Four-Mass Kinematics for Regge Crossing*

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We investigate the kinematics of two-particle–to–two-particle scattering for the purpose of Regge crossing. The boundaries of the physical regions for all three channels are given by a single equality, $|z_s|=|z_t|=|z_u|=1$, which is not the case for two different-mass and four equal-mass processes. In both the s and u physical regions, $|z_t|$ is larger than or equal to 1, and has a maximum whose value increases with increasing energy. At the same time, the position of this maximum approaches the forward direction as the energy increases. (Similar statements can be made about z_s in the t and u channels, and z_u in the s and t channels.) There exist three singular points in either the forward or backward directions for two channels at definite scattering energies for which one of z_s, z_t, or z_u is indeterminate.

I. INTRODUCTION

THE Regge analysis of scattering experiments in which all four particles have different masses is made difficult by the complexity of the kinematics. This problem was investigated some time ago,[1] and the general features of the kinematics are well known. Here we are interested mainly in those aspects which will be used in Regge crossing. In particular we discuss the behavior of the three angle variables in the s, t, and u channels on the Mandelstam plane. We have applied the results of this paper to a Regge analysis of the reaction $\pi^-+p\to\eta+n$.[2]

In the scattering of four equal masses the physical regions are defined by $s=t=u=0$. In the two-equal-mass case in which the t channel is $2m\to 2M$, they are given by $t=0$ and the two branches of the hyperbola $su=(M^2-m^2)^2$. In the four-unequal-mass case a single expression, a trinomial in s, t, and u defines the physical regions. In addition, on the boundaries of a particular channel not only is the cosine of the scattering angle of this channel ±1, but the cosines of the angles of the other two channels are also ±1. As mentioned before, this result does not hold if there are only one or two different masses. As a consequence of this feature, both in the forward and backward directions it is necessary to evaluate the Legendre functions $P_\alpha(z)$ which appear in the Regge representation at all energies rather than use their asymptotic expansions.

In Sec. II we define the variables and in particular the angle variables. We have included this section for the sake of completeness even though these formulas are well known. In Sec. III we discuss the boundaries of the physical regions in terms of the angle variables. We also investigate the indeterminacy points, their number, and the relation between the mass ordering and the distribution of these points among different channels. In Sec. IV we discuss the behavior of the angle variables inside and outside the physical regions. Of particular interest for Regge crossing is their behavior at fixed energy and the change in this behavior as the energy changes. In Sec. V we give a procedure for determining the behavior of the angle variables on the Mandelstam plane for any reaction.

II. DEFINITION OF SCATTERING VARIABLES

The 4-momenta of the particles taking part in the scattering are called p_i and P_i ($i=i, 2$) (see Fig. 1). They satisfy the following relations

$$p_1+p_2+P_1+P_2=0\,, \tag{2.1}$$

$$p_i^2=m_i^2,\quad P_i^2=M_i^2\quad (i=1,2)\,. \tag{2.2}$$

For convenience we define the quantity Σ to be $m_1^2+m_2^2+M_1^2+M_2^2$. The Lorentz invariant variables are given by

$$\begin{aligned} s&=(p_1+P_1)^2=(p_2+P_2)^2\,,\\ t&=(p_1+p_2)^2=(P_1+P_2)^2\,,\\ u&=(p_1+P_2)^2=(p_2+P_1)^2\,,\end{aligned} \tag{2.3}$$

$$s+t+u=\Sigma\,. \tag{2.4}$$

The s channel corresponds to the reaction

$$m_1+M_1\to m_2+M_2\,.$$

In this case the 4-momenta and the Mandelstam variables are

$$\begin{aligned} &p_1=(e_1,p_s,0,0)\,,\quad P_1=(E_1,-p_s,0,0)\,,\\ &-p_2=(e_2,q_s\cos\theta_s,q_s\sin\theta_s,0)\,,\\ &-P_2=(E_2,-q_s\cos\theta_s,-q_s\sin\theta_s,0)\,,\\ &s=m_1^2+M_1^2+2e_1E_1+2p_s^2=m_2^2+M_2^2+2e_2E_2+2q_s^2,\\ &t=m_1^2+m_2^2-2e_1e_2+2p_sq_sz_s\,,\\ &u=m_2^2+M_1^2-2E_1e_2-2p_sq_sz_s\,,\end{aligned} \tag{2.5}$$

where all quantities are defined in the center-of-mass system. $z_s=\cos\theta_s$ is the cosine of the scattering angle in the s channel, e_i and E_i are particle energies, and p_s and q_s are the magnitudes of the 3-momenta before and after the scattering which are given by

$$p_s^2=\frac{(s-s_2)(s-s_3)}{4s}\,,\quad q_s^2=\frac{(s-s_1)(s-s_4)}{4s}\,, \tag{2.6}$$

$$\begin{aligned} s_1&=(M_2-m_2)^2\,,\quad s_2=(M_1-m_1)^2\,,\\ s_3&=(M_1+m_1)^2\,,\quad s_4=(M_2+m_2)^2\,.\end{aligned} \tag{2.7}$$

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[1] T. W. B. Kibble, Phys. Rev. 117, 1159 (1960); D. A. Atkinson and V. Barger, Nuovo Cimento 38, 634 (1965).
[2] I. A. Sakmar and J. H. Wojtaszek (unpublished).

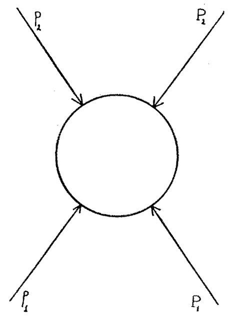

FIG. 1. Diagram defining 4-momenta of scattering particles.

The ordering of the s_i's is taken with the reaction $\pi^- + p \rightarrow \eta + n$ in mind. In this case $s_1 < s_2 < s_3 < s_4$. For other reactions the ordering may be different, but there is no loss of generality.

If p_s and q_s are eliminatedfro m the equation for u, one obtains

$$z_s = \frac{-s^2 - 2su + s\Sigma + (M_1^2 - m_1^2)(M_2^2 - m_2^2)}{[(s-s_1)(s-s_2)(s-s_3)(s-s_4)]^{1/2}}$$

$$= \frac{(u - \frac{1}{2}\Sigma)^2 - (t - \frac{1}{2}\Sigma)^2 + (M_1^2 - m_1^2)(M_2^2 - m_2^2)}{[(s-s_1)(s-s_2)(s-s_3)(s-s_4)]^{1/2}}. \quad (2.8)$$

The t-channel reaction is[3]

$$m_1 + m_2 \rightarrow M_1 + M_2 .$$

(Because the masses of particles and antiparticles are the same we make no distinction between M_1 and, say, $\bar{M}_1$. The results of this paper depend solely on the numerical values of the masses.) The variables now are

$$p_1 = (e_1', p_t, 0, 0), \quad p_2 = (e_2', -p_t, 0, 0),$$

$$-P_1 = (E_1', q_t \cos\theta_t, q_t \sin\theta_t, 0),$$

$$-P_2 = (E_2', -q_t \cos\theta_t, -q_t \sin\theta_t, 0),$$

$$\begin{aligned} t &= m_1^2 + m_2^2 + 2e_1'e_2' + 2p_t^2 \\ &= M_1^2 + M_2^2 + 2E_1'E_2' + 2q_t^2, \\ u &= m_2^2 + M_1^2 - 2e_2'E_1' - 2p_t q_t z_t, \\ s &= m_1^2 + M_1^2 - 2e_1'E_1' + 2p_t q_t z_t, \end{aligned} \quad (2.9)$$

$$p_t^2 = \frac{(t-t_2)(t-t_3)}{4t}, \quad q_t^2 = \frac{(t-t_1)(t-t_4)}{4t}, \quad (2.10)$$

$$\begin{aligned} t_1 &= (M_1 - M_2)^2, \quad t_2 = (m_1 - m_2)^2, \\ t_3 &= (m_1 + m_2)^2, \quad t_4 = (M_1 + M_2)^2. \end{aligned} \quad (2.11)$$

[3] The t-channel equations can be obtained from the s-channel ones by the replacements $s \rightarrow t \rightarrow u \rightarrow s$, $M_1 \rightarrow m_2 \rightarrow M_2 \rightarrow M_1$, $z_s \rightarrow -z_t$. These are more complicated than the usual ones ($s \leftrightarrow t$, $M_1 \leftrightarrow m_2$) in order that the ordering of s_i, t_i, and u_i be preserved, and for the sake of clarity the three sets of equations are written out completely. To get from t to u channels, the replacements are $s \rightarrow t \rightarrow u \rightarrow s$, $m_1 \rightarrow M_2 \rightarrow m_2 \rightarrow m_1$, $z_t \rightarrow z_u$.

Eliminating p_t and q_t in the equation for s, we find that

$$z_t = \frac{t^2 + 2ts - t\Sigma + (m_1^2 - m_2^2)(M_1^2 - M_2^2)}{[(t-t_1)(t-t_2)(t-t_3)(t-t_4)]^{1/2}}$$

$$= \frac{(u - \frac{1}{2}\Sigma)^2 - (s - \frac{1}{2}\Sigma)^2 + (m_1^2 - m_2^2)(M_1^2 - M_2^2)}{[(t-t_1)(t-t_2)(t-t_3)(t-t_4)]^{1/2}}. \quad (2.12)$$

The corresponding equations for the u channel,

$$m_2 + M_1 \rightarrow m_1 + M_2 ,$$

are

$$P_1 = (E_1'', -p_u, 0, 0), \quad p_2 = (e_2'', p_u, 0, 0),$$

$$-p_1 = (e_1'', q_u \cos\theta_u, q_u \sin\theta_u, 0),$$

$$-P_2 = (E_2'', -q_u \cos\theta_u, -q_u \sin\theta_u, 0),$$

$$\begin{aligned} u &= m_2^2 + M_1^2 + 2e_2''E_1'' + 2p_u^2 \\ &= m_1^2 + M_2^2 + 2e_1''E_2'' + 2q_u^2, \\ s &= m_1^2 + M_1^2 - 2e_1''E_1'' - 2p_u q_u z_u, \\ t &= m_1^2 + m_2^2 - 2e_1''e_2'' + 2p_u q_u z_u, \end{aligned} \quad (2.13)$$

$$p_u^2 = \frac{(u-u_1)(u-u_4)}{4u}, \quad q_u^2 = \frac{(u-u_2)(u-u_3)}{4u}, \quad (2.14)$$

$$\begin{aligned} u_1 &= (M_1 - m_2)^2, \quad u_2 = (M_2 - m_1)^2, \\ u_3 &= (M_2 + m_1)^2, \quad u_4 = (M_1 + m_2)^2. \end{aligned} \quad (2.15)$$

The relation between z_u, t, and u is

$$z_u = \frac{u^2 + 2tu - u\Sigma + (M_1^2 - m_2^2)(M_2^2 - m_1^2)}{[(u-u_1)(u-u_2)(u-u_3)(u-u_4)]^{1/2}}$$

$$= \frac{(s - \frac{1}{2}\Sigma)^2 - (t - \frac{1}{2}\Sigma)^2 + (M_1^2 - m_2^2)(M_2^2 - m_1^2)}{[(u-u_1)(u-u_2)(u-u_3)(u-u_4)]^{1/2}}. \quad (2.16)$$

III. BOUNDARIES OF THE PHYSICAL REGIONS

The physical region for the s channel is defined by

$$z_s^2 \leqslant 1, \quad s \geqslant s_4$$

(and similarly for the t and u channels). The equation for its boundary can be derived from Eq. (2.8). It is $z_s^2 = 1$:

$$stu + s(M_1^2 - m_2^2)(M_2^2 - m_1^2) + u(M_2^2 - m_2^2)(M_1^2 - m_1^2) - (M_1^2 M_2^2 - m_1^2 m_2^2)(M_1^2 + M_2^2 - m_1^2 - m_2^2) = 0. \quad (3.1)$$

The boundaries for the t and u channels are given by $z_t^2 = 1$:

$$stu - t(M_1^2 - m_1^2)(M_2^2 - m_2^2) - s(M_2^2 - M_1^2)(m_2^2 - m_1^2) - (M_1^2 m_2^2 - M_2^2 m_1^2)(M_1^2 - M_2^2 - m_1^2 + m_2^2) = 0; \quad (3.2)$$

$z_u^2 = 1$:

$$stu + u(M_2^2 - M_1^2)(m_2^2 - m_1^2) - t(M_1^2 - m_2^2)(M_2^2 - m_1^2) - (M_2^2 m_2^2 - M_1^2 m_1^2)(M_2^2 - M_1^2 + m_2^2 - m_1^2) = 0. \quad (3.3)$$

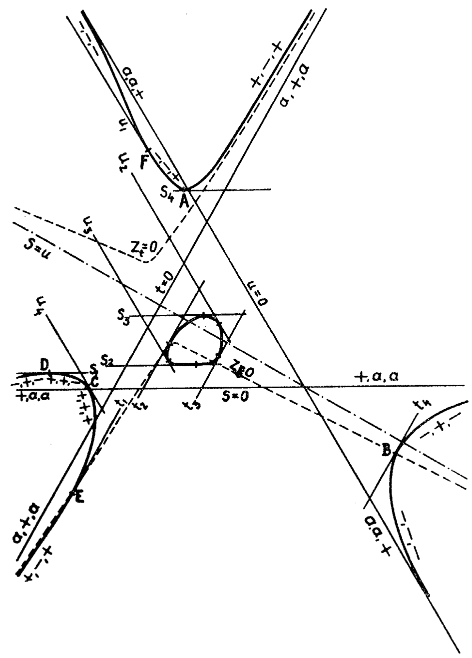

FIG. 2. Mandelstam diagram for four different mass particles. The three solid curves are the boundaries of the physical regions. The triplets specify the values of z_s, z_t, and z_u. (For example, $+, -, +$ means that $z_s=+1$, $z_t=-1$, and $z_u=+1$.) a means that the value of a particular variable is not constant. This figure is not drawn to scale, but is only meant to bring out the important features.

If the terms linear in u in Eqs. (3.1) and (3.3) are eliminated by using $u=\Sigma-s-t$, it is found that all three expressions are identical. That is, the boundaries of the physical regions for all three processes are given by the same equation, e.g., $|z_s|=|z_t|=|z_u|=1$. We shall now describe the behavior of this curve in the s, t, u plane.

Figure 2 is a graph of Eqs. (3.1), (3.2), and (3.3) corresponding to the case where the masses satisfy

$$m_1<m_2<M_1<M_2$$

(which are satisfied for the reaction $\pi^-+p\to\eta+n$).

The main features of Fig. 2 are the three open curves and a loop in the unphysical region where s, t, and u are all positive. These curves are tangent to the twelve lines

$$s=s_i,\quad t=t_i,\quad u=u_i\quad (i=1,2,3,4)$$

at twelve points. Three of these points labeled A, B, C on the diagram correspond to the thresholds for the three channels. Three other points labeled D, E, F are the extrema of the boundaries of the physical regions in s, t, or u directions.

It is seen that the s channel has only one such point (F), the t channel none, and the u channel two (D and E). From the form of the Eqs. (3.1), (3.2), and (3.3), as well as the ordering of the masses which is related to the definitions of the channels, one can show the following:

There can be at most three such points (D,E,F) on boundaries of the physical channels. If the t channel is defined as the channel in which the two lightest particles go into the two heaviest particles, then the t channel has no such extremum points.

The channel $m_1+M_1\to m_2+M_2$ for which $m_2>m_1$, $M_2>M_1$ has only one such point (F). ($\pi^-+p\to\eta+n$ is our example.)

The third channel $m_2+M_1\to m_1+M_2$ for which $m_2>m_1$ but $M_1<M_2$ has two points (D and E).

The values of the other variables at these points can be found by inserting the fixed values of the lines to which this curve is tangent into Eqs. (3.1), (3.2), and (3.3).

$$D:\ s=s_1,\quad t_D=\frac{m_2(M_2^2-M_1^2)+M_2(m_1^2-m_2^2)}{M_2-m_2};$$

$$s=s_2,\quad t=\frac{m_1(M_1^2-M_2^2)+M_1(m_2^2-m_1^2)}{M_1-m_1};$$

$$s=s_3,\quad t=\frac{m_1(M_2^2-M_1^2)+M_1(m_2^2-m_1^2)}{M_1+m_1};\tag{3.4a}$$

$$A:\ s=s_4,\quad t_A=\frac{m_2(M_1^2-M_2^2)+M_2(m_1^2-m_2^2)}{M_2+m_2};$$

$$E:\ t=t_1,\quad u_E=\frac{M_1(M_2^2-m_1^2)+M_2(m_2^2-M_1^2)}{M_2-M_1};$$

$$t=t_2,\quad u=\frac{m_1(m_2^2-M_1^2)+m_2(M_2^2-m_1^2)}{m_2-m_1};$$

$$t=t_3,\quad u=\frac{m_1(M_1^2-m_2^2)+m_2(M_2^2-m_1^2)}{m_2+m_1};\tag{3.4b}$$

$$B:\ t=t_4,\quad u_B=\frac{M_1(m_1^2-M_2^2)+M_2(m_2^2-M_1^2)}{M_2+M_1};$$

$$F:\ u=u_1,\quad s_F=\frac{m_2(M_1^2-m_1^2)+M_1(M_2^2-m_2^2)}{M_1-m_2};$$

$$u=u_2,\quad s=\frac{m_1(M_2^2-m_2^2)+M_2(M_1^2-m_1^2)}{M_2-m_1};$$

$$u=u_3,\quad s=\frac{m_1(m_2^2-M_2^2)+M_2(M_1^2-m_1^2)}{M_2+m_1};\tag{3.4c}$$

$$C:\ u=u_4,\quad s_C=\frac{m_2(m_1^2-M_1^2)+M_1(M_2^2-m_2^2)}{M_1+m_2}.$$

We do not name the remaining six points at which the loop is tangent to the lines mentioned above, because they are not in physical regions. The significance of these points is that at each of them, one of the angle variables is undefined. For example z_s is undefined at A and D; therefore all curves $z_s=$ constant pass through these points. This is a familiar result at the thresholds A, B, and C, where the scattering angles have no definite values, but it is something new at D, E, and F. These are points in either the backward or forward directions at definite scattering energies given by Eqs. (3.4a), (3.4b), and (3.4c).

For the case of two different masses,[4] e.g., when $m_1 \to m_2$ and $M_1 \to M_2$, F coincides with A and D coincides with C.

The points at which the curves cross the three axes are also of interest. These are:

$$s=0,\quad t=\frac{(M_2^2m_1^2-M_1^2m_2^2)(M_1^2-M_2^2-m_1^2+m_2^2)}{(M_1^2-m_1^2)(M_2^2-m_2^2)};$$

$$t=0,\quad u=\frac{(M_1^2m_1^2-M_2^2m_2^2)(M_1^2-M_2^2+m_1^2-m_2^2)}{(M_2^2-M_1^2)(m_2^2-m_1^2)};$$

$$u=0,\quad s=\frac{(M_1^2M_2^2-m_1^2m_2^2)(M_1^2+M_2^2-m_1^2-m_2^2)}{(M_1^2-m_2^2)(m_1^2-M_2^2)}. \qquad (3.5)$$

There are many reactions with only three masses different (for example $\pi+p\to\rho+p$ or $p+\bar{p}\to 2$ different particles), so it is of interest to investigate the case where $M_1\to M_2$. The only difference in Fig. 2 in that case is that $t_1\to 0$ and the two points E and the intersection of the curve with the $t=0$ line go to $s=-\infty$, as can be seen from Eqs. (3.4b) and (3.5).

IV. BEHAVIOR OF THE ANGLE VARIABLES ON THE MANDELSTAM PLANE

A. z_t

Because it is most often the case that crossing is performed from t to s channels, we shall discuss the behavior of z_t most carefully and then give summaries for z_s and z_u.

In discussing the values of z_t it is very important to keep in mind how the Mandelstam plane is divided by certain curves. The four lines

$$t=t_i \quad (i=i,\,2,\,3,\,4)$$

divide it into five strips. From Eq. (2.12) we see that these are the branch points of the denominator of z_t. We use the natural definition of the first Riemann surface, so that the square root has the phases $+$, i, $-$, $-i$, $+$ in the five strips as t varies from $+\infty$ to $-\infty$.

In addition to these four lines, the hyperbola

$$z_t=0=(u-\tfrac{1}{2}\Sigma)^2-(s-\tfrac{1}{2}\Sigma)^2+(M_1^2-M_2^2)(m_1^2-m_2^2) \qquad (4.1)$$

separates the plane into regions of positive and negative

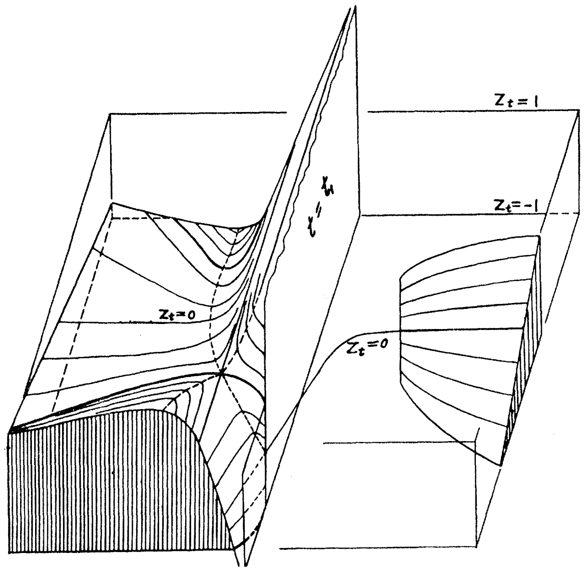

FIG. 3. Three-dimensional diagram of z_t in the Mandelstam plane.

z_t. This curve has asymptotes $s=u$ and $t=0$ and does not exist for

$$(s-\tfrac{1}{2}\Sigma)^2<(M_1^2-M_2^2)(m_1^2-m_2^2).$$

z_t is negative inside the upper branch of this hyperbola. Inside the lower branch the numerator of Eq. (2.12) is negative. Therefore z_t itself in the five strips has the phases $-$, $+i$, $+$, $-i$, $-$. Above the $z_t=0$ hyperbola it has opposite phases. As t approaches one of the lines t_i, z_t approaches ∞ with its phase determined according to the prescriptions given above.

The shape of the z_t surface on the Mandelstam plane can be seen as follows: We fix t at the value t_0. Let (t_0,s_0) be a point of the hyperbola $z_t=0$. Then, as can be seen from Eq. (2.12),

$$z_t(s,t_0)=f(t_0)(s-s_0),$$

where

$$f(t_0)=\frac{2t_0}{[(t_0-t_1)(t_0-t_2)(t_0-t_3)(t_0-t_4)]^{1/2}}.$$

Therefore the z_t surface when cut along lines of constant t has a linear cross section going through zero along the hyperbola and having a slope $f(t_0)$. This slope goes to zero at $t=0$ and $t=\pm\infty$ and to ∞ at $t=t_i$. At $t=0$ itself, $z_t=1$, as can be seen from Eq. (2.12). Thus the z_t surface is generated by a line, one point of which stays on the $z_t=0$ hyperbola, and whose slope depends only on t. Figure 3 is a three-dimensional drawing of the z_t surface on the Mandelstam plane. For the sake of clarity the central region between t_1 and t_4 is omitted. In addition we cut it off for $|z_t|>1$ in the physical t channel.

Keeping these points in mind, we now discuss the behavior of z_t in the three channels.

[4] I. A. Sakmar, Nuovo Cimento 40, 76 (1965).

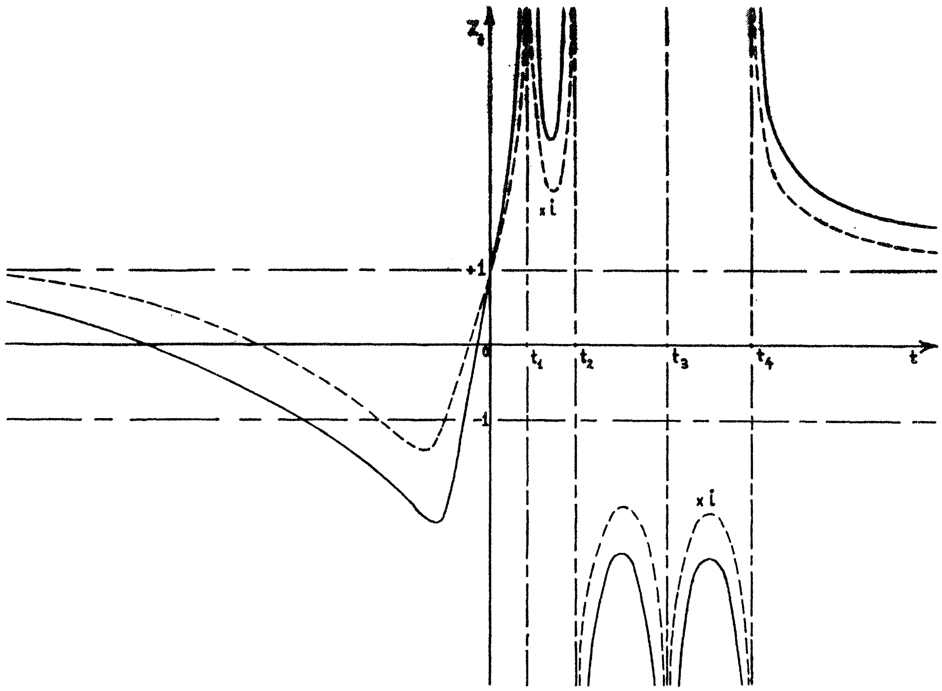

FIG. 4. Graphs of z_t at two different values of s. The solid curve corresponds to the higher energy. The physical regions are at negative t where $z_t \leqslant -1$.

In the t channel (i.e., $t=\text{const}>t_4$), z_t varies linearly with s going through ± 1 in the forward and backward directions.

For the s channel we show in Fig. 4 graphs of z_t for two energies. The main features are that inside the physical region $z_t < -1$, having a minimum which is close to the forward direction. The position of this minimum and of the three others shown in Fig. 4 can be obtained from Eq. (2.12).

$dz_t/dt=0$:

$$\begin{aligned} &st^4 - t^3[as + \tfrac{1}{2}(b-a^2) - c] + \tfrac{3}{2}t^2(d-ac) \\ &\quad + t[sd - c^2 + \tfrac{1}{2}(bc-ad)], \\ &\quad - c^2 s + \tfrac{1}{2}c(ac-d) = 0, \\ &a = \Sigma, \qquad (4.2) \\ &b = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4, \\ &c = (M_1^2 - M_2^2)(m_1^2 - m_2^2), \\ &d = \tfrac{1}{2}c^2(1/t_1 + 1/t_2 + 1/t_3 + 1/t_4). \end{aligned}$$

This equation has a finite, negative solution for t as s gets very large. This is the minimum in the physical region, which, because t is finite, becomes more and more forward at larger and larger scattering energies.

In the physical u channel, the behavior of z_t at constant u is different in the two regions $u>u_E$ and $u<u_E$. For $u>u_E$, z_t starts with the value -1 at $s=+\infty$, increases smoothly, and becomes $+1$ in the backward direction of the u channel. Inside the u channel it continues to increase, reaches a maximum value, and then drops to $+1$ on the line $t=0$. This maximum is larger for larger values of u and approaches the forward direction as u increases. Beyond the line $t=0$, z_t decreases to -1 in the forward direction and continues to decrease, approaching $-\infty$ at $t=t_1$.

For $u<u_E$, z_t has the same behavior up to the line $t=0$. But between $t=0$ and the forward direction of the u channel z_t has a valley. It first drops and then increases to $+1$ in the forward direction. Beyond the forward direction z_t increases, becoming $+\infty$ at $t=t_1$.

Along the line $u=u_E$, z_t behaves similarly except in the region between $t=0$ and the forward direction. Here it has neither a minimum as was the case for $u<u_E$, nor does it approach $-\infty$ as in the case of $u>u_E$. Instead it decreases smoothly and ends abruptly on the line $t=t_1$ where z_t equals zero. If point E is approached along a line other than $u=u_E$, a different value of z_t is obtained.

B. z_s

We now briefly summarize the behavior of z_s with the aid of Fig. 2. Along a line of constant $s>s_4$, z_s falls linearly from $+1$ to -1 in the physical region, going to 0 along the hyperbola

$$(t-\tfrac{1}{2}\Sigma)^2 - (u-\tfrac{1}{2}\Sigma)^2 = (M_1^2 - m_1^2)(M_2^2 - m_2^2).$$

Outside this region it approaches $+\infty$ for positive t and $-\infty$ for negative t. Along a line of constant $t>t_4$, z_s takes on the value -1 in both the backward and forward directions for the t channel and in between is less than -1 having a minimum which is peaked toward the line $s=0$. Outside the physical t region z_s rises asymptotically to $+1$ as $s\to-\infty$. As $s\to s_1$, z_s rises through $+1$ at $s=0$ and approaches $+\infty$.

In the three regions between s_1 and s_4 the z_s curve consists of a set of three peaked curves going to $+i\infty$,

$-\infty$, and $-i\infty$, respectively, in each of these regions as $s \to s_i$. For $s > s_4$, z_s drops from $+\infty$ and approaches $+1$ asymptotically.

The behavior of z_s in the u channel is complicated by the existence of the point D. In general z_s has a maximum $> +1$ peaked toward the line $s=0$. For u less than that corresponding to the point D, $z_s = +1$ in the backward direction of the u channel, and there is a valley between this and the line $s=0$ on which $z_s = +1$. For $u > u_D$, z_s rises from $+1$ in the forward direction, reaches a maximum, then falls again going to $+1$ at $s=0$. Finally it goes through zero within the backward cone and becomes -1 in the backward direction. Between s_1 and s_4, z_s has opposite signs to the case $t > t_4$. As $s \to \pm\infty$, z_s approaches -1.

C. z_u

Even more briefly, z_u falls from $+1$ to -1 in the physical u channel and approaches $\pm\infty$ as $s \to \mp\infty$. In the s channel z_u has a maximum > 1 peaked toward $u=0$, and takes on the value $+1$ in the forward direction and ± 1 in the backward direction if s is less than or greater than s_F.

In the t channel $z_u = -1$ on the backward and forward directions and has a minimum with value less than -1. Outside the physical regions $|z_u| \to \infty$ along the lines $u = u_i$ with phases determined in the same way as for z_t and z_s. Along constant s lines $z_u \to -1$ as $u \to \pm\infty$, and along constant t lines $z_u \to +1$ as $u \to \pm\infty$.

V. DISCUSSION

If the values of the angle variables $z_{s,t,u}$ are needed, they can be obtained by evaluating Eqs. (2.8), (2.12), and (2.16) anywhere on the Mandelstam plane. However, if only their general behavior is needed, the following prescription may be useful:

(1) Draw the twelve lines given by Eqs. (2.7), (2.11), and (2.15).

(2) Draw the three $z_{s,t,u}=0$ hyperbolas and label the regions where $z_{s,t,u}$ are positive and negative for a positive value of the denominators of Eqs. (2.8), (2.12), and (2.16).

(3) Label the thresholds and the three singular points A through F given by Eqs. (3.4a), (3.4b), and (3.4c). These are at the intersections of the lines $z_s=0$, $s=s_i$; $z_t=0$, $t=t_i$; and $z_u=0$, $u=u_i$.

(4) Sketch the boundaries of the physical regions. These are tangent to the lines of step 1 at the points given by step 3, and approach the axes for large values of s, t, and u.

(5) On these boundaries, $|z_{s,t,u}|=1$. Their signs can be determined by using the labeling found in step 2. In addition, note that $z_t=1$ at $t=0$ and that z_t changes sign on the boundaries at $t=t_i$ (similarly for s and u).

(6) In the three regions between the lines, $t=t_i$ at either constant s or constant u, z_t consists of peaked curves such as those shown in Fig. 4. In the physical regions of the s or u channels, $|z_t|$ has a maximum which is peaked toward the line $t=0$. It has been pointed out by Kibble in Ref. 1 that the loop which is tangent to six of the lines of part 1 with $i=2$ and 3 bounds the phase space for decays such as $M_1 \to M_2+m_1+m_2$ if $M_1 > M_2+m_1+m_2$. If this condition is not satisfied, such a loop still exists and corresponds to decay into three particles, some of which have negative kinetic energies.

There are two important features of four-mass kinematics. The first is the fact that all the angle variables have magnitudes 1 on the boundaries. Second is the fact that $|z_t|$ in the s or u channels has a maximum peaked toward the line $t=0$ (and similarly for z_s and z_u). As was discussed in Sec. IV, this maximum occurs at finite t and therefore becomes arbitrarily close to the forward direction if the energy is much larger than any of the masses. In this case, the asymptotic expansion of the Legendre functions for large arguments is valid except for two small regions near the backward and forward directions. Therefore the same results for differential cross sections are obtained as for identical particle scattering if the Regge representation is used. However, for reactions involving large mass differences and only moderate scattering energies, it presents no difficulty to experimentally break up the range of t values so that a peak in z_t, such as the one shown in Fig. 4, should be observable if it induces a peak in the differential cross section. We have checked ten cases and found that in four of them such a correspondence does exist. A more detailed analysis of one of these (see Ref. 2) which includes kinematical factors, residues, and a variable trajectory gives reasonably good agreement, not only for the position of the peak, but also for the entire differential cross section.