Inequality for Backward Electron- and Muon-Nucleon Scattering at High Momentum Transfer*

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(Received 13 July 1967)

From a sum rule for backward ν-p scattering, valid only in the limit of large four-momentum transfer q^2, we obtain an inequality for backward e-p inelastic scattering which depends upon the commutator of space components of isospin currents. Given chiral $U(6)\times U(6)$ current algebra, the total backward scattering at fixed large q^2 is predicted to be at least as great as that from a point Dirac particle with charge $\pm\frac{1}{2}e$.

RECENTLY, from Adler's sum rule for neutrino processes[1]

$$\lim_{E\to\infty}\left[\frac{d\sigma(\bar{\nu}p)}{dq^2}-\frac{d\sigma(\nu p)}{dq^2}\right]=\frac{G^2}{\pi}(\cos^2\theta_c+2\sin^2\theta_c)\,,\quad(1)$$

we have derived[2] an inequality for electron- and muon-nucleon scattering by isospin manipulaton:

$$\lim_{E\to\infty}\left[\frac{d\sigma_{ep}}{dq^2}+\frac{d\sigma_{en}}{dq^2}\right]\geq\frac{2\pi\alpha^2}{q^4}\,.\quad(2)$$

This inequality is of some interest inasmuch as it predicts a large amount of inelastic scattering at high momentum transfer q^2, something which can be experimentally tested. The magnitude is comparable to that resulting from scattering off point charges; this result can be traced back to the assumption of locality of the isospin current.

However, electron-nucleon scattering is described by two form factors, and the sum rule, Eq. (2), involves only one of them, the "charge" form factor which contributes to forward scattering. There arises the question of whether there is any such relation for the other form factor which describes backward scattering. The purpose of this paper is to provide a partial answer for large q^2. We write

$$\frac{\pi d\sigma_{ep}}{EE'd\Omega dE'}=\frac{d\sigma_{ep}}{dq^2dE'}=\frac{E'}{E}\times\{\cos^2(\tfrac{1}{2}\theta)\sigma_{1p}(q^2,\nu)+\sin^2(\tfrac{1}{2}\theta)\sigma_{2p}(q^2,\nu)\}\,.\quad(3)$$

Here E and E' are the incident and final lepton energy and θ the scattering angle; $q^2=-4EE'\sin^2(\frac{1}{2}\theta)$ and $\nu=E-E'$, the laboratory energy of the virtual photon. All hadron states of appropriate momentum have been summed over in writing Eq. (3).

The old inequality is[2]

$$\int_0^\infty d\nu\{\sigma_{1n}(q^2,\nu)+\sigma_{1p}(q^2,\nu)\}\geq\frac{2\pi\alpha^2}{q^4}\,.\quad(4)$$

The new inequality is (as $|q^2|\to\infty$ only)

$$\int_0^\infty\frac{d\nu}{\nu^2}\{\sigma_{2n}(q^2,\nu)+\sigma_{2p}(q^2,\nu)\}\geq\frac{4\pi\alpha^2}{|q^6|}\times\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\langle P|[j_x^+(\mathbf{x}),j_x^-(0)]|P\rangle\,,\quad(5)$$

and j_x^+ is the plus component of isovector current, normalized such that the commutator in Eq. (5) is unity for the $U(6)\times U(6)$ algebra.[3] Corresponding to Adler's old neutrino sum rule[1] [the β sum rule] for σ_1

$$\int_0^\infty d\nu[\sigma_1^{\bar{\nu}p}(q^2,\nu)-\sigma_1^{\nu p}(q^2,\nu)]=\frac{G^2}{\pi}\int d^3x\,\langle P|[J_0^+(\mathbf{x}),J_0^-(0)]|P\rangle e^{i\mathbf{q}\cdot\mathbf{x}}=\frac{G^2}{\pi}(\cos^2\theta_c+2\sin^2\theta_c)\,.\quad(6)$$

We also find (as $|q^2|\to\infty$ only)

$$\frac{|q^2|}{2}\int_0^\infty\frac{d\nu}{\nu^2}[\sigma_2^{\bar{\nu}p}(q^2,\nu)-\sigma_2^{\nu p}(q^2,\nu)]=\frac{G^2}{\pi}\int d^3x\,\langle P|[J_x^+(\mathbf{x}),J_x^-(0)]|P\rangle e^{i\mathbf{q}\cdot\mathbf{x}}=\begin{cases}(G^2/\pi)(\cos^2\theta_c+2\sin^2\theta_c) & [U(6)\times U(6)\text{ algebra}]\\ 0\ (\text{spin-0 constituents}).\end{cases}\quad(7)$$

$J_\mu^\pm(x)$ is now the full Cabibbo current ($V-A$, $\Delta S=0,1$). Although similar, Eq. (7) is *not* Adler's α sum rule,[1] which lacks the convergence factor q^2/ν^2.

As might be expected, the result depends upon the structure of the commutator of *space* components of isovector currents. With the chiral $U(6)\times U(6)$ algebra,[3] the commutator on the right-hand side of Eq. (5) is unity, and we expect relatively large scattering. How-

* Work supported by the U. S. Atomic Energy Commission.
[1] S. L. Adler, Phys. Rev. 143, 1144 (1966).
[2] J. D. Bjorken, Phys. Rev. Letters 16, 408 (1966).
[3] R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters 13, 678 (1964).

ever, one can imagine models in which the isospin current is carried by spinless objects; in this case the commutator vanishes and there is no lower bound to the backward scattering cross sections.

We start, as with the derivation of the forward-scattering inequality, Eq. (4), with the amplitude $M_{\mu\nu}$ for scattering an isovector current $j_\mu^+(x)$ from a proton in the forward direction[4] (see Fig. 1).

$$\begin{aligned}M_{\mu\nu}(q,P)=&[q^2P_\mu P_\nu-(q_\mu P_\nu+q_\nu P_\mu)q\cdot P+(q\cdot P)^2g_{\mu\nu}]\\&\times F_1(q^2,q\cdot P)+[q_\mu q_\nu-g_{\mu\nu}q^2]F_2(q^2,q\cdot P)\\&+[q_\mu P_\nu+q_\nu P_\mu-g_{\mu\nu}q\cdot P]/q^2\\&+[\text{polynomial in } q \text{ and } p].\end{aligned}\quad(8)$$

We include Born terms[5] in the definition of F_1 and F_2. $M_{\mu\nu}$ is defined (up to normalization factors) such that when lepton pairs are attached it is a piece of the S matrix. It is not necessarily the time-ordered product of currents. Notice

$$q_\mu M^{\mu\nu}=P^\nu+[\text{polynomial in } q \text{ and } P]\quad(9)$$

and

$$P_\mu=P_0\int\langle P|[j_0^+(\mathbf{x}),j_\mu^-(0)]|P\rangle d^3x.\quad(10)$$

The neutrino- (and antineutrino-) proton scattering cross section is proportional to $\mathrm{Im}F_1$ and $\mathrm{Im}F_2$. The backward-scattering cross sections σ_2 are proportional to the coefficient of $g_{\mu\nu}$:

$$\sigma_2\propto\mathrm{Im}\{(q\cdot P)^2F_1-q^2F_2\}.\quad(11)$$

Adler's sum rule is obtained by demanding, as is suggested by Regge theory,[6] asymptotic behavior for the coefficient of $q_\mu P_\nu$ less strong than constant. Thus

$$\frac{1}{\pi}\int d\nu'\,\mathrm{Im}F_1(q^2,\nu')=\frac{-1}{q^2}.\quad(12)$$

Regge behavior also suggests[6,7] that F_2 needs one subtraction. We shall assume this is the case:

$$F_2(q^2,\nu)=F_2(q^2,0)+\frac{\nu}{\pi}\int\frac{d\nu'\,\mathrm{Im}F_2(q^2,\nu')}{\nu'(\nu'-\nu)}.\quad(13)$$

We now study $M_{\mu\nu}$ as $q_0\to i\infty$, $\mathbf{q}$ fixed. As in Ref. 4, the coefficient of $1/q_0$ is an equal-time commutator. In the limit $q_0\to i\infty$,

$$F_1(q^2,\nu)=\frac{1}{\pi}\int_{|\nu'|\geq|q^2|/2m}^{\infty}\frac{d\nu'\,\mathrm{Im}F_1(q^2,\nu)}{\nu'-\nu}\to\frac{1}{\pi}\int^{\infty}\frac{d\nu'}{\nu'}\mathrm{Im}F_1(q^2,\nu').\quad(14)$$

The most reasonable estimate is that

$$\int\frac{d\nu'}{\nu'}\mathrm{Im}F_1(q^2,\nu')\sim\frac{\text{const}}{q^2}\int d\nu'\,\mathrm{Im}F_1\sim\frac{\text{const}}{q^4},\quad(15)$$

which would be rigorously true if $\mathrm{Im}F_1$ did not change sign. We assume that there are no delicate cancellations here and we may use Eq. (15). With this estimate, the terms involving F_1 are of order $1/q_0^2$ in the limit. Writing $\eta_\mu=(1,0,0,0)$, we find, barring pathological cancellations,

$$\begin{aligned}M_{\mu\nu}(q,P)\underset{q_0\to i\infty}{\longrightarrow}&[\text{polynomial}]+[\eta_\mu q_\nu+\eta_\nu q_\mu-2\eta\cdot q\eta_\mu\eta_\nu+(\eta_\mu\eta_\nu-g_{\mu\nu})q^2]F_2(q^2,0)\\&+(\eta_\mu\eta_\nu-g_{\mu\nu})\frac{q_0^3P_0}{\pi}\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\mathrm{Im}F_2(q^2,\nu')+\frac{[\eta_\mu P_\nu+\eta_\nu P_\mu-g_{\mu\nu}\eta\cdot P]}{q_0}+[\text{terms more convergent as } q_0\to\infty].\end{aligned}\quad(16)$$

(The axial part can be treated in a similar way.) On the other hand, the term $O(1/q_0)$ is

$$M_{\mu\nu}\to P_0\int\frac{\langle P|[j_\mu^+(x),j_\nu^-(0)]|P\rangle d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}}{q_0}+[\text{terms with different asymptotic behavior}].\quad(17)$$

Thus the term multiplying $F_2(q^2,0)$ contributes to any operator Schwinger terms involving $[j_0^+,j_i^-]$. A deviation of the commutator of space components of the currents from the chiral algebra prediction is measured by $\mathrm{Im}F_2$. Indeed

$$\begin{aligned}\int\langle P|[j_i^+(\mathbf{x}),j_i^-(0)]|P\rangle d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}&=\left[1+\frac{q^4}{\pi}\int\frac{d\nu'}{\nu'^2}\mathrm{Im}F_2(q^2,\nu')\right]=\frac{q^2}{\pi}\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\\&\times[-\nu'^2\,\mathrm{Im}F_1(q^2,\nu')+q^2\,\mathrm{Im}F_2(q^2,\nu')],\end{aligned}\quad(18)$$

[4] We follow the derivation outlined in J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); see also *Proceedings of the Third Coral Gable Conference Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1967).

[5] The apparent pole at $q^2=0$ in Eq. (8) is cancelled by another pole in the Born terms. We set the nucleon mass equal to unity.

[6] This assumption is, we believe, the least trustworthy in the derivation of that result. This is because non-Regge behavior has been shown to exist in the coefficient of $P_\mu P_\nu$ [J. Bronzan, I. Gerstein, B. Lee, and F. Low, Phys. Rev. Letters **18**, 32 (1967); V. Singh, Phys. Rev. Letters **18**, 36 (1967)]. If similar asymptotic behavior, corresponding to a fixed pole or Kronecker δ at $J=1$, also occurs in the coefficient of $q_\mu P_\nu$, we lose the sum rule Eq. (6) as well as the once-subtracted dispersion relation, Eq. (13). Arguments that this does not happen in simple models have been given by the above authors; there are other arguments by G. deAlfaro, S. Fubini, G. Furlan, and A. Rosetti (to be published).

[7] H. Harari, Phys. Rev. Letters **17**, 1303 (1966).

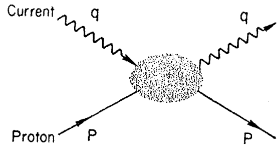

FIG. 1. Kinematics for forward scattering of a current from a nucleon.

where we have used Eq. (12). The quantity in brackets is proportional to the vector piece of $\sigma_2^{\bar{\nu}p}(q^2,\nu)$ or $\sigma_2^{\nu p}$ as defined in Eqs. (3) and (11). After a routine struggle with normalization factors (most simply done by considering free fields) one arrives at the sum rule Eq. (7). The same isospin manipulations[2] as used in obtaining Eq. (4) from Eq. (6) are sufficient to get Eq. (5) from Eq. (7).

It is tempting to assume the result Eq. (7) to be generally valid for all q^2. However, consideration of the limit as $q^2 \to 0$ gives

$$(\mu_p - \mu_n)^2 = 1\,, \tag{19}$$

in considerable disagreement with experiment.

The following physical picture of the result Eq. (5) suggests itself: If the "elementary constituents" (if any) of the nucleon, which couple to isospin, were spinless, there would be very little backward scattering at large q^2, because backward scattering demands helicity flip. If the constituents have spin $\frac{1}{2}$, the scattering should be incoherent and proportional to the sum of squares of the magnetic moments of the constituents.[8]

Experimental verification of the inequality Eq. (5) may be difficult because of the problems of radiative corrections.

The author thanks J. D. Walecka for asking the right question, his colleagues at SLAC for discussions, and Helen Quinn and Sam Berman for a reading of the manuscript.

[8] This picture is similar to that discussed for forward scattering by K. Gottfried [Phys. Rev. Letters **18**, 1174 (1967)]. There also exist sum rules of this kind in nuclear physics: for a review see deForest and Walecka, Advan. Physics **15**, 1 (1966).

PHYSICAL REVIEW VOLUME 163, NUMBER 5 25 NOVEMBER 1967

Necessary and Sufficient Conditions for the Current-Spectral-Function Sum Rules

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Necessary and sufficient conditions are given for the validity of the Weinberg-type current-spectral-function sum rules. For one class of the sum rules, the validity rests on the equality of the vacuum expectation values of the corresponding Schwinger terms. For the other class, the condition involves the triple commutator of the space component of the current with the Hamiltonian (of the world). Comments are made on the usual derivation of the sum rules and Lee, Weinberg, and Zumino's algebra of fields.

THE sum rules of Weinberg[1] for the spectral functions of the chiral $SU(2)\times SU(2)$ currents have been successfully used to relate the ρ and A_1 masses, and to calculate the electromagnetic pion mass difference.[2] Generalization to the case of $SU(3)\times SU(3)$ has led to a calculation of the ratio F_K/F_π,[3,4] in fair agreement with experiment. While all this demonstrates the usefulness of the Weinberg-type sum rules, their validity has not yet been rigorously established,[5] except in the context of a Lagrangian model of Lee, Weinberg, and Zumino,[6] which, however, has been brought into question by a recent consideration related to the electromagnetic corrections to the pion β decay.[7] It is therefore of interest to know exactly the conditions under which the Weinberg-type sum rules are valid. In this paper we shall give the necessary and sufficient conditions for their validity. For one class of sum rules, their validity rests on the equality of the vacuum expectation values of the corresponding Schwinger terms. For the other class, the condition involves the triple commutator of the space component of the current with the Hamiltonian (of the world).

Consider any local current density $J_\mu(x)$, which may or may not be conserved. The most general spectral representation for the current correlation function is

$$\langle J_\mu(x)J_\nu(0)\rangle_0 = (2\pi)^{-3}\int d^4p\;\theta(p^0)e^{ip\cdot x} \times[(g_{\mu\nu}-p_\mu p_\nu/p^2)\rho_1(-p^2)+p_\mu p_\nu\rho_0(-p^2)]\,, \tag{1}$$

[1] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[2] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Letters **18**, 759 (1967).
[3] H. T. Nieh, Phys. Rev. Letters **19**, 43 (1967).
[4] S. L. Glashow, H. J. Schnitzer, and S. Weinberg, Phys. Rev. Letters **19**, 139 (1967).
[5] A critical comment on the derivation of the spectral function sum rules is given by T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[6] See Ref. 5.

[7] K. Johnson, F. E. Low, and H. Suura, Phys. Rev. Letters **18**, 1224 (1967); N. Cabibbo, L. Maiani, and G. Preparata (to be published). We will come back to this point later.