m_{k^*}+m_\pi)^2-2m_k{}^2-2m_\pi{}^2}{2k^2}-1. \tag{2.8}$$

$B_l(s)$ is computed by projecting out the $\rho$ and $K^*$ exchanges in $t$ and $u$ channels, respectively, and then the next singularity in the $\cos\theta_s$ plane is replaced by the $K^*\pi$ threshold in the $u$ channel. This obtains for $\xi_2(s)$, the form given by (2.8). The computation of the Born terms is straightforward[7]; we just quote the relevant results. For the even-signature trajectory we choose the $K^{**}$ trajectory ($\alpha_2$ trajectory), and for the odd one, the $K^*$ trajectory ($\alpha_1$ trajectory).[8]

Then, for the $I=\frac{1}{2}$ and $\frac{3}{2}$ channels we can write down the final expressions.

$I=\frac{1}{2}$.

(i) Odd partial waves:

$$\begin{aligned}\ln S_{l-}{}^{1/2}(s)=\int_{\alpha_1(s)}^{\alpha_1{}^*(s)}dl'\frac{e^{(l'-l)\xi_2}}{l'-l}+\frac{i\sqrt{2}k}{\sqrt{s}}\frac{G_{\rho K\bar{K}}G_{\rho\pi\pi}}{4\pi}a(s)\\ \times\left[Q_l(\cosh\xi_3)-\frac{e^{-(l+1)\xi_2}}{l+1}\right]-\frac{ik}{3\sqrt{s}}\frac{G_{K^*K\pi}{}^2}{4\pi}\\ \times h(s)\left[Q_l(\cosh\xi_1)-\frac{e^{-(l+1)\xi_2}}{l+1}\right].\end{aligned} \tag{2.9}$$

(ii) Even partial waves:

$$\begin{aligned}\ln S_{l+}{}^{1/2}(s)=\int_{\alpha_2{}^*(s)}^{\alpha_2{}^*(s)}dl'\frac{e^{(l'-l)\xi_2}}{l'-l}+\frac{i\sqrt{2}k}{\sqrt{s}}\frac{G_{\rho K\bar{K}}G_{\rho\pi\pi}}{4\pi}\\ \times\left\{a(s)\left[Q_l(\cosh\xi_3)-\frac{e^{-(l+1)\xi_2}}{l+1}\right]-\delta_{l0}\right\}+\frac{ik}{3\sqrt{s}}\frac{G_{K^*K\pi}{}^2}{4\pi}\\ \times\left\{h(s)\left[Q_l(\cosh\xi_1)-\frac{e^{-(l+1)\xi_2}}{l+1}\right]-\delta_{l0}\right\}.\end{aligned} \tag{2.10}$$

$I=\frac{3}{2}$. Here we just take the Born terms, since there appear no well-established particles with $I=\frac{3}{2}$ (at least none seem strongly coupled to the elastic $K\pi$ system). We have for both odd- and even-parity cases the

[6] Methods to numerically evaluate the background integrals for $t$ values on the cut are presently under investigation.

[7] See, for instance, B. Diu, J. L. Gervais, and H. R. Rubinstein, Nuovo Cimento **31**, 27 (1964); **31**, 341 (1964); R. H. Capps, Phys. Rev. **131**, 1307 (1963).

[8] A possible trajectory due to the enigmatic $\kappa$ meson is ignored. Even if it were to exist, its width appears to be rather small and hence its coupling to this channel should be weak.

following solution (iii):

$$\ln S_l^{3/2}(s)=(-1)^{l+1}\frac{2ik}{3\sqrt{s}}G_{K^*K\pi}{}^2h(s)Q_l(\cosh\xi_1)-\frac{ik}{\sqrt{2}\sqrt{s}}\times\frac{G_{\rho K\bar{K}}G_{\rho\pi\pi}}{4\pi}a(s)Q_l(\cosh\xi_3)\,. \quad (2.11)$$

The various symbols in (2.9)–(2.11) are defined as follows:

$$\cosh\xi_1(s)\equiv\frac{s+m_{K^*}{}^2-2m_K{}^2-2m_\pi{}^2-1}{2k^2},$$

$$\cosh\xi_3(s)\equiv1+\frac{m_\rho{}^2}{2k^2},$$

$$a(s)\equiv\frac{2s-2m_K{}^2-2m_\pi{}^2+m_\rho{}^2}{2k^2}, \quad (2.12)$$

$$h(s)\equiv\frac{2s-2m_K{}^2-2m_\pi{}^2-(m_K{}^2-m_\pi{}^2)/m_{K^*}{}^2+m_{K^*}{}^2}{2k^2}.$$

The $G$'s and $m$'s stand for the various coupling constants and masses.

The Born terms due to spin-2 exchanges have not been included in the modification, since then contribution in the low-energy region is only a few percent of the vector exchange.[9]

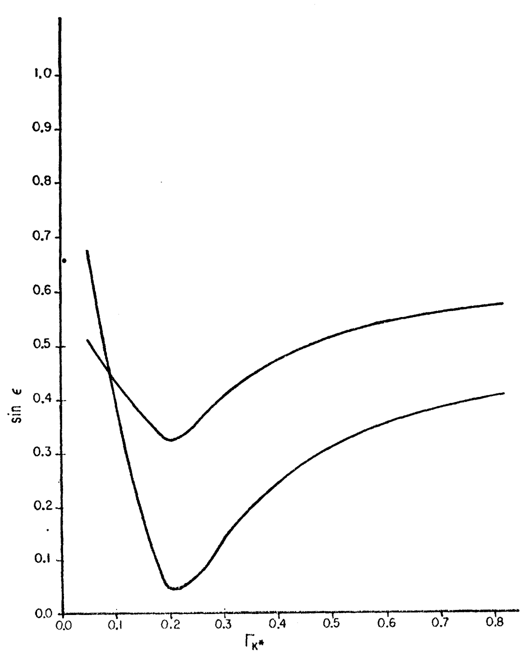


FIG. 1. sin ε is plotted against $\Gamma_{K^*}$ with a fixed value of $m_{K^*}{}^2=41.7m_\pi{}^2$ and $G_{\rho K\bar{K}}{}^2=0.5G_{\rho\pi\pi}{}^2$. The lower curve corresponds to the crossing for real part of the amplitude while the upper curve corresponds to the full amplitude.

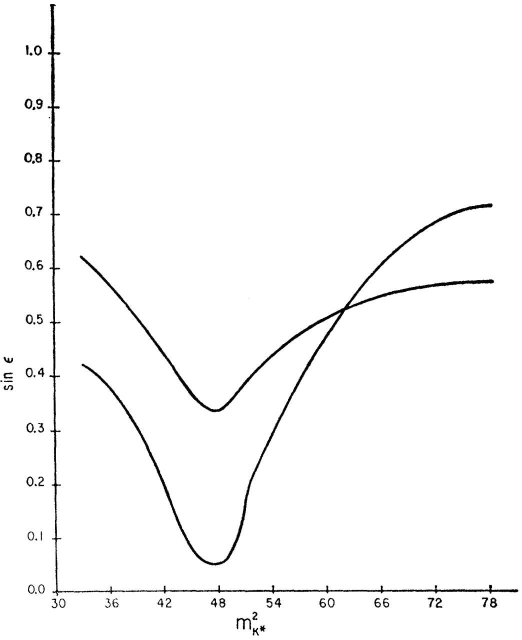


FIG. 2. sin ε is plotted against $m_{K^*}{}^2$ with a fixed value of $\Gamma_{K^*}=0.36m_\pi$ and $G_{\rho K\bar{K}}{}^2=0.5G_{\rho\pi\pi}{}^2$. The lower and the upper curves are as in Fig. 1.

Regarding the trajectories, we choose them to be straight lines, since the experimental data up to date seems to indicate that the trajectories in the meson-meson and meson-baryon system are remarkable straight lines.[10] In any case, for our present analysis, we need the trajectories only in the elastic region, and hence the detailed asymptotic behavior is not of great import. We choose for $\alpha$ (above thereshold):

$$\mathrm{Re}\alpha_i(k)=\alpha_{0i}+C_{1i}k^2, \quad (\alpha_0\geq\tfrac{1}{2})$$
$$=\alpha_{0i}+C_{1i}k^{2\alpha_{0i}+1}, \quad (\alpha_0<\tfrac{1}{2}) \quad (2.13)$$

and

$$\mathrm{Im}\alpha_i(k)=C_{2i}k^{2\alpha_i+1}.$$

This form for the trajectory ensures the correct threshold behavior.

The width of a resonance (Γ) is given by

$$\Gamma=\frac{\mathrm{Im}\alpha(S_r)}{(\sqrt{S_r})\,\mathrm{Re}\alpha(S_r)}. \quad (2.14)$$

Now the constants $C_{1i}$, $C_{2i}$ can be eliminated in favor of the masses and widths of $K^*$ and $K^{**}$ mesons. In the bootstrap for the $K^*$ parameters, we vary $\alpha_0$ over reasonable values and by direct computation find that the results are not very sensitive to its chosen value.

From Eqs. (2.9)–(2.13), we can construct the total

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[9] K. V. Vasavada, Phys. Rev. **144**, 1351 (1966).

[10] M. N. Focacci, W. Kienzle, B. Levrat, B. C. Maglić, and M. Martin, Phys. Rev. Letters **17**, 890 (1966).

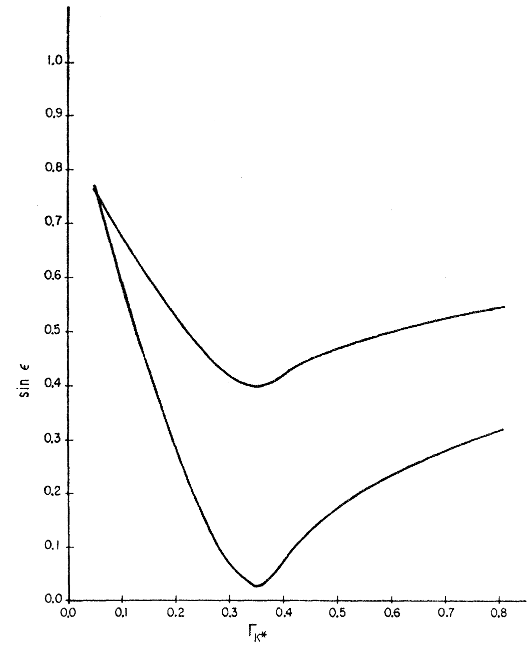


FIG. 3. Same as in Fig. 1 but with $G_{\rho K\bar{K}}^2 = 2.1 G_{\rho\pi\pi}^2$.

amplitudes $A^I(s,u)$ [Eq. (2.3) on which crossing (2.5) has to be imposed to determine the parameters]. This is performed in the next section.

## 3. CALCULATIONAL DETAILS

Various couplings that occur in Sec. 2 are obtained as follows: $G_{\rho\pi\pi}$ and $G_{K^*K\pi}$ are related to the widths of $\rho$ and $K^*$ in the equations

$$\Gamma_\rho = \frac{2}{3}\frac{G_{\rho\pi\pi}^2}{4\pi}\frac{k_\rho^3}{m_\rho^2}$$

and (3.1)

$$\Gamma_{K^*} = \frac{2}{3}\frac{G_{K^*K\pi}^2}{4\pi}\frac{k_{K^*}^3}{m_{K^*}^2},$$

where $k_\rho$ and $k_{K^*}$ are the c.m. momenta in $\pi\pi$ and $\pi K$ channels at the $\rho$ and $K^*$ meson masses, respectively.

$G_{\rho K\bar{K}}$ is *a priori* unknown. However, invoking $SU(3)$, we can relate it to $G_{\rho\pi\pi}$:

$$G_{\rho K\bar{K}}^2 = \tfrac{1}{2} G_{\rho\pi\pi}^2 . \qquad (3.2)$$

After the left-hand cut is shifted, our representation for $A(s,u)$ converges (via the partial-wave expansion) for $s$, $u$ values between the threshold, $(m_\kappa + m_\pi)^2 = 21.1 m_\pi^2$ and $m_\rho^2 = 30 m_\pi^2$ (see last section). Thus, we can construct $A(s,u)$ for $21.1 < s,u < 30$.

The crossing equations are explicitly given by (2.5) and (2.6). Eliminating $A^{3/2}(u,s)$ we obtain

$$4A^{3/2}(s,u) - A^{1/2}(s,u) = 3A^{1/2}(u,s) . \qquad (3.3)$$

For the satisfaction of crossing, we define a *figure of merit* as follows:

$$\sin\epsilon(s,u) = \frac{|A^L(s,u) - A^R(s,u)|}{\sqrt{2}\{[A^L(s,u)]^2 + [A^R(s,u)]^2\}^{1/2}}, \qquad (3.4)$$

where $A^L$ and $A^R$ denote the left- and the right-hand sides of the equation (3.3). Since both $A^L$ and $A^R$ are complex, we calculate sin $\epsilon_R$ and sin $\epsilon_I$ for the real and the imaginary parts of $A^{L,R}$ separately. We choose a fine mesh of points for the $s$ and $u$ in the region mentioned above and compute sin $\epsilon_R$ and sin $\epsilon_I$ for each $(s,u)$ pair. For crossing to be satisfied exactly, $\sin\epsilon_{R,I} = 0$. So, the average

$$\langle\sin\epsilon\rangle_{\rm av} = \tfrac{1}{2}(\langle\sin\epsilon_R\rangle_{\rm av} + \langle\sin\epsilon_I\rangle_{\rm av} , \qquad (3.5)$$

where $\langle\sin\epsilon_R\rangle_{\rm av}$ and $\langle\sin\epsilon_I\rangle_{\rm av}$ are themselves averages over the real and the imaginary parts, denotes the deviation from the satisfaction of crossing.

$\langle\sin\epsilon\rangle_{\rm av}$ is calculated from (3.5) for fixed values of $G_{\rho K\bar{K}}^2$ and $G_{\rho\pi\pi}^2$ and different choices for the masses and coupling constants for $K^*$. A minimum value of $\langle\sin\epsilon\rangle_{\rm av}$ corresponds to a maximum satisfaction of crossing symmetry and is in fact our bootstrapped solution.

## 4. RESULTS AND DISCUSSION

Initially, we choose the $SU(3)$ value for $G_{\rho K\bar{K}}^2$ $(= 0.5 G_{\rho\pi\pi}^2)$. Figures 1 and 2 show the figure of merit curves for the cases when $\Gamma_{K^*}$ is varied holding $m_{K^*}$ fixed and vice versa. The minima in the curves correspond to our bootstrapped values for the parameters.

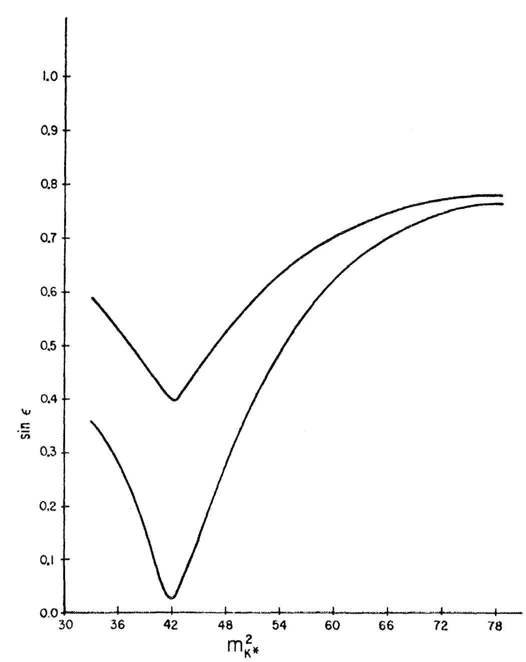


FIG. 4. Same as in Fig. 2 but with $G_{\rho K\bar{K}}^2 = 2.1 G_{\rho\pi\pi}^2$.

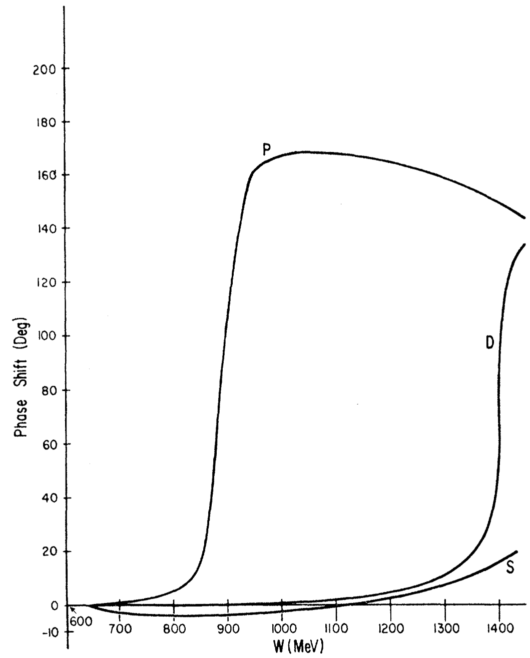


FIG. 5. $I=\frac{1}{2}$, $S$-, $P$-, and $D$-wave phase shifts for the $K$-$\pi$ system for $\Gamma_{K^*}=0.2m_\pi$, $m_{K^*}{}^2=41.7m_\pi{}^2$, and $G_{\rho K\bar{K}}{}^2=0.5G_{\rho\pi\pi}{}^2$.

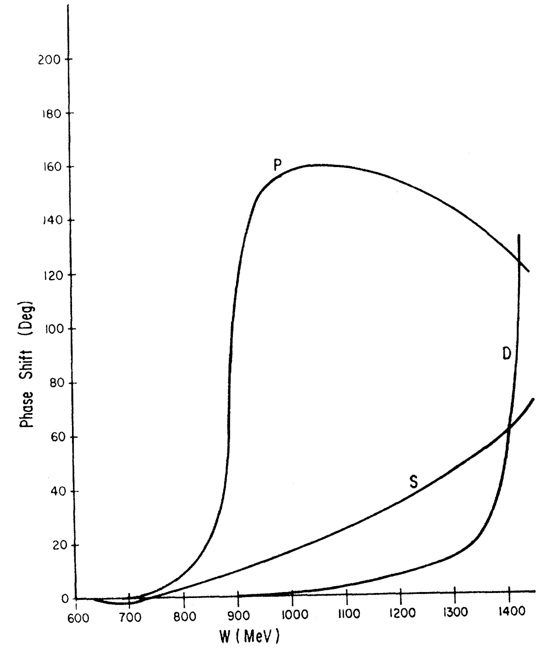


FIG. 6. Same as in Fig. 5 for $\Gamma_{K^*}=0.36m_\pi$, $m_{K^*}{}^2=41.7m_\pi{}^2$ (experimental values), $G_{\rho K\bar{K}}{}^2=2.1G_{\rho\pi\pi}{}^2$.

As is clear from the curves, rather steep minima are obtained indicating that indeed a good satisfaction of crossing has been obtained.

We remind the reader that no minima need have occurred. The bootstrapped values are

$$\Gamma_{K^*}=28 \text{ MeV (exp. value 50 MeV)},$$
$$m_{K^*}=960 \text{ MeV (exp. value 890 MeV)}.$$

We repeated the calculation for various values of the unknown coupling constant $G_{\rho K\bar{K}}{}^2$ and find that a *simultaneous self-consistency in* $m_{K^*}$ *and* $\Gamma_{K^*}$ *is achieved at the experimental value* when $G_{\rho K\bar{K}}{}^2=2.1G_{\rho\pi\pi}{}^2$. Figures 3 and 4 correspond to this case.

We also obtain the $S$-, $P$-, and $D$-wave phase shifts for the $I=\frac{1}{2}$, $K\pi$ system. Figures 5 and 6 depict these for the $SU(3)$ and the bootstrapped sets for the coupling constants. When $G_{\rho K\bar{K}}{}^2=2.1G_{\rho\pi\pi}{}^2$ (Fig. 6), the $S$-wave phase shift is rather large. It simply reflects the fact that a large $S$-wave contribution is needed for a good satisfaction of crossing in the region chosen.

Also, various scattering lengths are computed as shown below.[11] For $I=\frac{1}{2}$

$$S\text{-wave: } a_0=-0.078,$$
$$P\text{-wave: } a_1=+0.017.$$

One rather technical point needs mention. In the modification via the Born term [Eq. (2.10)], the Kronecker-$\delta$ term $\delta_{l0}$ has been retained, even though in the true spirit of angular-momentum analylicity such terms ought not to be included. As discussed in Ref. 12, in an approximately Reggeized theory the $\delta_{l0}$ term is consistent and must be retained. In this case, by actual computation, we find that a good consistency is obtained only when the $\delta_{l0}$ term *is* retained.

Finally, we remark that the present calculation along with the earlier one for the $\rho$ meson[4] seems to indicate that our formalism is capable of bootstrapping narrow mesonic resonances. Such attempts via $N/D$ techniques have so far proved inadequate.

## ACKNOWLEDGMENT

The computations were performed at the computing center of the University of Connecticut, which is supported in part by Grant No. GP-1819 of the National Science Foundation.

[11] Of course, $\pi K$ scattering is not directly accessible to experiment. However, the best $I=\frac{1}{2}$, $S$-wave scattering lengths, deduced from the low-energy $KN$ scattering data, is found by A. D. Martin and T. D. Spearman [Phys. Rev. **136**, B1480 (1964)] to be $a_0=-0.07\pm0.10$, which is consistent with our determination. However, an analysis of the $K_{l4}$ decay, coupled with the PCAC hypothesis and other assumptions, is made by B. R. Martin [Phys. Rev. **141**, 1571 (1966)], whose best scattering length is $a_0=0.06\pm0.03$, which disagrees with ours.

[12] Y. N. Srivastava and P. Nath, Phys. Rev. **142**, 982 (1966); E. Abers and V. Teplitz, Nuovo Cimento **39**, 739 (1965).