Electromagnetic Current in Strong-Coupling Theory*

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(Received 4 May 1967)

It is shown that the electromagnetic-current operator in the strong-coupling theories is unique.

IN this paper we discuss the behavior of electromagnetic (e.m.) current in static strong-coupling theories using the Lie-algebraic formulation of these models due to Cook, Goebel, and Sakita.[1] Of course, this problem has been studied before: Bose[2] and Singh[3] have discussed charge-independent static theory and these discussions were extended to $SU(3)$-symmetric theories.[4,5] But all these discussions assume a rather specific form of the e.m. current, namely, that it is proportional to a component of the "translation operator." One purpose of the present work is to show that this is not a separate assumption, but that its validity can be rigorously proved. We also make precise the meaning of this proportionality. Secondly, there seems to be no uniform treatment of the isoscalar part of e.m. current in charge-independent pseudoscalar theory. Thus this problem is not discussed at all in Ref. 2, while the discussion in Ref. 3 yields a magnetic-moment sum rule which does not agree with the results of an explicit calculation due to Pauli and Dancoff.[6] Therefore, the conclusions of Ref. 3 must, of necessity, be incorrect. We show in this paper that the isoscalar e.m. current in charge-independent, pseudoscalar theory is exactly zero.[7] Thus the Lie-algebraic treatment leads to an identical conclusion as Pauli and Dancoff.[6]

All our conclusions follow from two results: First we derive a condition which e.m. current must satisfy in the strong-coupling limit [Eq. (2)]. This condition, together with the behavior of the current under the "primitive invariance group," completely specifies its tensorial character under the strong-coupling group. The solution for the current operator is then implied by a uniqueness theorem. Finally, we discuss the application of this theorem to various cases.

Consider the photoproduction amplitude of mesons on static isobars: photon$+i\rightarrow\alpha+j$, where α denotes the state of meson, and i and j, respectively, the initial and final isobar. To lowest order in electromagnetism the Chew-Low equation for this process is written as

$$T_\alpha{}^{ji}(\omega)=\lambda\sum_k \frac{(\mathbf{J})^{jk}(A_\alpha)^{ki}}{M_k-M_i-\omega}+\frac{(A_\alpha)^{jk}(\mathbf{J})^{ki}}{M_k-M_j+\omega}$$
$$+\text{(one- or more-meson intermediate states)}, \quad (1)$$

where M_i is the energy of the ith isobar, and $\lambda(A)^{ij}$ is the matrix element of the meson source between ith and jth isobar. $\mathbf{J}$ is the e.m. current. λ denotes the strength of meson-isobar coupling. In the strong-coupling limit $\lambda\rightarrow\infty$, $T_\alpha{}^{ji}(\omega)$ must be finite in the physical region due to the unitarity condition. From this we get, proceeding as in Ref. 1, the condition that $\mathbf{J}$ must commute with A_α, i.e.,

$$[\mathbf{J},A_\alpha]=0. \quad (2)$$

Notice that Eq. (2) may also be viewed as a superconvergence condition.[8] Equation (2) has been noted independently by Biswas *et al.*[5] We now prove the uniqueness theorem.

Theorem: Consider a group G which is the semidirect product of a K and T, $G=KT$, where K is compact and T is Abelian, and an induced representation g of G for which the representation of the little group σ is taken to be one dimensional. Let O_i be a set of operators that transform like an irreducible representation of K, i.e., $G(k)O_iG^{-1}(k)=A_{ij}(k)O_j$, $k\epsilon K$. [The $A_{ij}(k)$ constitutes a representation of K.] The set of O_i is unique up to a multiplicative constant if (i) $[O_i,\tilde{T}_l]=0$, where the $\tilde{T}_l$ are the generators of T; and (ii) $A_{ij}(k)\Psi_j=\Psi_i$ for all $k\in\sigma$ implies that ψ_j is unique.

Proof: Let $|t\rangle$ be an eigenvector of $\tilde{T}_l$ with eigenvalues t_l. The set of all eigenvalues defines the orbit. Since the representation of the little group is assumed to be one dimensional, there exists only one vector with a given eigenvalue t_l. Thus, from $[O_i,\tilde{T}_l]|t\rangle=0$ we have $O_i|t\rangle=\lambda_i(t)|t\rangle$ and from this

$$g(k)O_ig^{-1}(k)g(k)|t\rangle=\lambda_i(t)g(k)|t\rangle,\quad k\in K.$$

From the assumed transformation properties of O_i, we have

$$A_{ij}(k)O_jg(k)|t\rangle=\lambda_i(t)g(k)|t\rangle.$$

* Work supported in part by the U. S. Atomic Energy Commission.

[1] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).
[2] S. K. Bose, Phys. Rev. **145**, 1247 (1966).
[3] V. Singh, Phys. Rev. **144**, 1275 (1966).
[4] C. J. Goebel, Phys. Rev. Letters **16**, 1130 (1966).
[5] S. N. Biswas, S. H. Patil, and R. P. Saxena, Ann. Phys. (N. Y.) **42**, 494 (1967); see also J. Kuriyan and E. C. G. Sudarshan (unpublished).
[6] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).
[7] The isoscalar current may, however, survive in intermediate-coupling models. For a discussion of these models see J. Kuriyan and E. C. G. Sudarshan, Phys. Letters **21**, 106 (1966).
[8] It has been shown elsewhere [S. K. Bose, P. C. De Celles, and W. D. McGlinn, Phys. Rev. Letters **18**, 873 (1967)] that for meson-baryon scattering in the static limit the requirements of superconvergence and strong-coupling (bootstrap) are equivalent in the approximation in which single-resonance intermediate states saturate the scattering amplitude. Analogous reasoning can be made for the photoproduction amplitude.

Now $g(k)$ rotates the vector $|t\rangle$ to a vector with eigenvalue $t_l'=K_{li}(k)t_i$ where $K_{li}(k)$ is the matrix representation of K which depends upon the assumed transformation properties of $\tilde{T}_l$. Thus,

$$A_{ij}(k)O_j|kt\rangle=\lambda_i(t)|kt\rangle$$
$$=A_{ij}(k)\lambda_j(kt)|kt\rangle.$$

This implies

$$\lambda_j(kt)=A_{ji}^{-1}(k)\lambda_i(t).$$

Thus if $\lambda_i(t)$ is known at any point on the orbit it is known everywhere and this determines O_i.

In particular let σ be the little group at the point t_l. Then

$$\lambda_j(t_l)=A_{ji}^{-1}(k)\lambda_i(t_l), \quad k\in\sigma;$$

i.e., $\lambda_i(t_l)$ is an eigenvector of $A_{ji}(k)$, $k\in\sigma$ with eigenvalue 1. If it is unique then O_i is unique. If such an eigenvector does not exist, then $O_i=0$.

Let us now apply this theorem to two cases:

(1) Charge-independent pseudoscalar theory. If one assumes that the electromagnetic current J_i is a linear combination of an isospin scalar J_1^0 and an isospin vector J_i^3, then Eq. (2) implies the two parts separately commute with A_α. The strong-coupling group for this theory is (in the notation of the theorem) $K=SU(2)\times SU(2)$, and T is a 9-parameter Abelian group. For the representation of this group, used to classify isobars,[1] the little group at one point on the orbit is the group generated by $\mathbf{L}+\mathbf{I}$ (where L_i are the generators of space rotations and I_i are the generators of isospin transformations) and the representation of σ chosen is the singlet. It is easy to see that the theorem implies $J_i^0=0$ and $J_i^3=\lambda A_i^3$, where A_i^3 is a generator of T. From this solution for J_i^3 and with knowledge of the explicit matrix-representation of the A's from Refs. 2 and 3, we derive the "old" Pauli-Dancoff result for total magnetic moment, i.e.,

$$\mu=I_3/(I+1)\times\text{const.}$$

In particular the neutron and proton moments are equal and opposite, a result in conflict with experiment.

(2) SU(3)-symmetric pseudoscalar theory. The strong-coupling group for this theory is $[SU(3)\times SU(2)]T_{24}$. The 24 translation generators, $A_{i\alpha}$ [$i=1$, 2, 3 are space indexes and $\alpha=1, 2, \cdots 8$ are $SU(3)$ indexes] correspond to a P-wave $SU(3)$-octet meson. For the representation of this group used to classify isobar states, advocated by Goebel[4] and Cook and Sakita,[9] the little group has for generators $\mathbf{J}+\mathbf{L}$, and H (H is the hypercharge operator). Again the representation of the little group is chosen to be one dimensional. (This representation is also the one arrived at by Dullemond and Van der Linde using conventional Hamiltonian methods.[10]) If now the electromagnetic current is assumed to transform like an octet, and commute with the charge operator which is a generator of $SU(3)$, then it is easy to see that the theorem implies

$$J_i=C[A_{i3}+(1/\sqrt{3})A_{i8}]. \tag{4}$$

Experimental consequences of Eq. (4) have been discussed in Refs. 4 and 5. The resulting relations for magnetic moments and e.m. mass difference seems to be in good agreement with experiment. Indeed, for the present case the "uniqueness theorem" was obtained by Goebel[4] starting with the assumption that the tensor operator under consideration is "a function of the coupling operators." No such assumption is utilized in the present derivation. Instead we use the strong-coupling condition Eq. (2) which the tensor operator must satisfy in this limit. Furthermore, our formulation of the "uniqueness theorem" is more general as it applies to a wider class of theories (such as the charge-independent pseudoscalar theory treated in the preceding section).

We wish to thank Dr. T. Cook for informative discussions.

Note added in proof. After writing this paper we have discovered that Eq. (2) has been noted previously by Sakita in his talk at The Third Coral Gables Conference, 1966 [see *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, California, 1966)].

[9] T. Cook and B. Sakita, J. Math. Phys. **8**, 708 (1967).

[10] C. Dullemond and E. J. M. van der Linde, Ann. Phys. (N. Y.) **41**, 372 (1967).