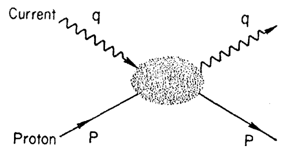

FIG. 1. Kinematics for forward scattering of a current from a nucleon.

where we have used Eq. (12). The quantity in brackets is proportional to the vector piece of $\sigma_2^{\bar{\nu}p}(q^2,\nu)$ or $\sigma_2^{\nu p}$ as defined in Eqs. (3) and (11). After a routine struggle with normalization factors (most simply done by considering free fields) one arrives at the sum rule Eq. (7). The same isospin manipulations[2] as used in obtaining Eq. (4) from Eq. (6) are sufficient to get Eq. (5) from Eq. (7).

It is tempting to assume the result Eq. (7) to be generally valid for all q^2. However, consideration of the limit as $q^2 \to 0$ gives

$$(\mu_p-\mu_n)^2=1\,, \tag{19}$$

in considerable disagreement with experiment.

The following physical picture of the result Eq. (5) suggests itself: If the "elementary constituents" (if any) of the nucleon, which couple to isospin, were spinless, there would be very little backward scattering at large q^2, because backward scattering demands helicity flip. If the constituents have spin $\frac{1}{2}$, the scattering should be incoherent and proportional to the sum of squares of the magnetic moments of the constituents.[8]

Experimental verification of the inequality Eq. (5) may be difficult because of the problems of radiative corrections.

The author thanks J. D. Walecka for asking the right question, his colleagues at SLAC for discussions, and Helen Quinn and Sam Berman for a reading of the manuscript.

[8] This picture is similar to that discussed for forward scattering by K. Gottfried [Phys. Rev. Letters **18**, 1174 (1967)]. There also exist sum rules of this kind in nuclear physics: for a review see deForest and Walecka, Advan. Physics **15**, 1 (1966).

PHYSICAL REVIEW VOLUME 163, NUMBER 5 25 NOVEMBER 1967

Necessary and Sufficient Conditions for the Current-Spectral-Function Sum Rules

H. T. NIEH
Institute for Theoretical Physics, State University of New York, Stony Brook, New York
(Received 14 July 1967)

Necessary and sufficient conditions are given for the validity of the Weinberg-type current-spectral-function sum rules. For one class of the sum rules, the validity rests on the equality of the vacuum expectation values of the corresponding Schwinger terms. For the other class, the condition involves the triple commutator of the space component of the current with the Hamiltonian (of the world). Comments are made on the usual derivation of the sum rules and Lee, Weinberg, and Zumino's algebra of fields.

THE sum rules of Weinberg[1] for the spectral functions of the chiral $SU(2)\times SU(2)$ currents have been successfully used to relate the ρ and A_1 masses, and to calculate the electromagnetic pion mass difference.[2] Generalization to the case of $SU(3)\times SU(3)$ has led to a calculation of the ratio F_K/F_π,[3,4] in fair agreement with experiment. While all this demonstrates the usefulness of the Weinberg-type sum rules, their validity has not yet been rigorously established,[5] except in the context of a Lagrangian model of Lee, Weinberg, and Zumino,[6] which, however, has been brought into question by a recent consideration related to the electromagnetic corrections to the pion β decay.[7] It is therefore of interest to know exactly the conditions under which the Weinberg-type sum rules are valid. In this paper we shall give the necessary and sufficient conditions for their validity. For one class of sum rules, their validity rests on the equality of the vacuum expectation values of the corresponding Schwinger terms. For the other class, the condition involves the triple commutator of the space component of the current with the Hamiltonian (of the world).

Consider any local current density $J_\mu(x)$, which may or may not be conserved. The most general spectral representation for the current correlation function is

$$\langle J_\mu(x)J_\nu(0)\rangle_0=(2\pi)^{-3}\int d^4p\,\theta(p^0)e^{ip\cdot x}\times[(g_{\mu\nu}-p_\mu p_\nu/p^2)\rho_1(-p^2)+p_\mu p_\nu\rho_0(-p^2)]\,, \tag{1}$$

[1] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[2] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters **18**, 759 (1967).
[3] H. T. Nieh, Phys. Rev. Letters **19**, 43 (1967).
[4] S. L. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 139 (1967).
[5] A critical comment on the derivation of the spectral function sum rules is given by T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[6] See Ref. 5.
[7] K. Johnson, F. E. Low, and H. Suura, Phys. Rev. Letters **18**, 1224 (1967); N. Cabibbo, L. Maiani, and G. Preparata (to be published). We will come back to this point later.

which implies

$$\langle T\{J_\mu(x)J_\nu(0)\}\rangle_0=\frac{i}{(2\pi)^4}\int_0^\infty dm^2\int d^4p\, e^{ip\cdot x}\times\left[\rho_1(m^2)\frac{g_{\mu\nu}+p_\mu p_\nu/m^2}{p^2+m^2-i\eta}+\rho_0(m^2)\frac{p_\mu p_\nu}{p^2+m^2-i\eta}\right]+ig_{\mu 0}g_{\nu 0}\delta^4(x)\int_0^\infty dm^2[\rho_1(m^2)/m^2+\rho_0(m^2)],\quad(2)$$

and

$$\langle T\{\partial^\mu J_\mu(x),J_\nu(0)\}\rangle_0=\frac{1}{(2\pi)^4}\int_0^\infty dm^2\int d^4p\times e^{ip\cdot x}m^2\rho_0(m^2)\frac{p_\nu}{p^2+m^2-i\eta}.\quad(3)$$

These last two equations in turn imply that

$$\langle\delta(x^0)[J_0(x),J\nu(0)]\rangle_0=(2\pi)^{-4}\int d^4p\, e^{ip\cdot x}(p_\nu+g_{\nu 0}p_0)\times\int_0^\infty dm^2[\rho_1(m^2)/m^2+\rho_0(m^2)].\quad(4)$$

It is clear from Eq. (4) that, for any two currents $J_\mu(x)$ and $J_\mu'(x)$, the necessary and sufficient condition for one class of the Weinberg-type sum rules

$$\int_0^\infty dm^2[\rho_1(m^2)/m^2+\rho_0(m^2)]=\int_0^\infty dm[\rho_1'(m^2)/m^2+\rho_0'(m^2)]\quad(5)$$

to hold is the equality of the vacuum expectation values of the corresponding Schwinger terms:

$$\langle\delta(x^0)[J_0(x),J_k(0)]\rangle_0=\langle\delta(x^0)[J_0'(x),J_k(0)]\rangle_0.\quad(6)$$

One notes that the validity of Eq. (5), in the case of $SU(2)\times SU(2)$ or $SU(3)\times SU(3)$, neither requires full knowledge of the current algebra nor depends upon whether the Schwinger terms are q numbers of C numbers. In Ref. 4, the condition (6) is proven for $SU(3)\times SU(3)$ by assuming the C-number nature of the Schwinger terms. This, however, is in contradiction with most of the $SU(3)\times SU(3)$ models. It seems that the equality of the vacuum expectation values of the $SU(3)\times SU(3)$ Schwinger terms is a considerably safer and less stringent assumption than that they are C numbers.

The condition for the validity of the other class of the Weinberg-type sum rules,

$$\int_0^\infty \rho_1(m^2)dm^2=\int_0^\infty \rho_1'(m^2)dm^2,$$

is obtained by considering the behavior of the T product $\langle T\{J_\mu(x)J_\nu(0)\}\rangle_0$ in the limit of $x^0\to 0$, or equivalently the behavior of its Fourier transform in the limit of $p^0\to\infty$. Either by expanding $J_\mu(x)$ around $x^0=0$,[8] or by truncating the sum over intermediate states,[9] one obtains

$$\int d^4x\, e^{ip\cdot x}\langle T\{J_\mu(x)J_\nu(0)\}\rangle_0 \xrightarrow[\mathbf{p}=0]{p^0\to\infty}\frac{1}{ip^0}\langle[Q_\mu,J_\nu(0)]\rangle_0+\frac{1}{(ip^0)^2}\langle[\dot{Q}_\mu,J_\nu(0)]\rangle_0+\frac{1}{(ip^0)^3}\langle[\ddot{Q}_\mu,J_\nu(0)]\rangle_0+\cdots,\quad(7)$$

where

$$Q_\mu\equiv\int d^3x\, J_\mu(0,\mathbf{x}),\quad \dot{Q}_\mu\equiv\partial_0 Q_\mu,\ \text{etc.}$$

On the other hand, the spectral representation (2) yields

$$\int d^4x\, e^{ip\cdot x}\langle T\{J_\mu(x)J_\nu(0)\}\rangle_0 \xrightarrow[\mathbf{p}=0]{p^0\to\infty}\frac{i}{(ip^0)^2}\times\int_0^\infty dm^2[\rho_1(m^2)(g_{\mu\nu}+g_{\mu 0}g_{\nu 0})+m^2\rho_0(m^2)g_{\mu 0}g_{\nu 0}]+O(p^0)^{-4}.\quad(8)$$

Comparing (7) and (8), one obtains

$$\langle[\dot{Q}_\mu,J_\nu(0)]\rangle_0=i\int_0^\infty dm^2[\rho_1(m^2)(g_{\mu\nu}+g_{\mu 0}g_{\nu 0})+m^2\rho_0(m^2)g_{\mu 0}g_{\nu 0}],\quad(9)$$

$$\langle[\partial_0{}^3Q_\mu,J_\nu(0)]\rangle_0=i^3\int_0^\infty dm^2m^2[\rho_1(m^2)(g_{\mu\nu}+g_{\mu 0}g_{\nu 0})+m^2\rho_0(m^2)g_{\mu 0}g_{\nu 0}],\ \text{etc.},\quad(10)$$

and[10]

$$\langle[Q_\mu,J_\nu(0)]\rangle_0=\langle[\ddot{Q}_\mu,J_\nu(0)]\rangle_0=\cdots=0.\quad(11)$$

Specifying $\mu=\nu=k$ $(k=1,2,3)$, one obtains from (9) that

$$\int_0^\infty \rho_1(m^2)dm^2=\left\langle\left[\left[H,\int d^3x\, J_k(0,\mathbf{x})\right],J_k(0)\right]\right\rangle_0,\quad(k\text{ not summed})\quad(12)$$

where H is the Hamiltonian of the world. It follows that the necessary and sufficient condition for the class of

[8] I am grateful to W. A. Bardeen for this suggestion.

[9] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[10] For $SU(3)\times SU(3)$, the conditions (11) imply that the relevant equal-time commutators are q numbers and contain no unitary singlet.

sum rules

$$\int_0^\infty \rho_1(m^2)dm^2=\int_0^\infty \rho_1'(m^2)dm^2 \qquad (13)$$

to hold is the equality of the vacuum expectation values of the triple commutators:

$$\left\langle\left[\left[H,\int d^3x\, J_k(0,\mathbf{x})\right],J_k(0)\right]\right\rangle_0 =\left\langle\left[\left[H,\int d^3x\, J_k'(0,\mathbf{x})\right],J_k'(0)\right]\right\rangle_0, \qquad (14)$$

where $J_\mu(x)$ and $J_\mu'(x)$ are any two currents.

While the condition (6) is satisfied by many $SU(3)\times SU(3)$ models, the condition (14) is a much more stringent dynamical requirement. It is therefore expected that sum rules of the type (5) are in general more reliable than those of the type (13). Although the recent successful calculation of the $\pi^+-\pi^0$ mass difference, which depends on the $SU(2)\times SU(2)$ Weinberg sum rules to eliminate divergences, *tends* to support the validity of the dynamical statement (14) (actually, only in conjunction with the partially conserved axial-vector current approximation) for $SU(2)\times SU(2)$ currents, one may feel reluctant to assume the same for $SU(3)\times SU(3)$. There are indeed signs that the $SU(3)\times SU(3)$ Weinberg sum rules work not as nicely as the $SU(2)\times SU(2)$ ones; the calculated F_K/F_π ratio[3,4] agrees with experiment only fairly well; the sum rule

$$\int_0^\infty \rho_1^{(\rho)}(m^2)dm^2=\int_0^\infty \rho_1^{(K^*)}(m^2)dm^2$$

together with the usual dominance assumptions[11] lead to the following K_{l3} coupling constant:

$$f_+(0)=G_{K^*K\pi}\sqrt{2}M_\rho F_\pi/M_{K^*}{}^2\simeq 0.82/\sqrt{2}\,,$$

which is to be compared with the $SU(3)$ symmetric value $\frac{1}{2}\sqrt{2}$. One can hardly consider a 20% renormalization effect to be compatible with the Ademollo-Gatto theorem.[12]

In Lee, Weinberg, and Zumino's Lagrangian model,[6] the conditions (6) and (14) are satisfied, if one identifies, as these authors did, the currents with the non-Abelian "gauge" fields of $SU(3)\times SU(3)$. These authors cite the calculation of the $\pi^+-\pi^0$ mass difference as a support for this identification. However, there exists an argument[13] pointing to the contrary; the finiteness of the radiative corrections, to the order of e^2, to the pion β decay requires the following equal-time commutation relation[7]:

$$[V_k^{\mathrm{em}}(\mathbf{x}),A_j^\alpha(\mathbf{x}')]=\mp i\epsilon_{kjl}V_l^\alpha(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,,$$

where V_k and A_k are the space components of the vector and axial-vector currents, respectively. But, according to the Lee-Weinberg-Zumino model,[6] the equal-time commutator of any two space components of the currents is identically zero.

Despite the usefulness of the Weinberg-type sum rules for the spectral functions of the currents, their validity remains to be established. We speculate that while one class of the sum rules, namely, of the type (5), have a good chance to be exactly true, it is doubtful that the dynamical condition (14) required for the validity of the other class, namely, of the type (13), can in general be satisfied.

It is a pleasure to thank W. A. Bardeen for clarifying discussions.

Note added in proof. It has by now been established that the $\pi^+-\pi^0$ mass-difference calculation of Das *et al.*[2] is not a real support for the field-algebra model of Lee, Weinberg, and Zumino,[6] since the field algebra implies a logarithmically divergent $\pi^+-\pi^0$ mass difference for the *physical* pion.[14] Sakurai[15] also indicated, on empirical grounds, that the Weinberg sum rules of the type (13) are not of good standing. All this agrees with the general contention of the present note.

[11] Namely, the dominance of $\rho_1^{(\rho)}$ by the ρ, and $\rho_1^{(K^*)}$ and f_+ by the K^*.

[12] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1964).

[13] This presumably is known to many people. A discussion of this point with W. A. Bardeen is gratefully acknowledged.

[14] M. B. Halpern and G. Segre, Phys. Rev. Letters **19**, 611 (1967); G. C. Wick and B. Zumino, CERN Report (1967) (unpublished); I. Gerstein, B. W. Lee, H. T. Nieh, and H. J. Schnitzer, Report, 1967 (unpublished); B. W. Lee and H. T. Nieh, Stony Brook Report, 1967 (unpublished).

[15] J. J. Sakurai, Phys. Rev. Letters **19**, 803 (1967).