threshold. A term of the form (19) may be thought of as arising from two n-plane poles, one at $n=\alpha$ and another at $n=-\alpha-2$. The symmetry conditions (13) make it plausible that poles do appear in such pairs. Although (19) is better behaved near threshold than (18), it could not be an accurate representation of the amplitude at low energy. Any s-plane analytic structure—resonances, inelastic thresholds, etc.—could arise only from a complicated set of n-plane singularities. In particular, we know that branch points are to be expected in the n plane.

Finally, we suggest that by representing asymptotic terms in the form (19) and by including direct-channel resonances in the fashion described by Barger and Cline,[6] it may be possible to obtain better fits to total cross sections and forward differential cross sections[7] at "intermediate" energies than would be possible with asymptotic terms different from (19). The idea is simply that asymptotic behaviors beneath the leading one, which may be moderately significant at these energies, will be properly handled in this way. The success of this approach depends, however, on the leading branch points in the n plane being sufficiently weak, the absence of other poles, and the effect of double counting, which is intrinsic to this approach, also being negligible. The ultimate test of whether (19)—or suitably modified forms in cases with spin—is better than other asymptotic expressions can really be determined only through a direct confrontation with experimental data.

[6] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966).

[7] As a practical matter it is preferable to consider total cross section differences or forward processes that do not contain vacuum contributions.

PHYSICAL REVIEW VOLUME 163, NUMBER 5 25 NOVEMBER 1967

Vacuum Polarization in Quantum Electrodynamics*

K. JOHNSON
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts
AND
R. WILLEY
University of Pittsburgh, Pittsburgh, Pennsylvania
AND
M. BAKER
University of Washington, Seattle, Washington
(Received 27 April 1967)

It is shown that if the photon propagator is set equal to $1/k^2$ in the expression for the vacuum polarization, then Z_3^{-1} (where Z_3 is the photon wave-function renormalization constant) diverges like a single power of the logarithm of an ultraviolet cutoff in all orders of perturbation theory. The implication of this result upon the possible finiteness of ordinary quantum electrodynamics is discussed.

I. INTRODUCTION AND SUMMARY OF RESULTS

In a previous paper[1] we developed a method to solve the Schwinger-Dyson equation for the unrenormalized electron propagator $S(p)$ in the asymptotic off-mass-shell region $p\gg m$. We found that the solution $1/S(p)=\gamma p+\Sigma(p)$ had the property $\Sigma(p)\to 0$ as $p\gg m$, and from Σ a finite electromagnetic mass for the particle was obtained. The following assumptions were made: (a) the mechanical (bare) mass of the particle was zero; (b)

$$\lim_{k2\to\infty} D(k^2)=\frac{1}{k^2}, \tag{1.1}$$

where $D(k^2)$ is the unrenormalized photon propagator. In this paper we wish to study the second requirement, Eq. (1.1) In order that (1.1) be consistent, the photon wave-function renormalization constant Z_3 calculated with the same hypothesis must be finite. Thus, if (1.1) is valid, then we may put $D(k^2)=1/k^2$ in order to calculate the dominant contribution to Z_3 (that is, the part which diverges in perturbation theory). In this paper we shall show that if $D(k^2)$ is set equal to $1/k^2$ in the calculation of Z_3^{-1}, then

$$\frac{1}{Z_3}=1+f(\alpha_0)\times\int\frac{dp^2}{p^2}+\text{finite part}, \tag{1.2}$$

where $f(\alpha_0)$ is a function of the unrenormalized fine-structure constant $\alpha_0=e_0^2/4\pi$. Thus, we shall prove that if $D(k^2)=1/k^2$, then $1/Z_3$ diverges like a single power

* Supported in part by the U. S. Atomic Energy Commission under Contract Nos. AT(30-1)-2098, AT(45-1)-1388B, and AT-(30-1)-3829.

[1] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136**, B1111 (1964).

of the logarithm of an ultraviolet cutoff in all orders of perturbation theory, just as it does in the lowest order. Consequently, if as $k^2\to\infty$, $D(k^2)\to 1/k^2$, and if there is a nonvanishing value of $\alpha_0>0$ for which $f(\alpha_0)=0$, then for that value of α_0, $1/Z_3$ is finite.[2]

Further, we shall show that when α_0 obeys this "eigenvalue" condition $f(\alpha_0)=0$, the leading correction to (1.1) is given by const.$\times(1/k^2)(m^2/k^2)^{k(\alpha_0)}$, where $k(\alpha_0)$ is also a function of α_0. If the solution to the equation $f(\alpha_0)=0$ has the property that $\alpha_0\ll 1$, then $k(\alpha_0)\cong-\alpha_0^2(d/d\alpha_0)[f(\alpha_0)/\alpha_0]$. Thus, if the additional requirement $k(\alpha_0)>0$ is satisfied, hypothesis (1.1) is self-consistent. In this case, namely, for α_0 such that $f(\alpha_0)=0$ and $k(\alpha_0)>0$, we find that the unrenormalized equations of quantum electrodynamics have completely finite solutions. The divergences would then simply be the result of using perturbation theory to compute renormalization constants, which is unjustified.

We have obtained the following formula for the function $f(\alpha_0)$:

$$f(\alpha_0)=(\alpha_0/2\pi)[\tfrac{2}{3}+g(\alpha_0)], \tag{1.3}$$

where

$$g(\alpha_0)\equiv\frac{f_1(\alpha_0)+2f_2(\alpha_0)+f_2^2(\alpha_0)}{1-f_1(\alpha_0)}+f_3(\alpha_0). \tag{1.4}$$

The functions $f_1(\alpha_0)$, $f_2(\alpha_0)$, and $f_3(\alpha_0)$ are defined in terms of the asymptotic Bethe-Salpeter kernel $\bar{K}^a(p,p'k)=\bar{K}^a(p+\frac{1}{2}k,p-\frac{1}{2}k,p'+\frac{1}{2}k,p'-\frac{1}{2}k)$ for electron-position scattering by the following formulas:

$$f_1(\alpha_0)=\frac{1}{48}\int\frac{d^4p'}{(2\pi)^4}\mathrm{Tr}\frac{(\gamma_\alpha\gamma p'\gamma_\mu-\gamma_\mu\gamma p'\gamma_\alpha)}{2p'^4}\times\bar{K}^a(p',p)(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu), \tag{1.5}$$

$$f_2(\alpha_0)=\frac{1}{48}\int\frac{d^4p'}{(2\pi)^4}\mathrm{Tr}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\bar{K}_\alpha{}^a(p',p)\times(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu), \tag{1.6}$$

$$f_3(\alpha_0)=\frac{1}{48}\int\frac{d^4p'}{(2\pi)^4}\mathrm{Tr}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\bar{K}_{\alpha\alpha}{}^a(p',p)\gamma p\gamma_\mu\gamma p, \tag{1.7}$$

where

$$\bar{K}^a(p',p)\equiv\bar{K}^a(p,p',k)_{k=0}, \tag{1.8}$$

$$\bar{K}_\alpha{}^a(p',p)\equiv\frac{\partial}{\partial k_\alpha}\bar{K}^a(p,p'k)_{k=0}, \tag{1.9}$$

$$\bar{K}_{\alpha\beta}{}^a(p',p)\equiv\frac{\partial^2}{\partial k_\alpha\partial k_\beta}\bar{K}^a(p,p'k)_{k=0}. \tag{1.10}$$

[2] The fact that quantum electrodynamics may be a finite theory only for certain values of the bare charge e_0 was understood by Gell-Mann and Low from the point of view of the renormalization group. See M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

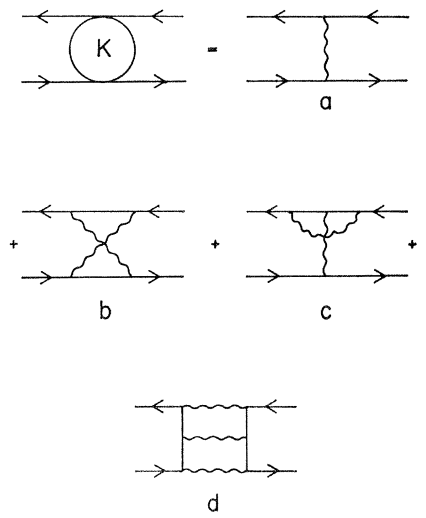


FIG. 1. Some perturbation-theory diagrams for $\bar{K}^a$.

The asymptotic kernel $\bar{K}^a$ is determined from the exact Bethe-Salpeter kernel K by setting all internal electron propagators $S(p)$ equal to $1/\gamma p$ and all internal photon propagators $D_{\mu\nu}$ equal to $D_{\mu\nu}{}^0$, where

$$D_{\mu\nu}{}^0=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}+\frac{bk_\mu k_\nu}{k^4}. \tag{1.11}$$

The gauge constant b is fixed by the condition that the vertex function Γ be finite.

The essential point of the proof of formula (1.2) is the demonstration that the integrals (1.6), (1.7), and (1.8) converge. We have shown this to be true if $\bar{K}^a$ is expanded in a power series in α_0 to any finite order of perturbation theory. Thus the generality of our proof of (1.2) is restricted to a perturbation treatment of $\bar{K}^a$.

The Feynman diagrams representing some of the first terms in the perturbation series for $\bar{K}^a$ are depicted in Fig. 1. Since by definition the kernel K determines the equation for the vertex function Γ_μ [see Eq. (3.1) of Sec. III], it does not contain one-photon annihilation diagrams. This is because such diagrams would give rise to nonproper vertex parts.

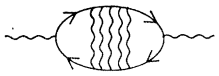
The $\frac{2}{3}$ in the brackets of formula (1.3) yields the weak-coupling limit for Z_3^{-1} (the Landau approximation[3]). Our result shows that if $D(k^2)$ is set equal to $1/k^2$, then no higher powers of $\ln\Lambda$ appear in the exact expression for Z_3^{-1}. Furthermore the coefficient of $\ln\Lambda$ is obtained by replacing $\frac{2}{3}$ by $\frac{2}{3}+g(\alpha_0)$. The self-consistency requirement

$$\tfrac{2}{3}+g(\alpha_0)=0 \tag{1.12}$$

means that the weak-coupling result is just cancelled by all the higher-order effects. We can calculate $g(\alpha_0)$

[3] L. D. Landau, A. A. Abrikosov, and I. M. Halatnikov, Dokl. Akad. Nauk. SSSR **95**, 497 (1954); **95**, 773 (1954); **95**, 1177 (1954).

FIG. 2. Simple ladder diagrams for Z_3^{-1}.



via Eq. (1.4) and formulas (1.5), (1.6), and (1.7) for f_1, f_2, and f_3.

If we use lowest-order perturbation theory for $\bar{K}^a$ [Fig. 1(a)], then the resulting expression (1.4) for $g(\alpha_0)$ represents the sum of the contributions to Z_3^{-1} arising from all uncrossed ladder graphs as depicted in Fig. 2. In this case we find $f_1=\alpha_0/2\pi$, $f_2=f_3=0$ (see Appendix D). Hence in this first approximation,

$$g(\alpha_0)=(\alpha_0/2\pi)/(1-\alpha_0/2\pi)\,, \tag{1.13}$$

and there is no positive α_0 for which $g(\alpha_0)=-\frac{2}{3}$. The values of f_1, f_2, and f_3 corresponding to the fourth-order diagrams for $\bar{K}^a$ [Figs. 1(b) and 1(c)] have been calculated by Rosner.[4] He obtained the results

$$f_1^{(4)}=\left(\frac{\alpha_0}{2\pi}\right)+\left(\frac{\alpha_0}{2\pi}\right)^2\left[-\tfrac{3}{4}\zeta(3)-\tfrac{3}{4}\right], \tag{1.14}$$

$$2f_2^{(4)}=\left(\frac{\alpha_0}{2\pi}\right)^2\left[-\tfrac{5}{2}\zeta(3)+\frac{10}{3}\right], \tag{1.15}$$

$$f_3^{(4)}=\left(\frac{\alpha_0}{2\pi}\right)^2\left[\frac{13}{4}\zeta(3)-\frac{23}{6}\right], \tag{1.16}$$

where

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

If we insert these results into Eq. (1.4) for $g(\alpha_0)$ and expand to order α_0^4, we obtain

$$g^{(4)}(\alpha_0)=\frac{\alpha_0}{2\pi}-\frac{1}{4}\left(\frac{\alpha_0}{2\pi}\right)^2. \tag{1.17}$$

That is, the $\zeta(3)$ terms in the f_i cancel when we calculate $g(\alpha)$ and we obtain a simple negative coefficient $-\frac{1}{4}$ for the $(\alpha_0/2\pi)^2$ contribution to $g(\alpha_0)$. This suggests that the structure of the function $g(\alpha_0)$ is much simpler than the structure of the functions $f_i(\alpha_0)$, and that any partial summation of diagrams such as Eq. (1.13) is likely to give misleading results. In any case the simple nature and negative sign of Rosner's 4th $g(\alpha_0)$ suggest the possibility that $g(\alpha_0)$ may be calculated in closed form and that the eigenvalue Eq. (1.12) may have a root for $\alpha_0>0$.

In the next section we discuss the ideas leading to our basic result, Eq. (1.2), in the context of perturbation theory. This crude discussion is for the most part independent of the detailed derivation which follows in later sections. It is included in the hope of adding clarity to the paper.

[4] J. L. Rosner, Phys. Rev. Letters **17**, 1190 (1966); Ann. Phys. (N. Y.) **44**, 11, (1967).

II. ELEMENTARY DISCUSSION OF RESULT (1.2)

We restrict ourselves to gauges for which the photon propagator $D_{\mu\nu}(k)$ can be written as

$$D_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)D(k^2)+\frac{b}{k^2}\frac{k_\mu k_\nu}{k^2}, \tag{2.1}$$

where b is an arbitrary constant, and the invariant function $D(k^2)$ is determined by the equations

$$D^{-1}(k^2)=k^2[1+\rho(k^2)], \tag{2.2}$$

where $\rho(k^2)$ is obtained from the polarization operator $\Pi_{\mu\nu}(k)$ by the equation

$$\begin{aligned}(k^2g_{\mu\nu}-k_\mu k_\nu)\rho(k^2)&=\Pi_{\mu\nu}(k^2)\\ &\equiv ie_0^2\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\gamma_\mu S(p+\tfrac{1}{2}k)\\ &\quad\times\Gamma_\nu(p+\tfrac{1}{2}k,p-\tfrac{1}{2}k)S(p-\tfrac{1}{2}k).\end{aligned} \tag{2.3}$$

In perturbation theory, superficially, the integrals for $\Pi_{\mu\nu}(k)$ diverge quadratically, but a correct gauge-invariant calculation gives $\Pi_{\mu\nu}(k)$ the tensor structure indicated in Eq. (2.3), and the resultant integrals for $\rho(k^2)$ diverge logarithmically. From Eqs. (2.2) and (2.3) we see that

$$Z_3^{-1}=1+\rho(0)=1+\frac{1}{24}\frac{2}{\partial k_\alpha}\frac{2}{\partial k_\alpha}\Pi_{\mu\nu}(k)\big|_{k=0}. \tag{2.4}$$

We rotate Eq. (2.3) to Euclidean coordinates, writing[5] $\int d^4p\,f(p)=i\int p^3dpd\Omega_p f(p)\equiv i\int p^3dp\,2\pi^2\langle f(p)\rangle_p$. Then, using Eq. (2.4), we obtain the following expression for Z_3^{-1}:

$$Z_3^{-1}=1+\int_0^\infty dp^2p^2\sigma(p^2)\,, \tag{2.5}$$

where

$$\begin{aligned}\sigma(p^2)=\frac{\alpha_0}{96\pi}\mathrm{Tr}\Big\langle&\frac{2}{\partial k_\alpha}\frac{2}{\partial k_\alpha}\gamma_\mu S(p+\tfrac{1}{2}k)\\ &\times\Gamma_\mu(p+\tfrac{1}{2}k,p-\tfrac{1}{2}k)S(p-\tfrac{1}{2}k)\Big\rangle_p\Big|_{k=0}.\end{aligned} \tag{2.6}$$

The full $S(p)$ appearing in Eq. (2.6) differs from $1/\gamma p$ by terms depending upon the electron physical mass m. From dimensional arguments the resulting dependence of $\sigma(p^2)$ upon m can be expressed as $p^2\sigma(p^2)=(1/p^2)$

[5] Since we consider spacelike values of k^2 ($k^2>0$), we can rotate the contour of integration in Eq. (2.3) so that the integral refers to a Euclidean four-vector d^4p and the functions $S(p)$ and $\Gamma_\mu(p+\frac{1}{2}k,p-\frac{1}{2}k)$ refer to external Euclidean variables. See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965), p. 314.

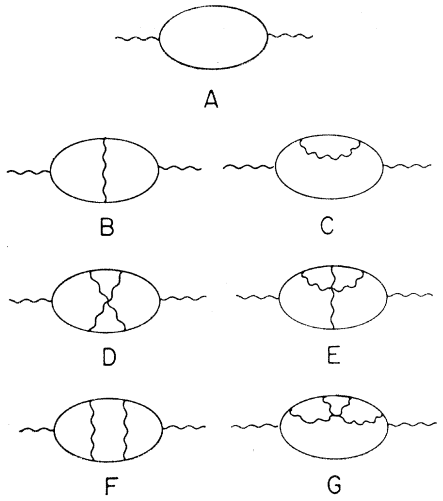


FIG. 3. Some perturbation-theory diagrams for $\Pi_{\mu\nu}(k)$.

$\times f(p^2/m^2)$, where f is a dimensionless function of p^2/m^2. Therefore we can obtain the high-p^2 limit of $p^2\sigma(p^2)$ by letting m approach zero, and result (1.2) follows if expression (2.6) for $\sigma(p^2)$ evaluated at $m=0$ is finite. We must thus show that in the perturbation-theory expansion of $\rho^2\sigma(p^2)$ the $m=0$ integrals are free of both ultraviolet and infrared divergences.

Some typical perturbation-theory graphs for $\Pi_{\mu\nu}(k)$ are depicted in Fig. 3. Since we are calculating $\Pi_{\mu\nu}(k)$ under the assumption $D(k^2)=1/k^2$, the photon propagator appearing in these diagrams is $D_{\mu\nu}^0$, given by Eq. (1.11). $\Pi_{\mu\nu}(k)$ does not depend upon the value of the gauge constant b. We choose b so that Γ is finite. We shall see that this greatly simplifies our work.

The diagram depicted in Fig. 4(A) is not included in $\Pi_{\mu\nu}$ because it represents a photon self-energy correction to the internal photon propagator. If Z_3^{-1} turns out to be finite, then the use of the full propagator for the internal photon line [Fig. 4(B)] will produce a contribution to the asymptotic form for $p^2\sigma(p^2)$ which is proportional to the contribution of diagram 3(B). In this case use of perturbation theory [Fig. 4(A)] would yield a spurious contribution to Z_3^{-1}.

From Eqs. (2.4) and (2.5) we see that in order to calculate Z_3 or $\sigma(p^2)$ all internal lines carrying momentum k must be differentiated twice. Figure 5 shows some of the contributions to Z_3^{-1} arising from differentiating the diagrams of Fig. 3(B). A line with one (two) dashes through it represents a propagator which has been differentiated once (twice).

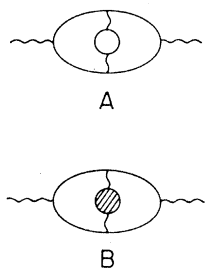


FIG. 4. Photon self-energy corrections to $\Pi_{\mu\nu}(k)$.

Graph A_1 of Fig. 5 gives the usual result of lowest-order perturbation theory, i.e.,

$$p^2\sigma(p^2)=\frac{\alpha_0}{3\pi}\frac{1}{p^2}. \tag{2.7}$$

In order to calculate $p^2\sigma(p^2)$ to order α_0^4, we choose $b=0$ (the Landau gauge). Diagrams B_1 of Fig. 5 yields an integral of the form

$$\frac{1}{p}\left\langle\int d^4p'\frac{1}{(p-p')^2}\frac{1}{p'^3}\right\rangle \tag{2.8}$$

for the quantity $p^2\sigma(p^2)$. Equation (2.8) is a once-differentiated vertex correction. We have kept track only of the correct powers of p and p', since factors,

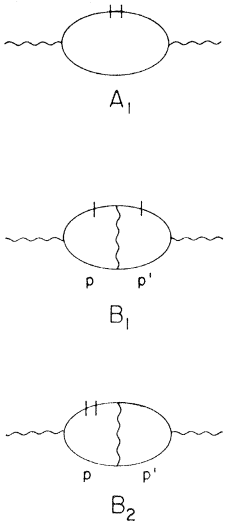


FIG. 5. Some second- and fourth-order diagrams for Z_3^{-1}.

spinor, and vector indices are irrelevant for our considerations. Now we see by inspection that the integral (2.8) contains neither infrared ($p'\to 0$) nor ultraviolet ($p'\to\infty$) divergences. Hence from dimensional arguments it follows that the integral (2.8) is equal to constant/p^2; i.e., graph B_1 yields the result

$$p^2\sigma(p)^2=\text{const.}/p^2, \tag{2.9}$$

as desired.

Diagram B_2 of Fig. 5 yields an integral of the form

$$\frac{1}{p^2}\int\frac{d^4p'}{(p-p')^2}\frac{1}{p'^2} \tag{2.10}$$

for $p^2\sigma(p^2)$. Integral (2.10) is a second-order vertex correction which is ultraviolet finite in the Landau gauge, and by inspection it is infrared finite in any gauge. Hence from dimensional arguments, it follows

that in the Landau gauge the integral (2.10) equals const./p^2; i.e., in the Landau-gauge graph B_2 also yields a result for $p^2\sigma(p^2)$ of the desired form (2.9).

The integral over the internal photon line in graph C of Fig. 3 converges in the Landau gauge with $m=0$. By dimensional arguments it follows that in this case the resulting correction to the internal electron propagator is of the form const./γp. Hence the contribution of graph C is proportional to the contribution of graph A and when differentiated yields a $p^2\sigma(p^2)$ of the desired form (2.9).

Thus in the Landau gauge the contributions of diagrams B_1, B_2, and C separately have the desired form

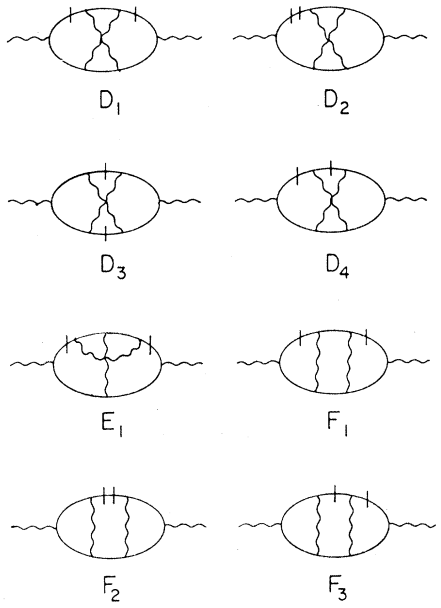


FIG. 6. Some sixth-order diagrams for Z_3^{-1}.

(2.9). In any other gauge the contributions of the diagrams B_2 and C would contain divergences which have to cancel when added because of the gauge independence of Z_3^{-1}.

Likewise, in order to simplify the sixth-order calculation of Z_3^{-1}, we choose the gauge $b=b^{(2)}=\frac{3}{2}\alpha_0/4\pi$, in which $\Gamma_\mu^{(4)}$ is finite. Then the sixth-order contributions to $\Pi_{\mu\nu}(k)$ arise not only from diagrams like D, E, F, and G of Fig. 3 calculated in the Landau gauge, but also from gauge corrections to diagrams B and C calculated with the photon propagator equal to $b^{(2)}k_\mu k_\nu/k^4$. Some of the sixth-order contributions which arise from differentiating diagrams D, E, and F are depicted in Fig. 6. By counting the powers of momentum in the integrands for $\sigma(p^2)$ corresponding to the contributions of diagrams D_1, D_3, D_4, and F_1, we find that these integrals [like integral (2.8) which corresponds to diagram B_1] contain neither ultraviolet nor infrared divergences. Hence by dimensional arguments they also give contributions to $p^2\sigma(p^2)$ of the desired form

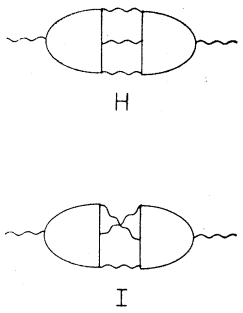


FIG. 7. Some eighth-order 3-photon diagrams for $\Pi_{\mu\nu}(k)$.

(2.9). The same is true for diagrams E_1 and F_2, once we realize that the second-order vertex insertions which they contain are finite in the Landau gauge. Diagram D_2 contains a divergent fourth-order vertex insertion. However, the sum of diagram D_2 and the gauge correction to diagram B_2 of Fig. 5 contains a finite vertex correction. This sum then yields a $p^2\sigma(p^2)$ of the desired form (2.9). Likewise the gauge correction to diagram C of Fig. 3 removes the infinite electron self-energy insertion contained in diagram G of Fig. 3.

Let us summarize this discussion of perturbation theory. In order to calculate $p^2\sigma(p^2)$ to order $2n$ we choose the gauge in which Γ is finite to order $2(n-1)$. In our expression for $p^2\sigma(p^2)$ there will be diagrams like B_1, D_1, and D_3 which do not contain any undifferentiated vertex or self-energy insertions. By a direct power-counting argument, it can be shown that the contributions of such diagrams to $p^2\sigma(p^2)$ do not contain any divergences. Hence they yield a $p^2\sigma(p^2)$ of the desired form (2.9). Dangerous diagrams like B_2 and D_2, which contain undifferentiated vertex insertions, are rendered harmless by the choice of gauge.

However, the above-mentioned power-counting argument is not completely general. Diagrams which con-

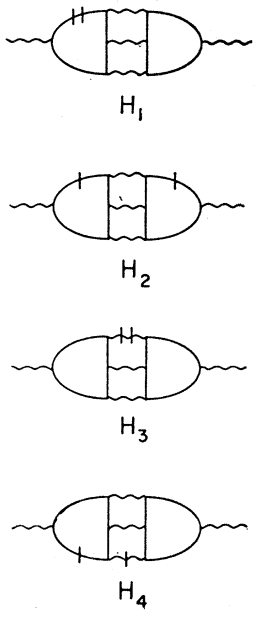


FIG. 8. Some eighth-order 3-photon diagrams for Z_3^{-1}.

tain three-photon intermediate states appear as exceptions (and the only exceptions). However, one can resort directly to gauge invariance in order to show that the contribution of such diagrams to $p^2\sigma(p^2)$ is of the form (2.9). Consider the simplest examples of such a case, diagrams H and I of Fig. 7. Some typical contributions to $p^2\sigma(p^2)$ which arise from differentiating diagram H are depicted in Fig. 8. To these contributions we must add the corresponding contributions obtained by differentiating diagram I. It is easy to show from direct power-counting arguments that diagram H_2 gives a contribution of the form (2.9) to $p^2\sigma(p^2)$. However, such an argument is not applicable to diagrams H_1 and H_3. (The integral over the doubly differentiated photon line in diagram H_3, for example, clearly diverges in the infrared region.) However, diagrams H_1, H_3, and H_4 each contain a photon-photon scattering-amplitude insertion, where one of the photons (the external one) has zero momentum. We know from gauge invariance that such an amplitude must vanish. Hence if the photon-photon scattering amplitude appearing in diagrams H_1, H_3, and H_4 is calculated in a gauge-invariant manner,[6] the contributions from these diagrams must vanish.

We now list the correspondence between the above perturbation-theory diagrams and the functions of f_1, f_2, and f_3 [which describe the exact behavior of $p^2\sigma(p^2)$ for large p^2]. Diagrams like B_1, D_1, and H_2 which contain only undifferentiated Bethe-Salpeter kernels contribute to f_1 [see Eq. (1.5)]. Diagrams like D_4 (D_3), which contain a once (twice) differentiated kernel contribute to f_2 (f_3). Diagrams like F_1 which contain iterations of an undifferentiated kernel give rise to contributions to $g(\alpha_0)$ via the f_1 in the denominator of Eq. (1.4). Because of Ward's identity, diagrams like B_2, which contain undifferentiated external vertex corrections, give contributions to $g(\alpha_0)$ which are exactly cancelled by the contributions of diagrams containing electron self-energy corrections. All electron propagators and the two external vertices thus appear uncorrected in the formulas for f_1, f_2, and f_3. Furthermore, the direct power-counting arguments previously mentioned in the discussion of perturbation-theory diagrams form the basis of the proof that the integrals defining f_1, f_2, and f_3 converge.

Having thus outlined the general nature of our result, we now give a detailed derivation of Eq. (1.2).

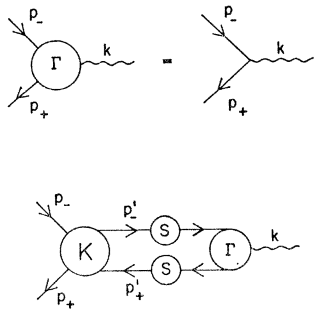


FIG. 9. Graphical representation of the integral equation for Γ_μ.

III. GENERAL EXPRESSION FOR Z_3^{-1}

In order to calculate Z_3^{-1} from Eqs. (2.3) and (2.4) we must know the vertex function $\Gamma_\mu(p+\frac{1}{2}k,p-\frac{1}{2}k)$. This function satisfies the integral equation

$$\Gamma_\mu(p_+,p_-)=\gamma_\mu+\int\frac{d^4p'}{(2\pi)^4}K(p_+,p_-,p_+',p_-')\times S(p_+')\Gamma_\mu(p_+',p_-')S(p_-'), \quad (3.1)$$

where

$$p_+=p+\tfrac{1}{2}k, \quad p_-=p-\tfrac{1}{2}k,$$
$$p_+'=p'+\tfrac{1}{2}k, \quad p_-'=p-\tfrac{1}{2}k.$$

The Bethe-Salpeter kernel K includes all diagrams which contain neither a single electron-positron pair nor a single-photon intermediate state. (See Fig. 1.) Equation (3.1) is depicted graphically in Fig. 9. It is convenient to write Eq. (2.3) for $\Pi_{\mu\nu}(k)$ and Eq. (3.1) for $\Gamma_\mu(p_+,p_-)$ in the following matrix notation:

$$\Pi_{\mu\nu}=-ie_0^2\,\mathrm{Tr}\gamma_\mu G\Gamma_\nu, \quad (2.3')$$

$$\Gamma_\mu=\gamma_\mu+KG\Gamma_\mu, \quad (3.1')$$

where

$$\langle p_+|G\Gamma_\mu|p_-\rangle\equiv S(p_+)\Gamma_\mu(p_+,p_-)S(p_-).$$

We designate the process of differentiation with respect to k_α by the index α, e.g., $\Gamma_{\mu\alpha}\equiv(\partial/\partial k_\alpha)\Gamma_\mu$, $G_\alpha\equiv(\partial/\partial k_\alpha)G$, etc. Differentiation of Eq. (3.1') then yields the following equation for $\Gamma_{\mu\alpha}$:

$$\Gamma_{\mu\alpha}=K_\alpha G\Gamma_\mu+KG_\alpha\Gamma_\mu+KG\Gamma_{\mu\alpha}. \quad (3.2)$$

Differentiating Eq. (2.3') and using Eqs. (3.1') and (3.2), we obtain the following expression for $\Pi_{\mu\nu,\alpha}(k)$:

$$\Pi_{\mu\nu,\alpha}(k)=-ie_0^2\,\mathrm{Tr}\{\Gamma_\mu G_\alpha\Gamma_\nu+\Gamma_\mu GK_\alpha G\Gamma_\nu\}. \quad (3.3)$$

The result (3.3) for $\Pi_{\mu\nu,\alpha}$ is easily understood if we expand $\Pi_{\mu\nu}$ in powers of G and K and then differentiate. $\Gamma_\mu G_\alpha\Gamma_\nu$ includes the contributions of all terms in which G is differentiated and $\Gamma_\mu GK_\alpha G\Gamma_\nu$ includes all terms in which K is differentiated. See Fig. 10 for a graphical representation of Eq. (3.3). One can keep track of the sign of k in the various terms in Eq. (3.3) by referring to Fig. 10. For example, the $\Gamma_\mu G$ which appears on the left in the expression $\Gamma_\mu GK_\alpha G\Gamma_\nu$ means $S(p_-)\Gamma_\mu(p_-,p_+)\times S(p_+)$. The equation for $\Gamma_\mu(p_-,p_+)$ can be written as

$$\Gamma_\mu(p_-,p_+)=\gamma_\mu+\int\frac{d^4p'}{(2\pi)^4}S(p_-')\Gamma_\mu(p_-',p_+')\times S(p_+')K(p_-',p_+',p_-,p_+),$$

[6] For example, use of a correct gauge-invariant current produces a counter term in the photon-photon scattering amplitude which guarantees that it has a gauge-invariant structure.

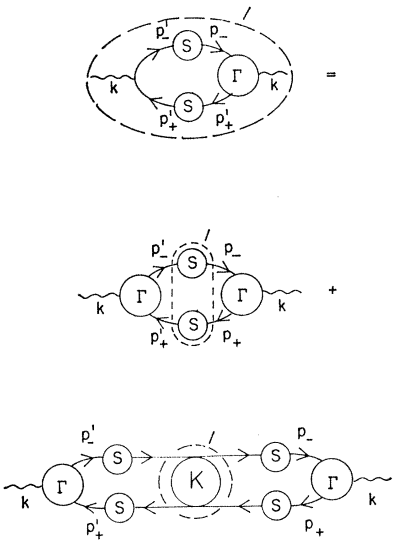


FIG. 10. Graphical representation of Eq. (33) for $\Pi_{\mu\nu,\alpha}$.

or in matrix notation

$$\Gamma_\mu=\gamma_\mu+\Gamma_\mu GK.$$

We now differentiate Eq. (3.3) and obtain the following expression for $\Pi_{\mu\mu\alpha\alpha}=(\partial/\partial k_\alpha)(\partial/\partial k_\alpha)\Pi_{\mu\mu}$:

$$\Pi_{\mu\mu\alpha\alpha}=-ie_0^2\,\mathrm{Tr}\{\Gamma_\mu G_{\alpha\alpha}\Gamma_\mu+\Gamma_\mu GK_{\alpha\alpha}G\Gamma_\mu+2\Gamma_{\mu\alpha}G_\alpha\Gamma_\mu +2\Gamma_\mu GK_\alpha\Gamma_\mu+2\Gamma_\mu GK_\alpha G\Gamma_{\mu\alpha}\}. \quad (3.4)$$

Equation (3.4), evaluated at $k=0$, gives us an exact expression for Z_3^{-1} and $\sigma(p^2)$ as defined in Eq. (2.5). Our problem is to obtain an exact expression for the high-p^2 behavior of $p^2\sigma(p^2)$ under the assumption that $D(k^2)=1/k^2$, i.e., $D_{\mu\nu}(k)=D_{\mu\nu}{}^0(k)$ given by (1.11). The individual terms which appear on the right-hand side of Eq. (3.4) depend upon the choice of the gauge constant b. We have shown in Ref. 1 that if $D=1/k^2$, a value of $b=b_m$ can be found such that Γ_μ is finite in perturbation theory. In order to make our present analysis of all the terms in Eq. (3.4) as complete and explicit as possible, in Sec. IV we shall give an independent proof of this result. In Sec. V we will analyze Eq. (3.4) in the gauge $b=b_m$.

IV. EQUATION FOR $\Gamma_\mu(p,p)$

In order to study the possible infinities in Eq. (3.1), it is sufficient to consider the equation for $\Gamma_\mu(p,p)$. We set $k=0$ in Eq. (3.1) and rotate to Euclidean coordinates, thereby obtaining the equation

$$\Gamma_\mu(p,p)=\gamma_\mu+\int_0^\infty\frac{dp'^2}{p'^2}F_\mu(p,p'), \quad (4.1)$$

where

$$\frac{F_\mu(p,p')}{p'^4}=(i/16\pi^2)\langle K(p,p')S(p')\Gamma_\mu(p',p')S(p')\rangle_{p'}. \quad (4.2)$$

Now in order that Γ_μ be finite it is necessary that $F_\mu\to 0$ as $p'\to\infty$. We have calculated $F_\mu(p,p')$ directly in lowest-order perturbation theory, where $S(p)=1/\gamma p$, $\Gamma_\mu=\gamma_\mu$, $K=-ie_0^2\gamma_\mu D_{\mu\nu}{}^0\gamma_\mu$, and find that

$$\lim_{p'\to\infty}F_\mu{}^{(2)}(p,p')=b\frac{\alpha_0}{4\pi}\gamma_\mu+O\left(\frac{p^2}{p'^2}\right). \quad (4.3)$$

From Eq. (4.3) we obtain the well-known result that the second-order vertex $\Gamma^{(2)}$ is finite in the Landau gauge ($b=0$). With $b=0$, Eqs. (4.1) and (4.2) yield the result

$$\Gamma_\mu{}^{(2)}(p,p)=(1+3\alpha_0/8\pi)\gamma_\mu. \quad (4.4)$$

We now want to show that if photon self-energy corrections are neglected (i.e., if $D=1/k^2$), then one can find a value of b for which

$$\lim_{p'\to\infty}F_\mu(p,p')=0.$$

We prove this by induction. We already know that in the gauge $b=b_0=0$,

$$\lim_{p'\to\infty}F_\mu{}^{(2)}(p,p')=O(p^2/p'^2)$$

and $\Gamma_\mu{}^{(2)}$ is finite. Let us assume that constants $a_0=0$, $a_1,\cdots,a_{n-1}$ can be found so that in the gauge $b_{n-1}\equiv a_0+a_1\alpha_0+\cdots+a_{n-1}\alpha_0{}^{n-1}$, $\Gamma^{(2n)}$ is finite. We will now show that one can find a constant a_n such that in the gauge $b_n\equiv b_{n-1}+a_n\alpha_0{}^n$,

$$\lim_{p'\to\infty}F_\mu{}^{[2(n+1)]}(p,p')\to 0.$$

$K^{[2(n+1)]}$, $\Gamma_\mu{}^{[2(n+1)]}$, and $F^{[2(n+1)]}$ depend upon the electron mass via their dependence upon the electron propagator $S^{(2n)}(p)$. The mass-dependent terms in $S^{(2n)}(p)$ must of course be calculated according to the scheme of Ref. 1. However, in order to calculate the high p' limit of $F_\mu{}^{[2(n+1)]}(p,p')$, one can neglect dependence upon electron mass. Let the values of $K(p_+,p_-,p_+',p_-')$, $\Gamma_\mu(p_+,p_-)$, and $F_\mu(p,p')$ obtained by setting the electron mass $m=0$ be written $K^a(p_+,p_-,p_+',p_-')$, $\Gamma_\mu{}^a(p_+,p_-)$, and $F_\mu{}^a(p',p)$. F^a then assumes the form

$$F_\mu{}^a(p,p')=\gamma_\mu F_1\left(\frac{p}{p'}\right)+\frac{p_\mu\gamma_p}{p^2}F_2\left(\frac{p}{p'}\right), \quad (4.5)$$

where F_1 and F_2 are dimensionless functions of the ratio p/p'. We can therefore determine the behavior of $F_\mu{}^a(p,p')$ for $p'\to\infty$ from its behavior for $p\to 0$. From Eq. (4.2) we see that the dependence of F_μ upon p arises solely from $K^a(p,p')$.

In Appendix A we study the infrared region of the perturbation-theory integrals and show that $K^{a[2(n+1)]}$ is finite at $p=0$. Hence

$$\lim_{p'\to\infty}F_\mu{}^{[2(n+1)]}(p,p')=F_\mu{}^{a[2(n+1)]}(0,p')\equiv F^{[2(n+1)]}\gamma_\mu, \quad (4.6)$$

where $F^{[2(n+1)]}$ is a finite constant. From Eq. (4.3) we see that the gauge term $a_n\alpha_0^n$ gives a contribution $(\alpha_0/4\pi)a_n\alpha_0^n$ to the constant $F^{[2(n+1)]}$.

We can thus choose a_n so that

$$F^{[2(n+1)]}=0. \tag{4.7}$$

[If $F_1(p/p')$ had contained terms of the form $[\ln(p/p')]^k$, it would clearly have been impossible to remove them with an appropriate choice of the constant a_n. The absence of such terms is guaranteed by the finiteness of $K^a(0,p')$.]

With a_n determined from Eq. (4.7), $\Gamma_\mu^{[2(n+1)]}(p,p)$ is finite and takes on the asymptotic form

$$\lim_{p\to\infty}\Gamma_\mu^{[2(n+1)]}(p,p)=\frac{1}{C^{[2(n+1)]}}\gamma_\mu + \frac{1}{(C')^{[2(n+1)]}}p_\mu\frac{\gamma p}{p^2}, \tag{4.8}$$

where the finite constants $C^{[2(n+1)]}$ and $C'^{[2(n+1)]}$ are determined by setting $m=0$ in the expression for $\Gamma_\mu^{[2(n+1)]}(p,p)$. At large p the corresponding electron propagator $S^{[2(n+1)]}(p)$ behaves like

$$\lim_{p\to\infty}S^{[2(n+1)]}(p)=\frac{(C'')^{[2(n+1)]}}{\gamma p}. \tag{4.9}$$

From Ward's identity $\partial S^{-1}/\partial p_\mu=\Gamma_\mu(p,p')$, it follows that

$$(C'')^{[2(n+1)]}=C^{[2(n+1)]}, \tag{4.10a}$$

$$\frac{1}{(C')^{[2(n+1)]}}=0. \tag{4.10b}$$

[Actually the value of the constant $(C'')^{[2(n+1)]}$ depends upon the choice of the variables of integration used in the perturbation-theory evaluation of $S^{[2(n+1)]}(p)$] For this reason, requirement (4.10a) just serves to remove this ambiguity in $(C'')^{[2(n+1)]}$.

Thus we see that in an appropriately chosen gauge

$$\lim_{p\to\infty,\,k\text{ fixed}}\Gamma_\mu(p_+,p_-)=\lim_{p\to\infty}\Gamma_\mu(p_+,p_-)=\frac{1}{C}\gamma_\mu, \tag{4.11}$$

$$\lim_{p\to\infty}S(p)=C\frac{1}{\gamma p}, \tag{4.12}$$

where C is a finite constant.

It is now convenient to rescale Eq. (3.1) for $\Gamma_\mu(p_+,p_-)$. We define

$$\bar{S}(\gamma,p)=(1/C)S(\gamma,p), \tag{4.13}$$

$$\bar{\Gamma}_\mu(p_+,p_-)=C\Gamma_\mu(p_+,p_-). \tag{4.14}$$

Then

$$\lim_{p\to\infty}\bar{S}(\gamma,p)=\frac{1}{\gamma p}, \tag{4.15}$$

$$\lim_{p\to\infty}\bar{\Gamma}_\mu(p_+,p_-)=\gamma_\mu, \tag{4.16}$$

and Eq. (3.1′) for $\bar{\Gamma}_\mu(p_+,p_-)=\bar{\Gamma}_\mu$ becomes

$$\bar{\Gamma}_\mu=C\gamma_\mu+\bar{K}\bar{S}\bar{\Gamma}_\mu\bar{S}, \tag{4.17}$$

where

$$\bar{K}\equiv C^2K. \tag{4.18}$$

Now

$$\bar{K}\equiv C^2K(S,\Gamma)=K((\bar{S},\bar{\Gamma})), \tag{4.19}$$

i.e., K is the same functional of $\bar{S}$ and $\bar{\Gamma}$ as K is of S and Γ.

Condition (4.7) determines a_n, and hence the gauge constant b is conveniently expressed in terms of

$$\bar{K}^a=C^2K^a=K\left(\frac{1}{\gamma p},\bar{\Gamma}^a\right) \tag{4.20}$$

as follows:

$$\left\langle\bar{K}^{a[2(n+1)]}(0,p')\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\right\rangle_{p'}=0. \tag{4.21}$$

Also the constant C, appearing in Eq. (4.17) for $\bar{\Gamma}_\mu$, is conveniently determined by setting $k=0$ in Eq. (4.17) and taking the limit $p\to\infty$. This yields the equation

$$(1-C)\gamma_\mu=\int\frac{d^4p'}{(2\pi)^4}\bar{K}^a(p,p')\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}. \tag{4.22}$$

In calculating the constant C to order $\alpha_0^{[2(n+1)]}$ from Eq. (4.22), we need know only $\bar{\Gamma}_\mu^{a(2n)}$. The latter is determined from Eq. (4.17) in terms of $C^{(2n)}$. [From Eq. (4.4) we see that $C^{(2)}=1-3\alpha_0/8\pi$.]

We repeat that the results of this section depend upon the fact that $K^{a[2(n+1)]}(0,p')$ is finite provided $\Gamma_\mu^{(2n)}$ is finite. The proof of this fact involves the study of the infrared behavior of the perturbation-theory integrals for $K^{a[2(n+1)]}(0,p')$. (See Appendix A.)

V. BASIC RESULT FOR Z_3^{-1}

We shall now evaluate Z_3^{-1} using Eqs. (3.4) and (2.6), and assuming that $D=1/k^2$. The sum of all the terms on the right-hand side of Eq. (3.4) is gauge-invariant, and for convenience we choose to evaluate Eq. (3.4) in the gauge for which Γ_μ is finite.

We begin by re-expressing $\Pi_{\mu\mu;\alpha\alpha}(0)$ in terms of $\bar{S}$, $\bar{\Gamma}$, and $\bar{K}$.

$$\Pi_{\mu\mu;\alpha\alpha}(0)=-ie_0^2\,\mathrm{Tr}\{\bar{\Gamma}_\mu\bar{G}_{\alpha\alpha}\bar{\Gamma}_\mu+\bar{\Gamma}_\mu\bar{G}\bar{K}_{\alpha\alpha}\bar{G}\bar{\Gamma}_\mu+2\bar{\Gamma}_{\mu\alpha}\bar{G}_\alpha\bar{\Gamma}_\mu +2\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha\bar{G}_\alpha\bar{\Gamma}_\mu+2\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha\bar{G}\bar{\Gamma}_{\mu\alpha}\}_{k=0}, \tag{5.1}$$

where $\bar{G}=G/C^2$.

The term $-ie_0^2\,\mathrm{Tr}\{\bar{\Gamma}_\mu\bar{G}_{\alpha\alpha}\bar{\Gamma}_\mu\}_{k=0}$ yields the contribution $(1/p^2)(\alpha_0/2\pi)f^{(0)}(p^2/m^2)$ to $p^2\sigma(p^2)$, where $f^{(0)}(p^2/m^2)$ is the following function of p^2/m^2 (the only available dimensionless parameter):

$$f^{(0)}\left(\frac{p^2}{m^2}\right)=\frac{p^4}{48}\mathrm{Tr}\Big\langle\bar{\Gamma}_\mu(p,p)\frac{2}{\partial k_\alpha}\frac{2}{\partial k_\alpha}\times\bar{S}\left(p+\frac{k}{2}\right)\bar{\Gamma}_\mu(p,p)\bar{S}\left(p-\frac{k}{2}\right)\Big\rangle_p|_{k=0}. \quad (5.2)$$

Now in the gauge we have chosen, we know [Eqs. (4.15) and (4.16)] that

$$\lim_{p\to\infty}\bar{\Gamma}_\mu(p,p)=\gamma_{\mu'},\quad \lim_{p\to\infty}\bar{S}(\gamma,p)=\frac{1}{\gamma p}.$$

Hence in the large-p limit, Eq. (5.2) coincides with lowest-order perturbation theory, i.e.,

$$\lim_{p\to\infty}f^{(0)}\left(\frac{p^2}{m^2}\right)=\tfrac{2}{3}. \quad (5.3)$$

The second term, $-ie_0^2\,\mathrm{Tr}(\bar{\Gamma}_\mu\bar{G}\bar{K}_{\alpha\alpha}\bar{G}\bar{\Gamma}_\mu)_{k=0}$, yields the contribution $(1/p^2)(\alpha_0/2\pi)f^{(3)}(p^2/m^2)$ to $p^2\sigma(p^2)$, where

$$f^{(3)}\left(\frac{p^2}{m^2}\right)=\frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\bar{S}(p')\bar{\Gamma}_\mu(p',p')\times\bar{S}(p')\bar{K}_{\alpha\alpha}{}^a(p',p)\bar{S}(p)\bar{\Gamma}_\mu(p,p)S(p)|_{k=0}. \quad (5.4)$$

Hence

$$\lim_{p2\to\infty}f^{(3)}\left(\frac{p^2}{m^2}\right)=\lim_{m\to0}f^{(3)}\left(\frac{p^2}{m^2}\right)=\frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\bar{K}_{\alpha\alpha}{}^a(p',p)\frac{1}{\gamma p}\gamma_\mu\frac{1}{\gamma p}\equiv f_3, \quad (5.5)$$

provided that integral (5.5) converges. In Appendix B we show that integral (5.5), defining the constant f_3, converges if $\bar{K}^a$ is expanded in a power series in α_0 to any finite order of perturbation theory. This result is derived from the same infrared properties of perturbation theory as are used in order to show that $\bar{K}^a(p,p')$ is finite at $p=0$. Hence the limit (5.5) exists, and the second term on the right-hand side of Eq. (5.1) gives a contribution $(1/p^2)(\alpha_0/2\pi)f_3$ to the high-p^2 behavior of $p^2\sigma(p^2)$.

In order to evaluate the contribution of the third and fifth terms of Eq. (5.1) to $\lim_{p^2\to\infty}p^2\sigma(p^2)$, we must calculate

$$\bar{\Gamma}_{\mu\alpha}(p,p)\equiv\frac{2}{\partial k_\alpha}\bar{\Gamma}_\mu(p_-,p_+)|_{k=0}$$

in the limit $m\to0$. The equation for $\bar{\Gamma}_{\mu\alpha}(p,p)$ is given by

$$\bar{\Gamma}_{\mu\alpha}(p,p)=\{\bar{\Gamma}_\mu\bar{G}_\alpha\bar{K}+\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha+\bar{\Gamma}_{\mu\alpha}\bar{G}\bar{K}\}_{k=0}. \quad (5.6)$$

If $\lim_{m\to0}\bar{\Gamma}_{\mu\alpha}(p,p)$ exists, then

$$\lim_{m\to0}\bar{\Gamma}_{\mu\alpha}(p,p)=\Gamma'\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{2p^2}, \quad (5.7)$$

where Γ' is a finite constant. The spinor structure of $\Gamma_{\mu\alpha}(p,p)$, given by Eq. (5.7), is an immediate consequence of PT invariance, as shown in Appendix C.

We will now show that the above limit (5.7) exists, and we will determine the constant Γ'. In the $m=0$ limit the inhomogeneous terms $\bar{\Gamma}_\mu\bar{G}_\alpha\bar{K}|_{k=0}$ and $\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha|_{k=0}$ in Eq. (5.6) take on the forms

$$\lim_{m\to0}(\bar{\Gamma}_\mu\bar{G}_\alpha\bar{K})_{k=0}=\int\frac{d^4p'}{(2\pi)^4}\frac{\gamma_\alpha\gamma p'\gamma_\mu-\gamma_\mu\gamma p'\gamma_\alpha}{2p'^4}\bar{K}^a(p',p)\equiv f_1(\alpha_0)\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{2p^2}, \quad (5.8)$$

$$\lim_{m\to0}(\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha)_{k=0}=\int\frac{d^4p'}{(2\pi)^4}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\bar{K}_\alpha{}^a(p',p)\equiv f_2(\alpha_0)\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{2p^2}. \quad (5.9)$$

In integrals (5.8) and (5.9) we used the definition

$$\bar{\Gamma}_\mu G_a|_{k=0}\equiv\frac{2}{\partial k_\alpha}\bar{S}(p-\tfrac{1}{2}k)\Gamma_\mu(p,p)S(p+\tfrac{1}{2}k)|_{k=0},$$

and the limits (4.15) and (4.16) to obtain

$$\lim_{p\to\infty\text{ or }m\to0}(\bar{\Gamma}_\mu\bar{G}_\alpha)_{k=0}=\frac{\gamma_\alpha\gamma p\gamma_\mu-\gamma_\mu\gamma p\gamma_\alpha}{2p^4}. \quad (5.10)$$

In Appendix B we show that limits (5.8) and (5.9) exist for $\bar{K}^a$ expanded to any finite order of perturbation theory. Hence they define finite constants $f_1(\alpha_0)$ and $f_2(\alpha_0)$. [These results follow almost as a direct consequence of the convergence of integral (5.5) for $f_3(\alpha_0)$.] Assuming that limit (5.7) exists, and using Eq. (5.8), we find that the $m=0$ limit of the homogeneous term $(\bar{\Gamma}_{\mu\alpha}\bar{G}\bar{K})_{k=0}$ in Eq. (5.6) is

$$\lim_{m\to0}(\bar{\Gamma}_{\mu\alpha}\bar{G}\bar{K})_{k=0}=\Gamma'\int\frac{d^4p'}{(2\pi)^4}\frac{1}{\gamma p'}\times\frac{(\gamma_\mu\gamma p'\gamma_\alpha-\gamma_\alpha\gamma p'\gamma_\alpha)}{2p'^2}\frac{1}{\gamma p'}\bar{K}^a(p'p)=\Gamma' f_1(\alpha_0)\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{2p^2}. \quad (5.11)$$

From Eqs. (5.6)–(5.9) and (5.11) we obtain for Γ' the equation

$$\Gamma' = f_1 + f_2 + \Gamma' f_1, \tag{5.12}$$

and hence

$$\Gamma' = (f_1+f_2)/1-f_1. \tag{5.13}$$

Thus in the limit $p\to\infty$, or equivalently $m\to 0$, Eq. (5.6) for $\bar{\Gamma}_{\mu\alpha}(p,p)$ has the finite solution (5.7) with Γ' given by Eq. (5.13).

We can now express the contribution of the last three terms in Eq. (5.1) to $\lim_{p^2\to\infty} p^2\sigma(p^2)$ in terms of the constants $f_1(\alpha_0)$ and $f_2(\alpha_0)$. The term $-ie_0^2\,\mathrm{Tr}2\times(\bar{\Gamma}_\mu \bar{G}_\alpha \bar{\Gamma}_\mu)_{k=0}$ yields the contribution $(1/p^2)(\alpha_0/2\pi)\times f^{(4)}(p^2/m^2)$ to $p^2\sigma(p^2)$, where $f^{(4)}(p^2/m^2)$ is the following function of the ratio p^2/m^2:

$$f^{(4)}(p^2/m^2)\equiv \frac{p^4}{48}\mathrm{Tr}\bar{\Gamma}_{\mu\alpha}(p,p)\frac{2}{\partial k_\alpha}\bar{S}(p+\tfrac{1}{2}k) \times\bar{\Gamma}_\mu(p,p)\bar{S}(p-\tfrac{1}{2}k). \tag{5.14}$$

Hence

$$\lim_{p^2\to\infty} f^{(4)}(p^2/m^2) = \lim_{m\to 0} f^{(4)}(p^2/m^2) = \frac{p^4\Gamma'}{48}\mathrm{Tr}\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma)}{2p^2} \times\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{2p^4}=\Gamma'. \tag{5.15}$$

Similarly, the contribution of the term

$$-ie^2\,\mathrm{Tr}2(\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha\bar{G}_\alpha\bar{\Gamma}_\mu)_{k=0}$$

to $p^2\sigma(p^2)$ can be written as $(1/p^2)(\alpha_0/2\pi)f^{(2)}(p^2/m^2)$, where

$$f^{(2)}(p^2/m^2)\equiv\frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\bar{S}(p')\bar{\Gamma}_\mu(p',p')\bar{S}(p')\bar{K}_\alpha(p',p) \times\frac{2}{\partial k_\alpha}\bar{S}(p+\tfrac{1}{2}k)\bar{\Gamma}_\mu(p,p)\bar{S}(p-\tfrac{1}{2}k)\big|_{k=0}. \tag{5.16}$$

Hence[7]

$$\lim_{p^2\to\infty} f^{(4)}(p^2/m^2) = \lim_{m\to 0} f^{(4)}(p^2/m^2) = \frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}\bar{K}_\alpha{}^a(p',p) \times\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{2p^4}\right) \tag{5.17}$$

$$= p^4 f_2(\alpha_0)\frac{\mathrm{Tr}}{48}\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{2p^2}\right) \times\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{2p^4}\right)=f_2(\alpha_0). \tag{5.18}$$

[7] When $\Gamma_\mu\bar{G}$ appears on the left, the sign of k is reversed compared to the case in which $\bar{G}\bar{\Gamma}_\mu$ appears on the right. Hence $(\bar{\Gamma}_\mu\bar{G}_\alpha)_{m=0}=-\bar{G}_\alpha\bar{\Gamma}_\mu$. This accounts for the reversed positions of the indices μ and α in Eqs. (5.10) and (5.17).

Finally, the contribution of $-ie_0^2\,\mathrm{Tr}2\bar{\Gamma}_\mu\bar{G}\bar{K}_\alpha\bar{G}\bar{\Gamma}_{\mu\alpha}|_{k=0}$ to $p^2\sigma(p^2)$ is $(1/p^2)(\alpha_0/2\pi)f^{(5)}(p^2/m^2)$, where

$$f^{(5)}(p^2/m^2)=\frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\bar{S}(p')\bar{\Gamma}_\mu(p',p')\bar{S}(p') \times\bar{K}_\alpha(p',p)\bar{S}(p)\bar{\Gamma}_{\mu\alpha}(p,p)\bar{S}(p). \tag{5.19}$$

Hence[8]

$$\lim_{p^2\to\infty} f^{(5)}\left(\frac{p^2}{m^2}\right)=\lim_{m\to 0} f^{(5)}\left(\frac{p^2}{m^2}\right) =\frac{p^4}{48}\mathrm{Tr}\int\frac{d^4p'}{(2\pi)^4}\frac{1}{\gamma p'}\gamma_\mu\frac{1}{\gamma p'}K_\alpha{}^a(p',p)\frac{1}{\gamma p}\Gamma' \times\frac{(\gamma_\alpha\gamma p\gamma_\mu-\gamma_\mu\gamma p\gamma_\alpha)}{2p^2}\frac{1}{\gamma p}$$

$$=\frac{p^4}{48}\mathrm{Tr}f_2(\alpha_0)\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{2p^2}\right)\Gamma' \times\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{2p^4}\right)=f_2(\alpha_0)\Gamma'. \tag{5.20}$$

Hence from (5.3), (5.5), (5.15), (5.18), and (5.20) we obtain

$$\lim_{p^2\to\infty} p^2\sigma(p^2)=\frac{1}{p^2}\frac{\alpha_0}{2\pi}[\tfrac{2}{3}+f_3(\alpha_0)+f_2(\alpha_0) +\Gamma'(1+f_2(\alpha_0))]. \tag{5.21}$$

Using Eq. (5.13) for Γ' we obtain our desired result

$$\lim_{p^2\to\infty} p^2\sigma(p^2)=\frac{1}{p^2}\frac{\alpha_0}{2\pi}[\tfrac{2}{3}+g(\alpha_0)], \tag{5.22}$$

where $g(\alpha_0)$ is defined by Eq. (1.5). Formulas (1.6), (1.7), and (1.8) for f_1, f_2, and f_3 follow immediately from Eqs. (5.5), (5.8), and (5.9).

We repeat that our result (5.22) depends upon the analysis of the perturbation expansion of K^a. This is carried out in Appendices A and B. We use the fact that K^a contains no photon self-energy insertions in order to obtain the results in these Appendices, and hence the assumption $D=1/k^2$ is essential for obtaining Eq. (5.22).

VI. DISCUSSION OF ASYMPTOTIC CORRECTIONS TO $D(k^2)$

In this section we will discuss the equation for the finite part of $D(k^2)$, supposing that there exists a solution of the self-consistency requirement, Eq. (1.12).

[8] We remember that the $\Gamma_{\mu\alpha}$, which appears on the right in Eq. (5.19), equals $(\partial/\partial k_\alpha)\Gamma_\mu(p_+,p_-)_{k=0}$. Hence its value for $m=0$ is opposite to that given by Eq. (5.7) for $(\partial/\partial k_\alpha)\Gamma_\mu \times(p_-,p_+)|_{k=0,m=0}$. (See Fig. 10.)

We first note that we can obtain the high-k^2 limit of $\rho(k^2)$ from our result, Eq. (1.3). Using Eq. (2.3) we express $\rho(k^2)$ as an integral of the form

$$\rho(k^2)=\int_0^\infty dp^2 p^2 \sigma(p^2,k^2), \tag{6.1}$$

where $\sigma(p^2,k^2)$ is defined in terms of S and Γ from Eq. (2.3). In perturbation theory the dominant contribution to $\rho(k^2)$ for large k^2 arises from that region of integration in (6.1) for which $p^2 \gg k^2$. Thus, since

$$\lim_{p^2\to\infty,k^2 \text{ fixed}} \sigma(p^2,k^2)=\lim_{p^2\to\infty}\sigma(p^2,0)=f(\alpha_0)/p^4, \tag{6.2}$$

it follows that

$$\lim_{k^2\to\infty}\rho(k^2)=f(\alpha_0)\int_{k^2}^\infty \frac{dp^2}{p^2}+\text{const.}+O\left(\frac{m^2}{k^2}\right). \tag{6.3}$$

The region of integration where $k^2 \sim p^2$ in Eq. (6.1) gives rise to the constant contribution to $\rho(k^2)$ in Eq. (6.3). From Eq. (2.2) we see that for $f(\alpha_0)=0$, the constant term in Eq. (6.3) produces a $D(k^2)$ which behaves like const./k^2 for large k^2. For this reason let us formulate our consistency requirement to allow for this contingency. We first note that the Schwinger-Dyson equations of the theory always involve $D(k^2)$ and α_0 in the combination $\alpha_0 D(k^2)$, aside from the equation which relates D to ρ, namely,

$$\frac{1}{\alpha_0 D(k^2)}=k^2\left(\frac{1}{\alpha_0}+\frac{1}{\alpha_0}\rho(k^2)\right). \tag{6.4}$$

Thus, $(1/\alpha_0)\rho(k^2)$ involves α_0 only in the combination $\alpha_0 D$, that is, we may write

$$(1/\alpha_0)\rho(k^2)\equiv\rho(k^2;\alpha_0 D(q^2)), \tag{6.5}$$

where in $\rho((k^2;\alpha_0 D(q^2))$ we exhibit a functional dependence on $D(q^2)$. In this case (6.4) becomes

$$\frac{1}{\alpha_0 D(k^2)}=k^2\left[\frac{1}{\alpha_0}+\rho(k^2;\alpha_0 D(q^2))\right]. \tag{6.6}$$

We have shown that with $D(q^2)=1/q^2$ and $k^2 \gg m^2$,

$$\rho(k^2)\to f(\alpha_0)\int_{k^2}^\infty \frac{dp^2}{p^2}+C+O\left(\frac{m^2}{k^2}\right), \tag{6.7}$$

or more precisely, with a cutoff $\Lambda^2 \gg k^2 \gg m^2$,

$$\rho(k^2)\to f(\alpha_0)\int_{k^2}^{\Lambda^2} \frac{dp^2}{p^2}+C+O\left(\frac{m^2}{k^2}\right). \tag{6.8}$$

In terms of the functional ρ defined above,

$$\rho(k^2;\alpha_0/q^2)\to\frac{f(\alpha_0)}{\alpha_0}\int_{k^2}^{\Lambda^2}\frac{dp^2}{p^2}+\frac{C}{\alpha_0}+O\left(\frac{m^2}{k^2}\right). \tag{6.9}$$

Consequently, we see that even if $f(\alpha_0)=0$,

$$\frac{1}{\alpha_0 D(k^2)}\to k^2\left(\frac{1}{\alpha_0}+\frac{C}{\alpha_0}\right), \tag{6.10}$$

so $k^2 D(k^2)\not\to 1$.

To take the above into account let us suppose that as $k^2\to\infty$,

$$\frac{1}{\alpha_0 D(k^2)}\to\frac{k^2}{\bar\alpha_0}, \tag{6.11}$$

where $\bar\alpha_0$ is a finite constant to be determined by the requirement of self-consistency. Replacing α_0 by $\bar\alpha_0$ in Eq. (6.9), we see that when $\Lambda^2 \gg k^2 \gg m^2$,

$$\rho(k^2;\bar\alpha_0/q^2)\to\frac{f(\bar\alpha_0)}{\bar\alpha_0}\int_{k^2}^{\Lambda^2}\frac{dp^2}{p^2}+\frac{\bar C}{\bar\alpha_0}+O\left(\frac{m^2}{k^2}\right), \tag{6.12}$$

where C is the same function of k^2/α_0 as C is of α_0. If we then take $\bar\alpha_0$ to satisfy our eigenvalue equation

$$f(\bar\alpha_0)=0, \tag{6.13}$$

we find that

$$\frac{1}{\alpha_0 D(k^2)}\to k^2\left(\frac{1}{\alpha_0}+\frac{\bar C}{\alpha_0}\right). \tag{6.14}$$

So with

$$\frac{1}{\alpha_0}=\frac{1}{\bar\alpha_0}-\frac{\bar C}{\bar\alpha_0}, \tag{6.15}$$

the assumed asymptotic behavior (6.11) is self-consistent. Indeed, let us define

$$\rho_R(k^2;\bar\alpha_0\bar D)=\rho(k^2;\alpha_0 D)-\frac{\bar C}{\bar\alpha_0}, \tag{6.16}$$

where $\bar\alpha_0\bar D\equiv\alpha_0 D$ defines $\bar D$ so that

$$\frac{1}{\alpha_0 D}=\frac{1}{\bar\alpha_0\bar D}=k^2\left(\frac{1}{\bar\alpha_0}+\rho_R(k^2;\bar\alpha_0\bar D)\right) \tag{6.17}$$

relates D to ρ_R. Then, as $k^2\to\infty$,

$$\rho_R(k^2;\bar\alpha_0/q^2)=\frac{f(\bar\alpha_0)}{\bar\alpha_0}\int_{k^2}^{\Lambda^2}\frac{dp^2}{p^2}+O\left(\frac{m^2}{k^2}\right). \tag{6.18}$$

So when $f(\bar\alpha_0)=0$,

$$\rho_R(k^2;\bar\alpha_0/q^2)\to O(m^2/k^2)$$

for $k^2 \gg m^2$. We now see that α_0 no longer plays any role in our equations since (6.17) involves only the parameter $\bar\alpha_0$ and the boundary condition that $\rho_R\to 0$ as $k^2\to\infty$.

With these refinements, we may turn to the calculation of the leading corrections to the asymptotic behavior of $\bar{D}$:

$$\bar{D}(k^2)\to 1/k^2 \quad \text{as} \quad k^2\gg m^2. \tag{6.19}$$

We have shown that $\rho_R(k^2;\bar{\alpha}_0/q^2)\sim O(m^2/k^2)$. However, $\rho_R(k^2;\bar{\alpha}_0\bar{D})$ may vanish much more slowly than m^2/k^2, because of the corrections to the asymptotic behavior of $\bar{D}$ given by ρ_R. If we assume that $\rho_R(k^2;\bar{\alpha}_0\bar{D})\to 0$ as $k^2\gg m^2$, then we may write

$$\bar{D}(k^2)\to (1/k^2)[1-\bar{\alpha}_0\rho_R(k^2;\bar{\alpha}_0\bar{D})] \tag{6.20}$$

when $k^2\gg m^2$. Now, we assume that to compute the leading term in $\rho_R(k^2;\bar{\alpha}_0\bar{D})$ for $k^2\gg m^2$, it is sufficient to use (6.20) in the functional dependence of ρ_R on $\bar{D}$[9]; that is, for $k^2\gg m^2$,

$$\rho_R(k^2;\bar{\alpha}_0\bar{D})\simeq\rho_R(k^2;(\bar{\alpha}_0/q^2)[1-\bar{\alpha}_0\rho_R(q^2;\bar{\alpha}_0\bar{D})]). \tag{6.21}$$

We will expand the functional to first order in ρ_R so that

$$\rho_R(k^2;\bar{\alpha}_0\bar{D})\simeq\rho_R(k^2;\bar{\alpha}_0/q^2)+\int\left.\frac{\delta\rho_R(k^2)}{\delta D(q^2)}\right|_{D=1/q}\times\left[-\frac{\bar{\alpha}_0}{q^2}\rho_R(q^2)\right]d^4q. \tag{6.22}$$

Equation (6.22) has been written in a Euclidean metric for q and k. The first term in (6.22) is of order m^2/k^2. We shall assume that the second term vanishes more slowly than the first, that is, we assume that

$$\rho_R(k^2;\bar{\alpha}_0\bar{D})\gg\rho_R(k^2;\bar{\alpha}_0/q^2)\sim O(m^2/k^2) \tag{6.23}$$

when $k^2\gg m^2$, and when the self-consistency requirement $f(\bar{\alpha}_0)=0$ is met. In this case ρ_R obeys a homogeneous equation asymptotically,

$$\rho_R(k^2)\simeq\bar{\alpha}_0^2\int K(k,q)\frac{1}{q^2}\rho_R(q^2)(d^4q), \tag{6.24}$$

where

$$K(k,q)\equiv-\left.\frac{\delta\rho_R(k^2;\bar{\alpha}_0\bar{D})}{\delta\bar{D}(q^2)}\right|_{\bar{D}=1/q^2}. \tag{6.25}$$

If $\bar{\alpha}_0^2\ll 1$, the only way that (6.24) can be satisfied is if the integral is very slowly convergent, so that it is of order $1/\bar{\alpha}_0^2\gg 1$. Therefore the only part of the kernel needed is the form when $q\gg k$. Now if we recall that

$$\rho_R(k^2;\bar{\alpha}_0/q^2)\sim\frac{f(\bar{\alpha}_0)}{\bar{\alpha}_0}\int_{k^2}^{\Lambda 2}\frac{dp^2}{p^2}+O\left(\frac{m^2}{k^2}\right),$$

we see that

$$\frac{\partial}{\partial\bar{\alpha}_0}\rho_R(k^2;\bar{\alpha}_0/q^2)=\int\left.\frac{\delta\rho_R(k^2)}{\delta\bar{D}(q^2)}\right|_{\bar{D}=1/q^2}\times\frac{1}{q^2}(d^4q)$$
$$=\frac{\partial}{\partial\bar{\alpha}_0}\left(\frac{f(\bar{\alpha}_0)}{\bar{\alpha}_0}\right)\int_{q^2}^{\Lambda 2}\frac{dp^2}{p^2}+O\left(\frac{m^2}{q^2}\right). \tag{6.26}$$

Consequently,

$$K(k^2;q^2)\sim-\frac{g'(\bar{\alpha}_0)}{2\pi^3}\frac{1}{q^2} \quad \text{for} \quad q^2\gg k^2\gg m^2. \tag{6.27}$$

In this case (6.24) becomes

$$\rho_R(k^2)\simeq-\frac{\bar{\alpha}_0^2}{2\pi}g'(\bar{\alpha}_0)\int_{k^2}^{\infty}\frac{dq^2}{q^2}\rho_R(q^2). \tag{6.28}$$

Solving (6.28) we obtain

$$\rho_R(q^2)\sim\text{const.}\times(m^2/q^2)^{-\bar{\alpha}_0^2/2\pi g'(\bar{\alpha}_0)} \tag{6.29}$$

when $\bar{\alpha}_0^2\ll 1$. The consistency of this solution requires that $g'(\bar{\alpha}_0)<0$. It is not unreasonable to suppose that this will happen, since g starts from 0 at $\alpha_0=0$, and falls to $-\frac{2}{3}$ at the self-consistent value of $\bar{\alpha}_0$.

If $\bar{\alpha}_0^2$ is not small, more detailed information about $K(k^2,q^2)$ than we can determine from (6.26) is required. Let us introduce the dimensionless function $k(x)$:

$$K(k^2,q^2)=(1/q^2)k(k^2/q^2)\times 1/\pi^2, \quad q>k$$
$$=(q^2/k^4)k(q^2/k^2)\times 1/\pi^2, \quad k>q,$$

where

$$k(0)=-g'(\bar{\alpha}_0)/2\pi. \tag{6.30}$$

This form for $K(k^2,q^2)$ is valid for $k^2\gg m^2$ and $q^2\gg m^2$. The validity of (6.30) under such conditions follows from the fact that $\delta\pi_{\mu\nu}(k^2)/\delta D_{\alpha\beta}(q^2)$ is the Bethe-Salpeter kernel for scattering of light by light and is symmetric in k and q. In this case the solution to (6.24) has the same form:

$$\rho_R(k^2)=\text{const.}\times(m^2/k^2)^{\epsilon}, \tag{6.31}$$

but with ϵ obtained from the solution of the equation

$$1=\bar{\alpha}_0^2\times\int_0^1 dx\,k(x)(x^{-1+\epsilon}+x^{1-\epsilon}). \tag{6.32}$$

Since $k(0)$ is finite, (6.32) requires that $\epsilon\to 0$, as $\bar{\alpha}_0^2\to 0$. Thus for $\bar{\alpha}_0^2\to 0$, (6.32) gives us the condition

$$1=\bar{\alpha}_0^2\left[\frac{k(0)}{\epsilon}+O(1)\right]. \tag{6.33}$$

Therefore, when $\bar{\alpha}_0^2\ll 1$,

$$\epsilon=\bar{\alpha}_0^2k(0)=-\bar{\alpha}_0^2[g'(\bar{\alpha}_0)/2\pi],$$

which reproduces our approximation discussed above. We also find that the fundamental function $g(\bar{\alpha}_0)$ can be

[9] This assumption can be justified to every order of the expansion of the functional ρ as a power series in D.

determined from the Bethe-Salpeter kernel for the scattering of light by light.

In order to determine the arbitrary constant in the solution (6.31), we must of course join the asymptotic solution to the solution of the equations for D in the nonasymptotic region. It is at least plausible that this can be done for any value of the constant in Eq. (6.31) if we appropriately adjust the renormalized charge α which characterizes the theory in the nonasymptotic region. That is, the free parameter in the theory, α, may remain. In this case, the equation $\bar{\alpha}_0=\alpha/Z_3(\alpha)$ would be an identity in α .

We may remark here that some years ago Gell-Mann and Low[2] reached a similar conclusion about the theory using what is now called the "renormalization group." We shall discuss the relation between their results and ours in a later publication.

VII. CONCLUSION

We have found that unrenormalized quantum electrodynamics may be a consistent and finite theory provided that the unrenormalized charge $\bar{\alpha}_0$ (defined in Sec. VI) obeys an eigenvalue equation $g(\bar{\alpha}_0)+\frac{2}{3}=0$, which must of course have a real, positive solution.[10] We have not been able to establish that such a root of the equation exists. An accurate calculation of $g(\bar{\alpha}_0)$, the fundamental function in quantum electrodynamics, remains to be done.

ACKNOWLEDGMENTS

We should like to thank J. Bjorken and K. A. Ter-Martirosyan for particularly helpful conversations. Two of us, M.B. and K.J., would like to thank the Alfred P. Sloan Foundation for support during the course of this work.

APPENDIX A. CONVERGENCE OF $K^a(0,p')$

We will show[10,11] that $K^{a[2(n+1)]}(0,p')$ is finite provided $\Gamma_\mu^{(2n)}$ is finite. The contribution of a given Feynman diagram to $K^{a[2(n+1)]}(0,p')$ can be written as

$$K^{a[2(n+1)]}(0,p')=\int d^4q_1\cdots d^4q_n f_n(q_1\cdots q_n,p')\,, \quad \text{(A1)}$$

where the integrand $f_n(q_1,q_2\cdots q_n p')$ is a product of numerators and denominators. We need only keep track of the correct powers of p' and the integration variables q_i in order to discuss the convergence of (A1). We perform a rotation of the contour of integration in (A1) so that all integrals d^4q_1, $d^4q_2\cdots d^4q_n$ refer to Euclidean four-vectors. This rotation is permitted since the external momenta are chosen to be spacelike vectors throughout.[5]

We know that the only ultraviolet divergences in the perturbation expansion for K^a arise from vertex and self-energy insertions. Since by definition $K^{a[2(n+1)]}$ contains no self-energy insertions and since by assumption the vertex insertion $\Gamma_\mu^{(2n)}$ is finite, the integrals for $K^{a[2(n+1)]}$ converge in the ultraviolet region. However, since K^a does not contain the electron mass, all the denominators in $f_n(q_1\cdots q_n p)$ which do not contain p vanish if some or all of the integration variables q_i vanish. Such vanishing denominators could give rise to infrared divergences. We now seek the precise conditions for the occurrence of this kind of divergence in a given integral.

For a given diagram the integrand $f_n(q_1\cdots q_n,p')$ depends upon the choice of the integration variables $q_1\cdots q_n$. We will consider all the functions f_n which correspond to the various possible choices of the q_i in a particular diagram. Then for a given integrand f_n we need only consider the possibility that the non-p-containing factors of f_n vanish. In order that the resulting integral (A1) converge (superficially) in the infrared region, it is necessary that

$$\tilde{n}-\tilde{d}>0\,, \quad \text{(A2)}$$

where $\tilde{n}(\tilde{d})$ is the number of powers of qi in the numerator (denominator) of those factors in the integrand of (A1) which do not contain p. In order that integral (A1) converge absolutely it is sufficient that all subintegrals of (A1) obtained by fixing any subset $qi_l\cdots qi_f$ of the variables qi also converge superficially. Before we examine these subintegrals we will first show that condition (A2) is satisfied for all K^a diagrams except those containing 3-photon intermediate states.

Let us calculate $\tilde{n}-\tilde{d}$ for a given diagram and a given choice of integration variables. The variable p' will follow a certain path L which in general contains both internal boson and internal fermion lines. Let $f(b)$ be the number of fermion (boson) lines which connect L to the

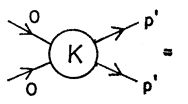
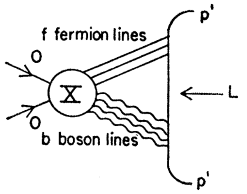


FIG. 11. Decomposition of graph for $K^a(0,p')$.



[10] For a short summary and discussion of the results of this paper, see K. Johnson, R. Willey, and M. Baker, Zh. Eksperim. i Teor. Fiz. 52, 318 (1967) [English transl.: Soviet Phys.—JETP 25, 205 (1967)].

[11] The ideas behind the proofs in Appendices A and B are due to J. D. Bjorken. J. D. Bjorken (private communication).

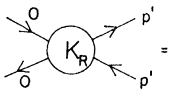
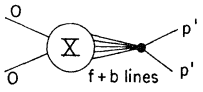


FIG. 12. A reduced diagram for $K^a(0,p')$.



remaining part of the diagram, which we denote by X. This decomposition is depicted in Fig. 11. By definition, the value of $\tilde{n}-\tilde{d}$ refers to the number of powers of the integration variables in that part of the integrand which corresponds to all lines except those forming the line L. Consider the reduced diagram K_R depicted in Fig. 12 obtained from the K^a diagram of Fig. 11 by contracting the line L to a point. The value of $n-d$ for the K^a diagram of Fig. 11 is then equal to the value of $\tilde{n}-\tilde{d}$ for the K_R diagram of Fig. 12, where $\tilde{n}$ and $\tilde{d}$ refer to the powers of the integration variables in the complete integrand. The value of $(\tilde{n}-\tilde{d})_{K_R}$ is the sum of the contribution $(\tilde{n}-\tilde{d})_X$ of the internal lines in X and the contribution $(\tilde{n}-\tilde{d})_{f+b}$ of the $f+b$ connecting lines. X is a diagram with $f+2$ external fermion lines and b external boson lines. Hence

$$(\tilde{n}-\tilde{d})_X=4-\tfrac{3}{2}(f+2)-b\,, \tag{A3}$$

while clearly

$$(\tilde{n}-\tilde{d})_{f+b}=-f-2b+4(f+b-1)\,, \tag{A4}$$

since there are $f+b-1$ independent integrations over the connecting lines. Adding (A3) and (A4) we find

$$(\tilde{n}-\tilde{d})_{K^a}=(\tilde{n}-\tilde{d})_{K_R}=1+\tfrac{3}{2}f+b-4\,. \tag{A5}$$

Result (A5) is valid even if X is a disconnected diagram such as that depicted in Fig. 13. Diagram X' of Fig. 13 has 4 fewer external fermion lines than diagram X. On the other hand, the corresponding reduced diagram K_R of Fig. 13 has 2 fewer integrations over internal connecting boson lines. These facts add terms $+(4)(\frac{3}{2})$ and $-(8-2)$, respectively, to (A3) and (A4), so that the sum (A5) remains unchanged.

We note that Eq. (A5) for $(\tilde{n}-\tilde{d})_{K^a}$ depends upon the number $f(b)$ of fermions (bosons) which appear in intermediate states. This dependence upon the internal

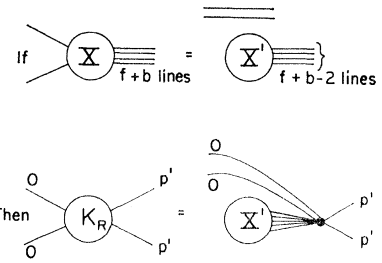


FIG. 13. An example of a reduced diagram with degenerate X.

structure of the diagram is to be contrasted with the structure-independent value -2 for $(n-d)_K$. From (A5) we thus conclude that if a given diagram contains an intermediate state in which

$$\tfrac{3}{2}f+b-4<0\,, \tag{A6}$$

then that diagram diverges in the infrared region.

Equation (A6) is satisfied if $f=0$, $b=3$. Thus diagrams containing 3-photon intermediate states possess infrared divergences. However, as we mentioned in Sec. II, such diagrams yield non-gauge-invariant results and the use of a properly gauge-invariant current yields terms which cancel these divergences. We now show that K^a contains no other diagrams for which (A6) is satisfied. We know that f has to be even. This means $f=0$ or 2.

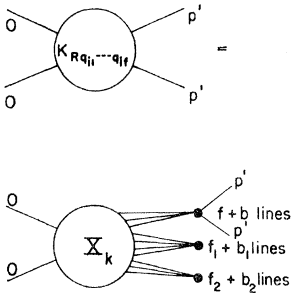


FIG. 14. A reduced diagram for $K^a(0,p')$ when some of the integration variables are held fixed.

If $f=0$, then Eq. (A6) is satisfied if $b=0$, 1, 2, or 3.

(a) The case $b=0$ does not occur since K^a does not contain disconnected diagrams.

(b) The case $b=1$ does not occur since K^a does not contain 1-photon annihilation diagrams.

(c) The case $b=2$ does not occur, because such diagrams do not contribute to the vertex by virtue of Furry's theorem.

(d) The case $b=3$ was just discussed.

If $f=2$, then Eq. (A6) is satisfied if $b=0$. This case also does not occur since K^a does not contain any 2-particle electron-positron intermediate state.

We thus conclude that K^a possesses no over-all infrared divergence. We now fix a certain subset $q_{i_1}\cdots q_{i_f}$ of the variables q_i and investigate the infrared properties of the resulting subintegral of (A1) over the remaining q_i. In order that this subintegral converge in the infrared region it is necessary that

$$(\tilde{n}-\tilde{d})_{q_{i_1}\cdots q_{i_f}}>0\,, \tag{A7}$$

where $(\tilde{n}-\tilde{d})_{q_{i_1}\cdots q_{i_f}}$ refers to the powers of the remaining q_i in those factors of f_n which do not contain $q_{i_1}\cdots q_{i_f}$ of p'. Let us suppose that the fixed subset $q_{i_1}\cdots q_{i_f}$ or variables form k independent loops L_i $(i=1,2,\cdots,k)$ in our diagram. To be considered independent a loop L must not overlap the line L containing the variable p. Then $(\tilde{n}-\tilde{d})_{q_{i_1}\cdots q_{i_f}}$ equals $(n-d)$ for the reduced diagram $K_{Rq_{i_1}\cdots q_{i_f}}$. This is the diagram

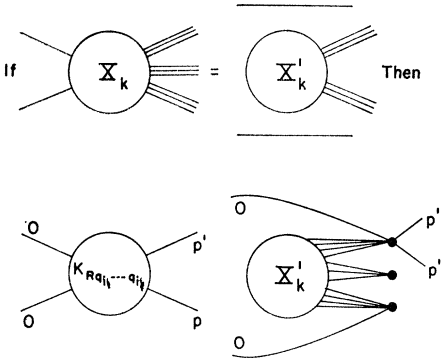


FIG. 15. An example of a reduced diagram with degenerate X_k.

(depicted in Fig. 14) in which the line L and the loops L_i are contracted to a point. Let $f_i(b_i)$ denote the number of fermion (boson) lines which connect the loop L_i to the remaining part of the diagram X_k. Then, following the same reasoning that led to Eq. (A5) we obtain the result

$$(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}=1+\tfrac{3}{2}f+b-4+\sum_{i=1}^{k}(\tfrac{3}{2}f_i+b_i-4). \quad \text{(A8)}$$

Equation (A8), like Eq. (A5), is valid even if X_k is a disconnected diagram such as depicted in Fig. 15.

We have already shown that $\frac{3}{2}f+b-4\geq 0$ for all relevant diagrams. We will now show that K^a contains no diagrams for which

$$\tfrac{3}{2}f_i+b_i-4<0. \quad \text{(A9)}$$

Equation (A9) is satisfied if $f_i=0$ and $b_i=0$, 1, 2, or 3 or if $f_i=2$ and $b_i=0$.

(a) The case $f_i=b_i=0$ does not occur since K^a does not contain disconnected diagrams.

(b) The case $f_i=0$, b_i odd, does not occur because of Furry's theorem.

(c) The case $f_i=0$, $b_i=2$, does not occur because K^a does not contain photon self-energy insertions.

(d) The case $b_i=0$, $f_i=2$, does not occur because K^a does not contain electron self-energy insertions.

We thus conclude that Eq. (A9) is never satisfied and hence that Eq. (A7) is satisfied for all choices of the fixed variables $q_{il}\cdots q_{if}$. Thus integral (A1) for $K^{a[2(n+1)]}$ converges absolutely (in the infrared region).

APPENDIX B. CONVERGENCE OF INTEGRALS (5.8), (5.9), AND (5.5)

We shall show that the integrals (5.8), (5.9), and (5.5), defining constants f_1, f_2, and f_3, converge when K^a is expanded to any finite order of perturbation theory. In order to obtain these integrals we must differentiate with respect to k once or twice various parts of the vertex integral depicted by the Feynman diagram of Fig. 16. After differentiation we set $k=0$ and

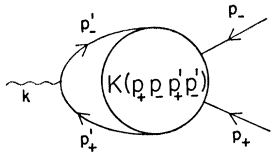


FIG. 16. Basic vertex diagram.

consider all possible choices of internal variables $\{q_i\}$ for the resulting integrals. Thus, as in the case of (A1), it is sufficient to investigate the convergence of these integrals in the infrared region where some or all of the integration variables q_i vanish.

The diagram of Fig. 16 contains an intermediate state consisting of a single electron-positron pair. If this pair of propagators were differentiated twice, the resulting integral $\int d^4p'/p'^4K(p',p)$ would clearly diverge for small p'. Of course none of the integrals (5.8), (5.9), or (5.5) is obtained by twice differentiating the electron-positron pair in Fig. 16. We will now show that differentiation of any other part of the integral of Fig. 16 yields convergent integrals.

We proceed precisely as in Appendix A. Let us fix a certain subset $qi_l\cdots qi_f$ of variables and calculate $(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}$ for the diagram depicted in Fig. 16 with $k=0$. As before, $(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}$ is equal to the value of $(\tilde{n}-\tilde{d})$ for the reduced diagram in which all the internal lines in Fig. 16 containing p or the fixed variables $qi_l\cdots qi_f$, are contracted to a point. This reduced diagram is depicted as Fig. 17. The reduced diagram of Fig. 17 differs from the reduced diagram of Fig. 14 in that a pair of external zero-momentum electron lines has been replaced by an external zero-momentum photon line. This simply means that in $(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}$ a term (-3) has been replaced by (-1). This difference of $+2$ is just what is needed in order to differentiate twice the integral depicted in Fig. 16 without encountering difficulties. That is, adding 2 to the value of $(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}$ given by Eq. (A8), we immediately obtain the value of $(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}$ for the desired integral of Fig. 16:

$$(\tilde{n}-\tilde{d})_{qi_l\cdots qi_f}=1+\tfrac{3}{2}f+b-4+\sum_{i=1}^{k}(\tfrac{3}{2}f_i+b_i-4)+2, \quad \text{(B1)}$$

where f, b, f_i, and b_i are the number of connecting lines in Fig. 17. Equation (B1) is reduced by 1 when we differentiate a propagator corresponding to one of the internal lines of the reduced diagram in Fig. 17. Hence from (B1) we obtain the following expressions for $(\tilde{n}-\tilde{d})'_{qi_l\cdots qi_f}$ and $(\tilde{n}-\tilde{d})''_{qi_l\cdots qi_{f'}}$ corresponding to the once and twice differentiated integrals (5.8), (5.9),

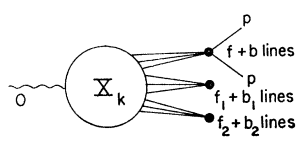


FIG. 17. Reduced vertex diagram.

FIG. 18. Reduced vertex diagram with $f=2$, $b=0$.

and (5.5):

$$(\tilde{n}-\tilde{d})'_{qi_1\cdots qi_f}\geq 1+\tfrac{3}{2}f+b-4+\sum_{i=1}^{k}(\tfrac{3}{2}f_i+b_i-4)+1\,, \tag{B2}$$

$$(\tilde{n}-\tilde{d})''_{qi_1\cdots qi_f}\geq 1+\tfrac{3}{2}f+b-4+\sum_{i=1}^{k}(\tfrac{3}{2}f_i+b_i-4)\,. \tag{B3}$$

Equations (B2) and (B3) become equalities when the differentiated propagators are not contracted to a point in Fig. 17.

Thus if there are no diagrams for which

$$\tfrac{3}{2}f+b-4<0\,, \tag{B4}$$

or

$$\tfrac{3}{2}f_i+b_i-4<0\,, \tag{B5}$$

then

$$(\tilde{n}-\tilde{d})''_{qi_1\cdots qi_f}>0\,, \tag{B6}$$

and integral (5.5) converges in the infrared region. The

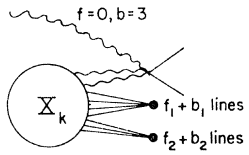


FIG. 19. Reduced vertex diagram with $f=0$, $b=3$.

requirement

$$(\tilde{n}-\tilde{d})'_{qi_1\cdots qi_f}>0\,, \tag{B7}$$

which guarantees the convergence of integrals (5.8) and (5.9), is met if there are no diagrams for which (B5) is satisfied or for which

$$\tfrac{3}{2}f+b-4<-1\,. \tag{B8}$$

The arguments following Eq. (A9) can be applied without modification to the diagram of Fig. 17 in order to show that K^a contains no diagrams for which (B5) is satisfied. In considering (B4) or (B8), the arguments following (A6) can be repeated with the following trivial qualifications.

(a) Since the diagram of Fig. 16 contains one electron-positron intermediate state, the reduced diagram of Fig. 17 can take on the form depicted in Fig. 18 for which $(\tilde{n}-\tilde{d})_{qi_1\cdots qi_f}=2$. However, since all the differentiated propagators in integral (5.5) are contracted

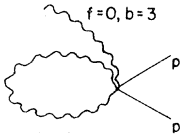


FIG. 20. A degenerate case of Fig. 19 with $f=0$, $b=3$, $f_i=b_i=0$.

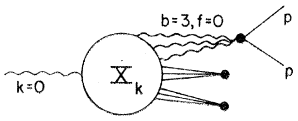


FIG. 21. Nondegenerate $f=0$, $b=3$, reduced diagram.

to a point, $(\tilde{n}-\tilde{d})''$ is also equal to 2. Besides this trivial diagram of Fig. 18, there are no other reduced diagrams for which $f=2$ and $b=0$.

(b) The case $b=3$, $f=0$, can occur for diagrams which are not true 3-photon intermediate states—for example, when X_k is the disconnected diagram giving rise to the reduced diagram depicted in Fig. 19. In general such a diagram contains a photon self-energy insertion and hence does not contribute to K^a. However, when $f_i=b_i=0$, the degenerate reduced diagram of Fig. 20 does not contain a photon self-energy insertion. [It is trivial to show that our general formula (B1) for $(\tilde{n}-\tilde{d})_{qi_1\cdots qi_f}$ also applies to such a degenerate case.] If the denominator of the internal photon line in the diagram of Fig. 20 were differentiated twice, $(\tilde{n}-\tilde{d})''$ would equal zero, thus giving rise to an infrared divergence. However, in all diagrams for K^a not containing multiphoton intermediate states, it is always possible to choose the integration variables so that no photon line carries the external photon momentum k. Thus this possible way of obtaining $(\tilde{n}-\tilde{d})''=0$ can be realized only by photon-annihilation diagrams. However, in such diagrams, there always appears an undifferentiated multiphoton scattering amplitude with one zero-momentum external photon. Hence by gauge invariance the contribution of such a diagram to f_3 must vanish.

(c) Finally we consider nondegenerate reduced diagrams such as that depicted in Fig. 21 for which $b=3$, $f=0$. But again from gauge invariance it follows that the contribution of such diagrams to f_2 and f_3 must vanish.

APPENDIX C. DERIVATION OF EQ. (5.7)

When m is set equal to zero, the only terms which contribute to $\Gamma_\mu(p_-,p_+)$ are those containing an odd number of γ matrices. Hence

$$\Gamma_\mu{}^a(p_-,p_+)=F_{\mu\lambda}(p,k)\gamma_\lambda+F_{\mu\lambda}{}^5(p,k)\gamma_\lambda\gamma_5\,, \tag{C1}$$

where $F_{\mu\lambda}(p,k)$ is a tensor function of the vectors p and k and $F_{\mu\lambda}{}^5$ is a pseudotensor function of p and k. Now CPT invariance implies

$$\Gamma_\mu(p_-,p_+)=\gamma_5\Gamma_\mu(-p_-,-p_+)\gamma_5\,. \tag{C2}$$

Also C invariance implies

$$-\Gamma_\mu{}^T(-p_+,-p_-)=C^{-1}\Gamma_\mu(p_-,p_+)C\,, \tag{C3}$$

where $\Gamma_\mu{}^T$ represents the transposed matrix, and the charge-conjugation matrix C satisfies

$$C^{-1}\gamma_\mu C=-\gamma_\mu{}^T\,,$$
$$C^{-1}\gamma_5 C=\gamma_5{}^T\,, \tag{C4}$$

where $\gamma_5^\dagger=-\gamma_5$, $\gamma_5^2=-1$. Then PT invariance gives us the condition

$$\Gamma_\mu(p_+,p_-)=-\gamma_5 C^{-1}\Gamma_\mu{}^T(p_-,p_+)C\gamma_5. \quad \text{(C5)}$$

Using Eqs. (C1), (C4), and (C5) we immediately obtain the conditions

$$F_{\mu\lambda}(p,-k)=F_{\mu\lambda}(p,k), \quad \text{(C6)}$$

$$F_{\mu\lambda}{}^5(p,-k)=-F_{\mu\lambda}{}^5(p,+k). \quad \text{(C7)}$$

Hence from (C1), (C6), and (C7), we obtain

$$\Gamma_{\mu\alpha}{}^a=\frac{\partial}{\partial k_\alpha}\Gamma_\mu{}^a(p_-,p_+)\,|_{k=0}=\frac{\partial}{\partial k_\alpha}F_{\mu\lambda}{}^5(p,k)\,|_{k=0}\gamma_\lambda\gamma_5. \quad \text{(C8)}$$

Now $(\partial/\partial k_\alpha)F_{\mu\lambda}{}^5(p,k)\,|_{k=0}$ is a pseudotensor function of the vector p. The only such function is of the form const.$\times\epsilon_{\alpha\mu\lambda\kappa}p_\kappa/p^2$. Hence

$$\Gamma_{\mu\alpha}{}^a\propto\epsilon_{\alpha\mu\lambda\kappa}p_\kappa\gamma_\lambda\gamma_5=\text{const.}\left(\frac{\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu}{p^2}\right). \quad \text{(C9)}$$

APPENDIX D. LOWEST-ORDER CALCULATION OF f_1

In lowest-order perturbation theory

$$K^{(2)}(p,p',k)$$
$$=-ie_0{}^2\gamma^a\left(g_{ab}-\frac{(p-p')_a(p-p')_b}{(p-p')^2}\right)\gamma_b\frac{1}{(p-p')^2}. \quad \text{(D1)}$$

Since $K^{(2)}$ does not depend upon k, $f_2=f_3=0$; and from Eq. (1.6)

$$f_1=\frac{-ie_0{}^2}{48}\int\frac{d^4p'}{(2\pi)^4}\mathrm{Tr}\frac{(\gamma_\alpha\gamma p'\gamma_\mu-\gamma_\mu\gamma p'\gamma_\alpha)}{2p'^4}$$
$$\times\gamma_a\frac{(\gamma_\mu\gamma p\gamma_\alpha-\gamma_\alpha\gamma p\gamma_\mu)}{(p-p')^2}\gamma_b$$
$$\times\left(g_{ab}-\frac{(p-p')_a(p-p')_b}{(p-p')^2}\right). \quad \text{(D2)}$$

If we evaluate the trace of the g_{ab} contribution to (D2), we obtain

$$f_1{}^{g_{ab}}=\frac{\alpha_0}{2\pi}\int_0{}^\infty\frac{dp'^2}{p'^2}\left\langle\frac{p\cdot p'}{(p-p')^2}\right\rangle_{p'}. \quad \text{(D3)}$$

The angular average $\langle p\cdot p'/(p-p')^2\rangle_{p'}$ is easily carried out with the result

$$\left\langle\frac{p\cdot p'}{(p-p')^2}\right\rangle=\tfrac{1}{2}pp'\frac{1}{p_>{}^4}\left(\frac{p_<}{p_>}\right), \quad \text{(D4)}$$

where

$$p_<=p \quad \text{if} \quad p<p',$$
$$=p' \quad \text{if} \quad p'<p,$$

and

$$p_>=p' \quad \text{if} \quad p>p'$$
$$=p' \quad \text{if} \quad p'>p.$$

From (D3) and (D4) we obtain

$$f_1{}^{g_{ab}}=\alpha_0/2\pi. \quad \text{(D5)}$$

Carrying out the trace of the $(p-p')_a(p-p')_b/(p-p')^2$ contribution to (D2), we find

$$f_1\left(\text{due to}\ \frac{(p-p')_a(p-p')_b}{(p-p')^2}\right)=-\frac{\alpha_0}{2\pi}\int_0{}^\infty\frac{dp'^2}{p'^2}$$
$$\times\left\{\left\langle\frac{p\cdot p'}{(p-p')^2}\right\rangle-\frac{2}{3}\left\langle\frac{p^2p'^2-(p\cdot p')^2}{(p-p')^4}\right\rangle\right\}. \quad \text{(D6)}$$

However,

$$\left\langle\frac{p'^2p^2-(p\cdot p')^2}{(p-p')^4}\right\rangle=\frac{3p'^2p^2}{4p_>{}^4}. \quad \text{(D7)}$$

Hence from (D4), (D6), and (D7) we find that

$$f_1\left(\text{due to}\ \frac{(p-p')_a(p-p')_b}{(p-p')^2}\right)=0. \quad \text{(D8)}$$

The fact that we can obtain the correct value of f_1 from a calculation in the g_{ab} gauge is just an accident of the lowest-order calculation.