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Derivation of the $SU(3)\times SU(3)$ Space-Time Local Current Commutators

DAVID J. GROSS* AND ROMAN JACKIW*
Jefferson Laboratory of Physics, Harvard University, Cambridge, Massachusetts
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The commutators of the charge with the current density of vector and axial-vector currents are derived, and restrictions are placed on the Schwinger terms present in the charge-density–current-density equal-time commutators. In order to prove these results, the commutator of the time component of a current with the energy density is derived. The following assumptions are made: (1) Equal-time commutation relations between time components of vector and axial-vector currents satisfy the local $SU(2)$, $SU(2)\times SU(2)$, or $SU(3)\times SU(3)$ algebra; (2) the transformation properties of the divergence of the axial current are assumed to be known. The second assumption is shown to be necessary as well as sufficient. It is shown that the Schwinger terms involve at most one derivative of a δ function and have definite symmetry properties. Symmetry properties frequently conjectured for the Schwinger terms are examined in the context of the present investigation, and the consequences of these conjectures are explored. The current-density–current-density equal-time commutation is also studied with the present techniques, and it is found that only very mild restrictions can be imposed in a model-independent fashion.

I. INTRODUCTION

IT has been proposed by Gell-Mann[1] that the time components of the vector and axial-vector currents ($V_\mu{}^a$ and $A_\mu{}^a$, respectively), obey the equal-time commutation relations (ETCR) of $SU(3)\times SU(3)$:

$$[V_0{}^a(\mathbf{x},t),V_0{}^b(\mathbf{y},t)]=if_{abc}V_0{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\,, \quad (1a)$$

$$[V_0{}^a(\mathbf{x},t),A_0{}^b(\mathbf{y},t)]=if_{abc}A_0{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\,, \quad (1b)$$

$$[A_0{}^a(\mathbf{x},t),A_0{}^b(\mathbf{y},t)]=if_{abc}V_0{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})\,. \quad (1c)$$

Here f_{abc} are the usual $SU(3)$ structure constants. Many authors have proposed ETCR of the time components with the space components

$$[V_0{}^a(\mathbf{x},t),V_i{}^b(\mathbf{y},t)]=if_{abc}V_i{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})+S_{VV,i}{}^{ab}(\mathbf{x},\mathbf{y},t)\,, \quad (2a)$$

$$[V_0{}^a(\mathbf{x},t),A_i{}^b(\mathbf{y},t)]=if_{abc}A_i{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})+S_{VA,i}{}^{ab}(\mathbf{x},\mathbf{y},t)\,, \quad (2b)$$

$$[A_0{}^a(\mathbf{x},t),V_i{}^b(\mathbf{y},t)]=if_{abc}A_i{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})+S_{AV,i}{}^{ab}(\mathbf{x},\mathbf{y},t)\,, \quad (2c)$$

$$[A_0{}^a(\mathbf{x},t),A_i{}^b(\mathbf{y},t)]=if_{abc}V_i{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})+S_{AA,i}{}^{ab}(\mathbf{x},\mathbf{y},t)\,. \quad (2d)$$

The $S_i{}^{ab}$'s are the notorious Schwinger terms (ST), involving gradients of δ functions, which upon integration over $\mathbf{x}$ vanish, and which must be present in the VV and AA ETCR.[2,3] Adler and Callan, on the basis of explicit calculation of the commutators for the σ model,[4] conjectured that

$$S_{VV,i}{}^{ab}=S_{VV,i}{}^{ba},S_{AA,i}{}^{ab}=S_{AA,i}{}^{ba},S_{AV,i}{}^{ab}+S_{VA,i}{}^{ab}=S_{AV,i}{}^{ba}+S_{VA,i}{}^{ba}\,. \quad (3)$$

Such symmetry properties of the ST's have been used widely to justify their neglect in appropriate situations.

In this paper, we utilize the fact that the commutator of the energy density with the time component of a vector or axial-vector current is given in terms of the space components of the current and its divergence. This enables us, using the Jacobi identity,[5] to derive the ETCR of the space-time components (2) from the ETCR of the time-time components (1), which we assume to hold. We also derive restriction on the ST, proving that they can involve at most only one derivative of a δ function. Finally we discuss various symmetry properties of the ST, and show under what conditions the symmetries (3) conjectured by Adler and Callan can hold.

In Sec. II, we establish the ETCR of the energy density with the time component of a vector current or axial-vector current. In Sec. III, we derive in detail the time-space ETCR for the case of the isotopic-spin currents as well as the general form of the ST, and such symmetries of the ST which are model independent. Then we extend the results to $SU(2)\times SU(2)$ and to $SU(3)\times SU(3)$, where the axial-vector currents are not conserved. Section IV concerns itself with the various further conjectures that can be made concerning the symmetries of the ST, and the consequences of these assumptions. Finally, in Sec. V, we examine what our methods have to say about the ETCR of the space components of current densities.

* Junior Fellow, Society of Fellows.
[1] M. Gell-Mann, Physics 1, 63 (1964).
[2] T. Goto and T. Imanura, Progr. Theoret. Phys. (Kyoto) 14, 296 (1955); J. Schwinger, Phys. Rev. Letters 3, 296 (1959).
[3] S. Okubo, Nuovo Cimento 44 A, 1015 (1966).
[4] S. L. Adler and C. G. Callan, CERN report, 1965 (unpublished).

[5] It has been shown recently that unless one is careful about the interchange of limiting procedures implicit in the Jacobi identity, contradictions may arise. A contradiction between canonical commutation relations and the Jacobi identity has been found in the case of 3 space components of quark currents, by F. Bucalla, G. Veneziano, R. Gatto, and S. Okubo [Phys. Rev. 149, 1268 (1966)]. It is, however, unlikely that this problem arises unless one is dealing with more than one space component. In our application of the Jacobi identities, we use at most one space component, and we shall assume that no problem arises in the use of these identities.

II. DERIVATION OF $[H_{00}(\mathbf{x},t), J_0(\mathbf{y},t)]$

In the present section, we derive the ETCR of the time component of a vector or axial-vector current J_0, with the energy density H_{00}. In simple field-theoretic models, direct application of the canonical-commutation relations leads to the result

$$[H_{00}(\mathbf{x},t),J_0(\mathbf{y},t)]=iJ_k(\mathbf{x},t)\partial_k\delta(\mathbf{x}-\mathbf{y})-D(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}), \quad (4)$$

where $D(\mathbf{x},t)$ is the (possibly vanishing) divergence of the current

$$D(\mathbf{x},t)=i\partial_\mu J^\mu(\mathbf{x},t). \quad (5)$$

(We employ the summation convention with repeated indices.) This commutator, upon integration over $\mathbf{x}$, as well as $\mathbf{x}$ and $\mathbf{y}$, gives the following commutation relations with the generator of time translations, viz., with the Hamiltonian $H\equiv\int d^3x\, H_{00}(\mathbf{x},t)$, and with the generator of time rotations $M^{0i}\equiv\int d^3x(tH^{i0}-x^iH^{00})$.

$$[H,J_0(y,t)]=-i\partial_0J_0(y,t), \quad (6a)$$

$$[M^{0i},J_0(y,t)]=-it\partial_iJ_0+ix^i\partial_0J_0+iJ_i, \quad (6b)$$

$$\left[H,\int d^3yJ_0(y,t)\right]=-\int D(y,t)d^3y. \quad (6c)$$

These commutators are of course required by general principles. Further ST could be present in (4), which integrate to zero in such a way such that (6) still holds. However, no examples of ST in ETCR of time-time components of currents are known, and it is usually assumed that ST arise only in ETCR of space-time components. Thus (4) may be expected to hold as written.

It is also possible to justify (4) in a quite rigorous fashion. This proof follows that given by Schwinger[6] in one of his derivations of the $[J_0,J_0]$ and $[H_{00},H_{00}]$ ETCR. According to Schwinger, ETCR result from the action principle, which implies that

$$\int d^3x[A(\mathbf{x},t),\delta\mathcal{L}(\mathbf{x}',t)]=i[\partial_0\delta'A(x)-\delta'B(x)]. \quad (7)$$

Here $\mathcal{L}$ is the Lagrangian density and A is any operator satisfying

$$\partial_0A(x)=B(x).$$

δ signifies a total variation of the Lagrangian with respect to arbitrary parameters, while δ' is a variation of the operators A and B arising from their explicit dependence on these parameters. We vary $\mathcal{L}$ with respect to a prescribed gravitational field, which is governed by metric $g_{\mu\nu}(y)$ satisfying $-g_{00}(y)\neq1$, $g_{0i}=0$, $g_{ij}=\delta_{ij}$.

$$\frac{\delta\mathcal{L}(\mathbf{x}',t)}{\delta g_{00}(y)}=\tfrac{1}{2}H_{00}(\mathbf{x}',t)\delta^4(x'-y). \quad (8)$$

[6] J. Schwinger Phys. Rev. **130**, 406 (1963).

For A we choose $(-g)^{1/2}J_0$, where

$$i\partial_\mu[(-g)^{1/2}J^\mu]=(-g)^{1/2}D,\quad g=\det\{g_{\mu\nu}\}=g_{00}. \quad (9)$$

Hence B is given by

$$B=-\partial_i[(-g_{00})^{1/2}J^i]-i(-g_{00})^{1/2}D,$$
$$=-J^i\partial_i(-g_{00})^{1/2}-(-g_{00})^{1/2}\partial_iJ^i-i(-g_{00})^{1/2}D. \quad (10)$$

A can depend on the varying parameter g_{00} only if B on the time derivative of g_{00}, since $\partial_0A=B$. We make the assumption that B does not depend on the time derivative of g_{00}, hence, $\delta'A=0$. For $\delta'B$ we have (we assume that J^i and D do not depend explicitly on g_{00}).

$$\frac{\delta'B(x)}{\delta g_{00}(y)}=J^i(x)\partial_i\left(\frac{\delta^4(x-y)}{2(-g_{00}(y))^{1/2}}\right)-\frac{\delta^4(x-y)}{2(-g_{00}(y))^{1/2}}\partial_iJ^i(x)+i\frac{\delta^4(x-y)}{2(-g_{00}(y))^{1/2}}D(x). \quad (11)$$

Therefore from (7) and (8) we have, in the limit of zero external field, $g_{00}=-1$, the desired result [Eq. (4)].

These arguments obviously apply to each member of a triplet or octet of currents. Hence the ETCR which form the basis of this paper are, in momentum space

$$[H_{00}(\mathbf{q}),V_0{}^a(\mathbf{p})]=p_iV_i{}^a(\mathbf{p}+\mathbf{q}),$$
$$[H_{00}(\mathbf{q}),A_0{}^a(\mathbf{p})]=p_iA_i{}^a(\mathbf{p}+\mathbf{q})-D^a(\mathbf{p}+\mathbf{q}), \quad (12)$$

where our Fourier transforms are defined by $O(\mathbf{p})=\int d^3x\, e^{-\mathbf{p}\cdot\mathbf{x}}O(\mathbf{x},t)$.

III. DERIVATION OF THE SPACE-TIME LOCAL COMMUTATORS

A. $SU(2)$, Derivation of $[V_0{}^a(\mathbf{x},t)V_i{}^b(\mathbf{y},t)]$

We shall assume that the time components of the isotopic-spin currents satisfy a local $SU(2)$ algebra, i.e., in momentum space

$$[V_0{}^a(\mathbf{p}),V_0{}^b(\mathbf{q})]=i\epsilon_{abc}V_0{}^c(\mathbf{p}+\mathbf{q}),\quad a,b,c=1,2,3. \quad (13)$$

By evaluating the Jacobi identity

$$[H_{00}(\mathbf{p}),[V_0{}^a(\mathbf{q}),V_0{}^b(\mathbf{k})]]+[V_0{}^b(\mathbf{k}),[H_{00}(\mathbf{p}),V_0{}^a(\mathbf{q})]]+[V_0{}^a(\mathbf{q}),[V_0{}^b(\mathbf{k}),H_{00}(\mathbf{p})]]=0, \quad (14)$$

we will learn about the ETCR of the space and time components. The above identity gives

$$k_i[V_0{}^a(\mathbf{q}),V_i{}^b(\mathbf{p}+\mathbf{k})]-q_i[V_0{}^b(\mathbf{k}),V_i{}^a(\mathbf{p}+\mathbf{q})]=i(q_i+k_i)\epsilon_{abc}V_i{}^c(\mathbf{p}+\mathbf{q}+\mathbf{k}). \quad (15)$$

Setting $\mathbf{q}=\mathbf{0}$ (or $\mathbf{k}=\mathbf{0}$), we derive the commutator of $V_i{}^b(\mathbf{p})$ with the isotopic charge $V_0{}^a(\mathbf{0})=\int d^3x\, V_0{}^a(\mathbf{x},t)$.

$$[V_0{}^a(\mathbf{0}),V_i{}^b(\mathbf{p})]=i\epsilon_{abc}V_i{}^c(\mathbf{p}). \quad (16)$$

This expresses the fact that $V_i{}^b$ transforms as an isovector. The local ETCR may be written (without loss of generality) as

$$[V_0^a(\mathbf{q}),V_i^b(\mathbf{p})]=i\epsilon_{abc}V_i^c(\mathbf{p}+\mathbf{q})+S_{VV,i}^{ab}(\mathbf{q},\mathbf{p}). \quad (17)$$

This serves to define $S_{VV,i}^{ab}$, which will satisfy by virtue of (15), the equation

$$k_iS_{VV,i}^{ab}(\mathbf{q},\ \mathbf{p}+\mathbf{k})=q_iS_{VV,i}^{ba}(\mathbf{k},\ \mathbf{p}+\mathbf{q}). \quad (18)$$

Clearly, $S_{VV,i}^{ab}(\mathbf{q},\mathbf{p})$ vanishes as $\mathbf{q}\rightarrow\mathbf{0}$, which reflects the fact that (16) holds.

Equation (18) places certain model-independent restrictions on the ST $S_{VV,i}^{ab}$. To extract these, differentiate (18) with respect to k_j and then set $k=0$. Defining

$$R_{VV,ij}^{ab}(\mathbf{p}+\mathbf{q})=\frac{\partial}{\partial k_j}S_{VV,i}^{ab}(\mathbf{k},\ \mathbf{p}+\mathbf{q})\,|_{\mathbf{k}=\mathbf{0}}, \quad (19a)$$

we have from (18)

$$S_{VV,j}^{ab}(\mathbf{q},\mathbf{p})=q_iR_{VV,j}^{ab}(\mathbf{p}+\mathbf{q}),\qquad S_{VV,j}^{ab}(\mathbf{0},\mathbf{p})=\mathbf{0}. \quad (19b)$$

Substituting (19b) into 18 yields

$$q_ik_jR_{VV,ij}^{ab}(\mathbf{p}+\mathbf{q}+\mathbf{k})=q_jk_iR_{VV,ij}^{ba}(\mathbf{p}+\mathbf{q}+\mathbf{k}). \quad (19c)$$

Differentiating this with respect to q_i and k_j and then setting $\mathbf{q}=\mathbf{0}=\mathbf{k}$, gives finally

$$R_{VV,ij}^{ab}(\mathbf{p})=R_{VV,ji}^{ba}(\mathbf{p}). \quad (19d)$$

Thus the local ETCR of V_0^a with H_{00} and with V_0^b imply that the ETCR of V_0^a and V_i^b is (in position space)

$$[V_0^a(\mathbf{x},t),V_i^b(\mathbf{y},t)]=i\epsilon_{abc}V_i^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})-iR_{VV,ji}^{ab}(\mathbf{y},t)\partial_j\delta(\mathbf{x}-\mathbf{y}), \quad (20a)$$

and the following symmetry holds:

$$R_{VV,ij}^{ab}(x)=R_{VV,ji}^{ba}(x). \quad (20b)$$

Equations (20) exhibit our first result: The derivation of the $[V_0^a,V_i^b]$ ETCR, the fact that the ST involves at most one derivative of a δ function, and model-independent symmetry restrictions on the ST.

B. $SU(2)\times SU(2)$, Derivation of $[V_0^a(\mathbf{x},t),A_i^b(\mathbf{y},t)]$, $[A_0^a(\mathbf{x},t),V_i^b(\mathbf{y},t)]$, and $[A_0^a(\mathbf{x},t),A_i^b(\mathbf{y},t)]$

We assume the $SU(2)\times SU(2)$ algebra for the time components of the vector and axial-vector charge densities

$$[K_0^a(\mathbf{p}),L_0^b(\mathbf{q})]=i\epsilon_{abc}(K\cdot L)_0^c(\mathbf{p}+\mathbf{q}),\qquad a,\ b,\ c=1,\ 2,\ 3, \quad (21a)$$

where K and L represent V or A, with the multiplication law

$$A\cdot A=V,\quad V\cdot V=V,\quad A\cdot V=V\cdot A=A. \quad (21b)$$

The axial-vector current is not conserved; its divergence will be defined as

$$i\partial^\mu A_\mu^a(x)=D^a(x). \quad (22)$$

Because of this lack of conservation, we shall have to make additional assumptions in order to derive even the once integrated space-time ETCR. It will be shown that these additional assumptions are necessary as well as sufficient.

We write (without loss of generality)

$$[K_0^a(\mathbf{p}),L_i^b(\mathbf{q})]=i\epsilon_{abc}(K\cdot L)_i^c(\mathbf{p}+\mathbf{q})+S_{KL,i}^{ab}(\mathbf{p},\mathbf{q}). \quad (23)$$

The Jacobi identity for $H_{00}(\mathbf{p})$, $V_0^a(\mathbf{q})$, and $A_0^b(\mathbf{k})$ leads to

$$k_iS_{VA,i}^{ab}(\mathbf{q},\ \mathbf{p}+\mathbf{k})-q_iS_{AV,i}^{ba}(\mathbf{k},\ \mathbf{p}+\mathbf{q})=[V_0^a(\mathbf{q}),D^b(\mathbf{p}+\mathbf{k})]-i\epsilon_{abc}D^c(\mathbf{p}+\mathbf{q}+\mathbf{k}). \quad (24)$$

Setting $\mathbf{k}=\mathbf{0}$, we derive the local ETCR

$$[V_0^a(\mathbf{q}),D^b(\mathbf{p})]=i\epsilon_{abc}D^c(\mathbf{p}+\mathbf{q})+q_iS_{VD,i}^{ab}(\mathbf{p}+\mathbf{q}), \quad (25a)$$

where $S_{VD,i}^{ab}$ is given, according to (24), by

$$S_{VD,i}^{ab}(\mathbf{p})=-S_{AV,i}^{ba}(\mathbf{0},\mathbf{p}). \quad (25b)$$

Equation (25a), when $\mathbf{q}=\mathbf{0}$, gives the not unexpected result that $D^b(\mathbf{p})$ transforms as an isovector. Combining (25) with (24) yields

$$k_iS_{VA,i}^{ab}(\mathbf{q},\ \mathbf{p}+\mathbf{k})-q_iS_{AV,i}^{ba}(\mathbf{k},\ \mathbf{p}+\mathbf{q})=-q_iS_{AV,i}^{ba}(\mathbf{0},\ \mathbf{p}+\mathbf{q}+\mathbf{k}). \quad (26)$$

This condition on the ST replaces the simpler condition of $SU(2)$, Eq. (18).

To exploit this condition, we proceed as before. Differentiate by k_j, set $\mathbf{k}=\mathbf{0}$ and, define

$$R_{KL,ij}^{ab}(\mathbf{p}+\mathbf{x})=\frac{\partial}{\partial k_j}S_{LK,i}^{ba}(\mathbf{k},\ \mathbf{p}+\mathbf{q})\,|_{\mathbf{k}=\mathbf{0}},\qquad \mathcal{R}_{KL,ij}^{ab}(p)=\frac{\partial}{\partial p_j}S_{LK,i}^{ba}(\mathbf{0},\mathbf{p}). \quad (27a)$$

We then have from (26a)

$$S_{VA,j}^{ab}(\mathbf{q},\mathbf{p})=q_iR_{VA,ij}^{ab}(\mathbf{p}+\mathbf{q})-q_i\mathcal{R}_{VA,ij}^{ab}(\mathbf{p}+\mathbf{q}). \quad (27b)$$

Next we differentiate (26) by q_j and set $\mathbf{q}=\mathbf{0}$, yielding

$$S_{AV,j}^{ab}(\mathbf{k},\mathbf{p})=q_iR_{AV,ij}^{ab}(\mathbf{p}+\mathbf{q})+S_{AV,j}^{ab}(\mathbf{0},\ \mathbf{p}+\mathbf{q}). \quad (27c)$$

The results (27b) and (27c) are to be compared to (19b). Finally, we obtain the symmetries of $R_{VV,ij}^{ab}$. Differentiate (27b) or (27c) by q_i and set $\mathbf{q}=\mathbf{0}$. The result is

$$R_{AV,ij}^{ab}(\mathbf{p})=R_{VA,ij}^{ba}(\mathbf{p})-\mathcal{R}_{VA,ij}^{ba}(\mathbf{p}), \quad (27d)$$

which can be compared to the previous (20b).

It is easy to verify that (26) places no further restrictions on the form of the ST. We note that in general, the present results (26), (27b), (27c), and (27d) differ from the corresponding previous results (18), (19b), and (20b), by an additional term which is present if there is a ST in the $[V_0^a,D^b]$ ETCR.

Equation (27b) shows that $S_{VA,j}{}^{ab}(\mathbf{0},\mathbf{p})=0$. Hence the once integrated $SU(2)\times SU(2)$ ETCR holds for $V_0{}^a$ and $A_i{}^b$. It further follows from (27b) that the local $SU(2)\times SU(2)$ ETCR of $V_0{}^a$ and $A_i{}^b$ contains a ST, with at most one derivative of the δ function. No further assumptions are necessary to arrive at this result, which shows that (2b) holds for $SU(2)\times SU(2)$. The same is *not* true for $A_0{}^a$ and $V_i{}^b$. The once integrated ETCR between these two operators is of the $SU(2)\times SU(2)$ form if and only if $S_{AV,i}{}^{ab}(\mathbf{0},\mathbf{p})=0$. Thus it is seen from (25b) that a necessary and sufficient condition for the validity of (2c) is that the ETCR between $V_0{}^a$ and D^b contain no ST. We therefore make this assumption. Consequently, $S_{AV,j}{}^{ab}(\mathbf{0},\mathbf{p})$ and $\mathcal{R}_{VA,ij}{}^{ab}(\mathbf{p})$ vanish, and the ST present in the local $[A_0{}^a,V_j{}^b]$ ETCR, contains at most one derivative of the δ functions.

Next, we examine the $[A_0{}^a,A_i{}^b]$ ETCR. The Jacobi identity for $H_{00}(\mathbf{p})$, $A_0{}^a(\mathbf{q})$, $A_0{}^b(\mathbf{k})$ yields

$$k_iS_{AA,i}{}^{ab}(\mathbf{q},\,\mathbf{p}+\mathbf{k})-q_iS_{AA,i}{}^{ba}(\mathbf{k},\,\mathbf{p}+\mathbf{q})=[A_0{}^a(\mathbf{q}),D^b(\mathbf{p}+\mathbf{k})]-[A_0{}^b(\mathbf{k}),D^a(\mathbf{p}+\mathbf{q})]. \quad (28)$$

Setting $\mathbf{k}$ and $\mathbf{q}$ to zero and introducing σ^{ab} by the definition

$$[A_0{}^a(\mathbf{0}),D^b(\mathbf{p})]=\sigma^{ab}(\mathbf{p}), \quad (29a)$$

we have by virtue of (28)

$$\sigma^{ab}(\mathbf{p})=\sigma^{ba}(\mathbf{p}). \quad (29b)$$

The local ETCR between $A_0{}^a$ and D^b may, therefore, be taken to be of the form

$$[A_0{}^a(\mathbf{q}),D^b(\mathbf{p})]=\sigma^{ab}(\mathbf{p}+\mathbf{q})+S_{AD}{}^{ab}(\mathbf{q},\mathbf{p}), \quad (30a)$$

$$S_{AD}{}^{ab}(\mathbf{0},\mathbf{p})=0. \quad (30b)$$

Therefore combining (28) with (30) gives

$$k_iS_{AA,i}{}^{ab}(\mathbf{q},\,\mathbf{p}+\mathbf{k})-q_iS_{AA,i}{}^{ab}(\mathbf{k},\,\mathbf{p}+\mathbf{q})=S_{AD}{}^{ab}(\mathbf{q},\,\mathbf{p}+\mathbf{k})-S_{AD}{}^{ba}(\mathbf{k},\,\mathbf{p}+\mathbf{q}), \quad (31a)$$

or

$$S_{AD}{}^{ab}(\mathbf{q},\mathbf{p})=-q_iS_{AA,i}{}^{ba}(\mathbf{0},\,\mathbf{p}+\mathbf{q}). \quad (31b)$$

Reinserting (31b) into (31a) yields an equation which implies, by an analysis similar to the previous, that

$$S_{AA,j}{}^{ab}(\mathbf{q},\mathbf{p})=q_iR_{AA,ij}{}^{ab}(\mathbf{p}+\mathbf{q})+S_{AA,j}{}^{ab}(\mathbf{0},\,\mathbf{p}+\mathbf{q}), \quad (32a)$$

$$R_{AA,ij}{}^{ab}(\mathbf{p})=R_{AA,ij}{}^{ba}(\mathbf{p}). \quad (32b)$$

Since the once integrated $[A_0{}^a,A_i{}^b]$ ETCR is of the $SU(2)\times SU(2)$ form if and only if $S_{AA,j}{}^{ab}(\mathbf{0},\mathbf{p})=0$, according to (31b), a necessary and sufficient condition for (2d) to hold within $SU(2)\times SU(2)$ is that no ST be present in the $[A_0{}^a,D^b]$ ETCR. We now make this assumption. We have thus shown that

$$[K_0{}^a(\mathbf{x},t),L_i{}^b(\mathbf{y},t)]=i\epsilon_{abc}(K\cdot L)_i{}^c(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})-iR_{KL,ji}{}^{ab}(\mathbf{y},t)\partial_j\delta(\mathbf{x}-\mathbf{y}),a,\,b=1,\,2,\,3, \quad (33a)$$

$$R_{KL,ij}{}^{ab}(x)=R_{LK,ji}{}^{ba}(x). \quad (33b)$$

Note that the ST may also be written in the form

$$\frac{\partial}{\partial x^i}(R_{KL,ji}{}^{ab}(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y}))=\frac{\partial}{\partial x^i}\times(R_{KL,ji}{}^{ab}(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y})). \quad (33c)$$

Hence the space-time current ETCR is always given by a term which is the simple quark commutator plus a ST which can be written as a total three-divergence.

C. Extension to $SU(3)\times SU(3)$

It is clear that for the case of $SU(3)\times SU(3)$, arguments identical to those given for $SU(2)\times SU(2)$ yield identical results. Thus, we may conclude that the $SU(3)\times SU(3)$ ETCR are of the form (33) with f_{abc} replacing ϵ_{abc} and $a,\,b,\,c=1,\,\cdots,\,8$.

D. Discussion

The results of this investigation followed from the ETCR of the energy density with the charge density (12). Although we were able to derive (12) from the action principle, we now wish to inquire whether it is possible to derive our results from relations less stringent than (12). Specifically, we now assume only the ETCR dictated by Poincaré invariance Eqs. (6).

First, we assume only (6a). Then we evaluate the Jacobi identity for H, $V_0{}^a(\mathbf{q})$, and $V_0{}^b(\mathbf{k})$ as in (14) with $\mathbf{p}=\mathbf{0}$. Equation (15) holds with $\mathbf{p}=\mathbf{0}$, and we may still use (17), since that entails no loss of generality. The restriction on $S_{VV,i}{}^{ab}(\mathbf{q},\mathbf{p})$, which now replaces (18), is

$$k_iS_{VV,i}{}^{ab}(\mathbf{q},\mathbf{k})=q_iS_{VV,i}{}^{ba}(\mathbf{k},\mathbf{q}). \quad (34a)$$

[Note that (34a) follows directly from (18) by setting $\mathbf{p}=\mathbf{0}$.] This then implies that

$$S_{VV,j}{}^{ab}(\mathbf{q},\mathbf{0})=q_iR_{VV,ij}{}^{ab}(\mathbf{q}), \quad (35a)$$

$$p_iS_{VV,i}{}^{ab}(\mathbf{0},\mathbf{p})=0. \quad (35b)$$

These equations contain less information than the corresponding Eq. (19b). In particular, we cannot conclude that $S_{VV,j}{}^{ab}(\mathbf{0},\mathbf{p})=0$, i.e., that the once integrated $[V_0{}^a,V_i{}^b]$ ETCR is of the form (16). Obviously, the ETCR involving the axial current also cannot be determined. Thus, we conclude that (6a) alone is insufficient to yield any useful information about the space-time current components ETCR.

Next we assume the validity of (6b) in addition to (6a). The Jacobi identity for M^{0i}, $V_0{}^a(\mathbf{q})$, and $V_0{}^a(\mathbf{k})$, together with (17), yields the restriction on the ST $S_{VV,i}{}^{ab}$

$$k_i\frac{\partial}{\partial k_j}S_{VV,i}{}^{ba}(\mathbf{q},\mathbf{k})=q_i\frac{\partial}{\partial q_j}S_{VV,i}{}^{ba}(\mathbf{k},\mathbf{q}). \quad (35c)$$

[Note that this condition may be obtained directly from (18) by differentiating (18) by p_j and setting $\mathbf{p}=\mathbf{0}$.]

Eq. (35c) then implies that

$$p_i \frac{\partial}{\partial p_j} S_{VV,i}{}^{ab}(\mathbf{0},\mathbf{p})=0. \tag{36}$$

By using the identity

$$S_{VV,j}{}^{ab}(\mathbf{0},\mathbf{p})=\frac{\partial}{\partial p_j}\{p_i S_{VV,i}{}^{ab}(\mathbf{0},\mathbf{p})\} - p_i \frac{\partial}{\partial p_j} S_{VV,i}{}^{ab}(\mathbf{0},\mathbf{p}), \tag{37a}$$

we may conclude that

$$S_{VV,j}{}^{ab}(\mathbf{0},\mathbf{p})=0, \tag{37b}$$

since each of the two terms on the right-hand side of (37a) vanishes by virtue of (35b) and (36). Thus Poincaré invariance is sufficient to prove that the once integrated $[V_0{}^a, V_i{}^b]$ ETCR is of the form (16). Equation (35c) further implies that

$$\frac{\partial}{\partial p_l} S_{VV,j}{}^{ab}(\mathbf{q},\mathbf{p})\big|_{\mathbf{p}=0} = q_i \frac{\partial}{\partial q_l} R_{VV,ij}{}^{ab}(\mathbf{q}). \tag{38}$$

It is seen that (35a) and (38) are the first two terms in an expansion in powers of $\mathbf{p}$ of (19b). However, we cannot obtain from the present considerations, the full Eq. (19b), and therefore, cannot conclude that the ST contains only one derivative of a δ function. Moreover, we have no way of deriving the symmetry properties of (19d). Similar considerations apply to the ETCR involving axial currents.

To summarize: Poincaré invariance is sufficient to determine the once integrated ETCR, while information about ST can be arrived at only when more detailed information is available about the ETCR of the energy density with the charge density.

IV. FURTHER STUDY OF THE SCHWINGER TERMS

(a) In the present section, we examine further the ST $S_{KL,i}{}^{ab}(\mathbf{p},\mathbf{q})$. [We restrict ourselves for simplicity to $SU(2)\times SU(2)$.] We have already derived the restrictions

$$S_{KL,i}{}^{ab}(\mathbf{p},\mathbf{q})=p_j R_{KL,ji}{}^{ab}(\mathbf{p}+\mathbf{q}), \tag{39a}$$

$$R_{KL,ji}{}^{ab}=R_{LK,ij}{}^{ba}. \tag{39b}$$

The Jacobi identity for $K_0{}^a(\mathbf{p})$, $L_0{}^b(\mathbf{q})$, and $M_i{}^c(\mathbf{k})$ (where K, L, $M=A$, V) leads to

$$\begin{aligned} q_j[K_0{}^a(\mathbf{p}),\, R_{LM,ji}{}^{bc}(\mathbf{q}+\mathbf{k})] - p_j[L_0{}^b(\mathbf{q}),\, R_{KM,ji}{}^{ac}(\mathbf{p}+\mathbf{k})] \\ = i\epsilon_{abd}(p_j+q_j)R_{K\cdot L\ M,ji}{}^{dc}(\mathbf{p}+\mathbf{q}+\mathbf{k}) \\ - i\epsilon_{bcd}p_j R_{K\ L\cdot M,ji}{}^{ad}(\mathbf{p}+\mathbf{q}+\mathbf{k}) \\ + i\epsilon_{acd}q_j R_{L\ K\cdot M,ji}{}^{bd}(\mathbf{p}+\mathbf{q}+\mathbf{k}). \end{aligned} \tag{40}$$

An immediate consequence of this equation is (set $\mathbf{p}=\mathbf{0}$, differentiate by q_j, and set $\mathbf{q}=\mathbf{0}$)

$$[K_0{}^a(0), R_{LM,ji}{}^{bc}(\mathbf{k})] = i\epsilon_{abd}R_{K\cdot L\ M,ji}{}^{dc}(\mathbf{k}) + i\epsilon_{acd}R_{L\ K\cdot M,ji}{}^{db}(\mathbf{k}). \tag{41a}$$

Using the same techniques as in the derivation of (39), it is easy to see that the local version of (41a) may possess a further ST which can involve at most one derivative of a δ function

$$[K_0{}^a(\mathbf{p}), R_{LM,ji}{}^{bc}(\mathbf{k})] = i\epsilon_{abd}R_{K\cdot L\ M,ji}{}^{dc}(\mathbf{p}+\mathbf{k}) + i\epsilon_{acd}R_{L\ K\cdot M,ji}{}^{bd}(\mathbf{p}+\mathbf{k}) + p_l R_{KLM,lji}{}^{abc}(\mathbf{p}+\mathbf{k}). \tag{41b}$$

The symmetries of $R_{LM,ji}{}^{bc}$, (39b), together with (40) then impose a symmetry restriction on $R_{KLM,lji}{}^{abc}$. This object is invariant under permutations of the three quantities (aKl), (bLj), *and* (cMi).

We may decompose $R_{KL,ji}{}^{ab}$ into the following:

$$R_{LM,ij}{}^{ab}=\delta_{ij}R_{LM}{}^{ab}+\epsilon_{ijk}R_{LM,k}{}^{ab}+\tilde{R}_{LM,ij}{}^{ab}, \tag{41c}$$

$$\begin{aligned} R_{LM}{}^{ab} &= \tfrac{1}{3}R_{LM,ii}{}^{ab}, \\ R_{LM,k}{}^{ab} &= \tfrac{1}{2}\epsilon_{ijk}R_{LM,ij}{}^{ab}, \\ \tilde{R}_{LM}{}^{ab} &= \tfrac{1}{2}(R_{LM,ij}{}^{ab}+R_{LM,ji}{}^{ab}-\tfrac{2}{3}\delta_{ij}R_{LM,kk}{}^{ab}). \end{aligned} \tag{41d}$$

Evidently, $R_{LM,k}{}^{ab}$ is a vector and $R_{LM,ij}{}^{ab}$ is a symmetric and traceless tensor in position space. The symmetries (39b) require now that

$$\begin{aligned} R_{LM}{}^{ab} &= R_{ML}{}^{ba}, \\ R_{LM,k}{}^{ab} &= -R_{ML,k}{}^{ba}, \\ \tilde{R}_{LM,ij}{}^{ab} &= \tilde{R}_{LM,ji}{}^{ab}=\tilde{R}_{ML,ji}{}^{ba}. \end{aligned} \tag{42}$$

Note that the position-space scalar part of $R_{LM,ij}{}^{ab}$, $R_{LM}{}^{ab}$ is symmetric in a and b (i.e., contains no isospin one, vector part) when $L=M=A$, V; however, $R_{AV}{}^{ab}$ may contain an isospin one part.

It follows from (42) that

$$S_{KL,i}{}^{ab}(\mathbf{p},\mathbf{q})+S_{LK,i}{}^{ab}(\mathbf{p},\mathbf{q})-S_{KL,i}{}^{ba}(\mathbf{p},\mathbf{q})-S_{LK,i}{}^{ba}(\mathbf{p},\mathbf{q}) = 3p_j\epsilon_{jil}[R_{KL,l}{}^{ab}(\mathbf{p}+\mathbf{q})+R_{LK,l}{}^{ab}(\mathbf{p}+\mathbf{q})]. \tag{43}$$

On the other hand, the Adler-Callan hypothesis (3) requires the left-hand side (43) to vanish. Thus, that hypothesis can hold if and only if

$$R_{KL,l}{}^{ab}+R_{LK,l}{}^{ab}=0. \tag{44}$$

Therefore, when $R_{KL,ij}{}^{ab}$ contains no (position-space) vector component, the Adler-Callan hypothesis is satisfied. Specifically, when the ST is of the particularly simple form $S_{KL,ij}{}^{ab}(\mathbf{p}+\mathbf{q})=p_i R_{KL}{}^{ab}(\mathbf{p}+\mathbf{q})$, i.e.,

$$R_{KL,ij}{}^{ab}=\delta_{ij}R_{KL}{}^{ab},$$

(as in the σ model) (3) holds.

(b) In a scalar-meson theory, i.e., where the basic operators are position-space scalars, vector and tensor operators can arise only through some c-number operation. A particularly simple and obvious operation which can yield tensor operators from scalar operators is (in momentum space) multiplication by c-number vectors. (In position space, this corresponds to differentiation.) Thus a natural and simple assumption, for a scalar-meson theory, is that

$$R_{KL,ij}{}^{ab}(p)=\delta_{ij}\mathcal{S}_{KL}{}^{ab}(\mathbf{p})+\epsilon_{ijk}p_k\mathcal{T}_{KL}{}^{ab}(\mathbf{p})+p_ip_j\mathcal{U}_{KL}{}^{ab}(\mathbf{p}). \quad (45)$$

We shall now show, that this simplest assumption implies that $\mathcal{T}_{KL}{}^{ab}=0$ and $\mathcal{U}_{KL}{}^{ab}=\delta_{ab}\mathcal{U}_{KL}$, $\mathcal{U}_{AV}=\mathcal{U}_{VA}$, $\mathcal{U}_{AA}=\mathcal{U}_{VV}$. To establish this, we return to (40), substitute (45) for $R_{KL,ij}{}^{ab}$, set $k_i=\lambda\epsilon_{inm}p_nq_m$, multiply (40) by k_i/λ, and set $\lambda=0$. This then leads to

$$\epsilon_{bcd}\mathcal{T}_{K\ L\cdot M}{}^{ab}+\epsilon_{acd}\mathcal{T}_{L\ K\cdot M}{}^{bd}=0, \quad (46)$$

which upon multiplication by ϵ_{bce} shows that $\mathcal{T}_{KL}{}^{ab}=0$. Since $\mathcal{T}_{KL}{}^{ab}$ vanishes, the only terms in (40) which survive are proportional to p_i, q_i, and k_i. Equating to zero, the coefficients of these arbitrary vectors give

$$[K_0{}^a(\mathbf{p}),\mathcal{U}_{LM}{}^{bc}(\mathbf{q})]=i\epsilon_{abd}\mathcal{U}_{K\cdot L\ M}{}^{dc}(\mathbf{p}+\mathbf{q})+i\epsilon_{acd}\mathcal{U}_{L\ K\cdot M}{}^{bd}(\mathbf{p}+\mathbf{q})=0, \quad (47)$$

$$[K_0{}^a(\mathbf{p}),\mathcal{S}_{LM}{}^{bc}(\mathbf{q})]=i\epsilon_{abd}\mathcal{S}_{K\cdot L\ M}{}^{dc}(\mathbf{p}+\mathbf{q})+i\epsilon_{acd}\mathcal{S}_{L\ K\cdot M}{}^{bd}(\mathbf{p}+\mathbf{q}). \quad (48)$$

Equation (47), upon multiplication by ϵ_{abe}, has the consequence that

$$\mathcal{U}_{KL}{}^{ab}=\delta_{ab}\mathcal{U}_{KL},\quad \mathcal{U}_{AA}=\mathcal{U}_{VV},\quad \mathcal{U}_{AV}=\mathcal{U}_{VA}. \quad (49)$$

Thus, when the vector and tensor parts of $R_{KL,ij}{}^{ab}(\mathbf{p})$ are proportional to p_i, the Adler-Callan hypothesis is satisfied. Also in that instance the isospin-one vector component can occur only in $\mathcal{S}_{AV}{}^{ab}$. Finally, it is seen that the assumption (45) shows that there are no further ST in the $[K_0{}^a,R_{LM,ij}{}^{bc}]$ ETCR; i.e., $R_{KLM,ijk}{}^{abc}=0$. Conversely, if $R_{KL,ij}{}^{ab}$ contains a nonvanishing position-space vector part, then the vector and tensor structure of $R_{KL,ij}{}^{ab}$ must be more complicated than (45).

In a vector-meson theory, there is no reason to expect (45) to hold since the theory provides vector and tensor operators which are not obtained by a c-number operation on scalar operators.

(c) We now decompose the isospin structure of $R_{KL,ij}{}^{ab}$.

$$R_{KL,ij}{}^{ab}=\delta_{ab}S_{KL,ij}+\epsilon_{abc}T_{KL,ij}{}^c+U_{KL,ij}{}^{ab}, \quad (50a)$$

$$\begin{aligned} S_{KL,ij}&=\tfrac{1}{3}R_{KL,ij}{}^{aa},\\ T_{KL,ij}{}^c&=\tfrac{1}{2}\epsilon_{cab}R_{KL,ij}{}^{ab},\\ U_{KL,ij}{}^{ab}&=\tfrac{1}{2}(R_{KL,ij}{}^{ab}+R_{KL,ij}{}^{ba}-\tfrac{2}{3}\delta_{ab}R_{KL,ij}{}^{cc}).\end{aligned} \quad (50b)$$

$S_{KL,ij}$, $T_{KL,ij}{}^c$, and $U_{KL,ij}{}^{ab}$ are the isospin parts zero, one, and two of $R_{KL,ij}{}^{ab}$. The symmetries on these are

$$\begin{aligned} S_{KL,ij}&=S_{LK,ji},\\ T_{KL,ij}{}^c&=-T_{LK,ji}{}^c,\\ U_{KL,ij}{}^{ab}&=U_{KL,ij}{}^{ba}=U_{LK,ji}{}^{ab}.\end{aligned} \quad (50c)$$

Next we decompose the ETCR $[K_0{}^a(\mathbf{0}),R_{LM,ij}{}^{bc}(\mathbf{p})]$, Eq. (41a), and exhibit explicitly the various terms. The ETCR with $V_0{}^a$ are (we suppress the momentum arguments which are the same throughout the remainder of this section: Zero for $K_0{}^a$ and p for $R_{LM,ij}{}^{bc}$)

$$[V_0{}^a,S_{KL,ij}]=0, \quad (51a)$$

$$[V_0{}^a,T_{KL,ij}{}^b]=i\epsilon_{abc}T_{KL,ij}{}^c, \quad (51b)$$

$$[V_0{}^a,U_{KL,ij}{}^{bc}]=i\epsilon_{abd}U_{KL,ij}{}^{dc}+i\epsilon_{acd}U_{KL,ij}{}^{bd}. \quad (51c)$$

These relations are not unexpected. They merely reflect the fact that $V_0{}^a(\mathbf{0})$ is a generator of isospin rotations and that $S_{KL,ij}$, $T_{KL,ij}{}^b$, $U_{KL,ij}{}^{bc}$ transform like $I=0$, 1, 2 objects, respectively.

The ETCR with $A_0{}^a(\mathbf{0})$ are more complicated. These are, explicitly

$$\begin{aligned} [A_0{}^a,S_{VV,ij}]&=\tfrac{2}{3}i(T_{VA,ij}{}^a-T_{AV,ij}{}^a),\\ [A_0{}^a,S_{AA,ij}]&=-\tfrac{2}{3}i(T_{VA,ij}{}^a-T_{AV,ij}{}^a),\\ [A_0{}^a,S_{AV,ij}]&=\tfrac{2}{3}i(T_{AA,ij}{}^a-T_{VV,ij}{}^a),\end{aligned} \quad (52a)$$

$$\begin{aligned} [A_0{}^a,T_{VV,ij}{}^b]&=\tfrac{1}{2}i\epsilon_{abc}(T_{VA,ij}{}^c+T_{AV,ij}{}^c)+i\delta_{ab}(S_{AV,ij}-S_{VA,ij})+\tfrac{1}{2}i(U_{VA,ij}{}^{ab}-U_{AV,ij}{}^{ab}),\\ [A_0{}^a,T_{AA,ij}{}^b]&=\tfrac{1}{2}i\epsilon_{abc}(T_{AV,ij}{}^c+T_{VA,ij}{}^c)+i\delta_{ab}(S_{AV,ij}-S_{VA,ij})+\tfrac{1}{2}i(U_{AV,ij}{}^{ab}-U_{VA,ij}{}^{ab}),\\ [A_0{}^a,T_{AV,ij}{}^b]&=\tfrac{1}{2}i\epsilon_{abc}(T_{AA,ij}{}^c+T_{VV,ij}{}^c)+i\delta_{ab}(S_{VV,ij}-S_{AA,ij})+\tfrac{1}{2}i(U_{AA,ij}{}^{ab}-U_{VV,ij}{}^{ab}),\end{aligned} \quad (52b)$$

$$\begin{aligned} [A_0{}^a,U_{VV,ij}{}^{bc}]&=\tfrac{1}{2}i\epsilon_{abd}(U_{AV,ij}{}^{dc}+U_{VA,ij}{}^{dc})+\tfrac{1}{2}i\epsilon_{acd}(U_{AV,ij}{}^{bd}+U_{VA,ij}{}^{bd})+\tfrac{1}{2}i\delta_{ac}(T_{AV,ij}{}^b-T_{VA,ij}{}^b)\\&\quad+\tfrac{1}{2}i\delta_{ab}(T_{AV,ij}{}^c-T_{VA,ij}{}^c)-\tfrac{1}{3}i\delta_{bc}(T_{AV,ij}{}^a-T_{VA,ij}{}^a),\\ [A_0{}^a,U_{AA,ij}{}^{bc}]&=\tfrac{1}{2}i\epsilon_{abd}(U_{VA,ij}{}^{dc}+U_{AV,ij}{}^{dc})+\tfrac{1}{2}i\epsilon_{acd}(U_{VA,ij}{}^{bd}+U_{AV,ij}{}^{bd})+\tfrac{1}{2}i\delta_{ac}(T_{VA,ij}{}^b-T_{AV,ij}{}^b)\\&\quad+\tfrac{1}{2}i\delta_{ab}(T_{VA,ij}{}^c-T_{AV,ij}{}^c)-\tfrac{1}{3}i\delta_{bc}(T_{VA,ij}{}^a-T_{VA,ij}{}^a),\\ [A_0{}^a,U_{AV,ij}{}^{bc}]&=\tfrac{1}{2}i\epsilon_{abd}(U_{VV,ij}{}^{dc}+U_{AA,ij}{}^{dc})+\tfrac{1}{2}i\epsilon_{acd}(U_{VV,ij}{}^{bd}+U_{AA,ij}{}^{bd})+\tfrac{1}{2}i\delta_{ac}(T_{VV,ij}{}^b-T_{AA,ij}{}^b)\\&\quad+\tfrac{1}{2}i\delta_{ab}(T_{VV,ij}{}^c-T_{AA,ij}{}^c)-\tfrac{1}{3}i\delta_{bc}(T_{VV,ij}{}^a-T_{AA,ij}{}^a).\end{aligned} \quad (52c)$$

Examining this array of relations and comparing it to (51), it is seen that $A_0{}^a(\mathbf{0})$ does *not* act as a generator of chiral isospin rotations. By this we mean that commutating $A_0{}^a(\mathbf{0})$ with an object of definite isospin (zero, one, two) does not produce an object of the same isospin (zero, one, two, respectively), but introduces quantities with different isospin.

If we were to demand that (52a) be of the same form as (51a), we would have to have

$$T_{VA,ij}{}^{a}=T_{AV,ij}{}^{a}, \quad T_{AA,ij}{}^{a}=T_{VV,ij}{}^{a}. \tag{53}$$

Equations (53) and (52b) together then imply that

$$S_{AV,ij}=S_{VA,ij}, \quad S_{AA,ij}=S_{VA,ij}, \tag{54}$$

$$U_{AV,ij}{}^{ab}=U_{VA,ij}{}^{ab}, \quad U_{AA,ij}{}^{ab}=U_{VV,ij}{}^{ab}, \tag{55}$$

which assures that (52b) is of the form (51b). Finally, (52c) is seen to be of the form (51c) by virtue of (53). [It is easily seen that a similar chain of argument can be carried out by requiring (52b) to be of the form (51b), or by requiring that (52c) to be of the form (51c).] In particular, note that if the $I=1$ component of the ST satisfies the relation (53), then the AA ST is equal to the VV ST ,and the AV ST is equal to the VA ST, as operator identities. [The same conclusion follows if (54) or (55) holds.] This result is a generalization of Weinberg's recent calculation[7] which proved that if there is no $I=1$ ST, then the vacuum expectation value of $S_{AA,ij}$ equals that of $S_{VV,ij}$. It is also seen that the minimal assumption, which is necessary to prove Weinberg's result (rather than our more general result), is that the vacuum expectation value of $[A_0{}^a,T_{AV,ii}{}^b]$ vanishes. Finally, we note that if the symmetric state of affairs exists, so that (53), (54), and (55) hold, then we have from (50c) that

$$S_{AA,ij}=S_{AA,ji}=S_{VV,ij}=S_{VV,ji}, \quad S_{AV,ij}=S_{VA,ij}=S_{VA,ji}=S_{AV,ji}, \tag{56a}$$

$$T_{AA,ij}{}^{c}=-T_{AA,ji}{}^{c}=T_{VV,ij}{}^{c}=-T_{VV,ji}{}^{c}, \quad T_{AV,ij}{}^{c}=-T_{VA,ji}{}^{c}=T_{VA,ij}{}^{c}=-T_{AV,ji}{}^{c}, \tag{56b}$$

$$U_{AA,ij}{}^{ab}=U_{AA,ji}{}^{ab}=U_{VV,ij}{}^{ab}=U_{VV,ji}{}^{ab}, \quad U_{AV,ij}{}^{ab}=U_{VA,ji}{}^{ab}=U_{VA,ij}{}^{ab}=U_{AV,ji}{}^{ab}. \tag{56c}$$

These equations show that $S_{Kl,ij}$ and $U_{Kl,ij}{}^{ab}$ are symmetric in the position-space indices ij, while $T_{KL,ij}{}^{c}$ is antisymmetric. Therefore in the notation of (41), we have

$$R_{LM,k}{}^{ab}=\tfrac{1}{2}\epsilon_{abc}\epsilon_{ijk}T_{LM,ij}{}^{c}, \tag{56d}$$

and the Adler-Callan hypothesis is satisfied if and only if $T_{LM,ij}{}^{c}=0$.

Although it is very attractive to assume that (53) holds on the grounds of symmetry, model field theories, such as the σ theory, do not possess this property.[4]

V. THE SPACE-SPACE CURRENT COMMUTATOR

In this section, we study to what extent the previous techniques can be used to determine in a model-independent fashion, the ETCR of the spatial components of the current densities. We begin by a few definitions. The ETCR, in momentum space is of the form

$$[K_i{}^a(\mathbf{p}),L_j{}^b(\mathbf{q})]=i\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{p},\mathbf{q}). \tag{57}$$

Evidently, the following symmetry holds:

$$\mathcal{C}_{LK,ji}{}^{ba}(\mathbf{q},\mathbf{p})=-\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{p},\mathbf{q}). \tag{58}$$

We may consider $\mathcal{C}_{LK,ji}{}^{ab}$ to be a function of

$$\mathbf{P}\equiv\tfrac{1}{2}(\mathbf{p}+\mathbf{q}) \quad\text{and}\quad \mathbf{Q}\equiv\tfrac{1}{2}(\mathbf{p}-\mathbf{q}), \quad \mathcal{C}_{KL,ij}{}^{ab}(\mathbf{p},\mathbf{q})=\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q}). \tag{59}$$

Locality and causality require that $C_{LK,ij}{}^{ab}(\mathbf{P},\mathbf{Q})$ be polynomial in $\mathbf{Q}$, with coefficients that depend only on $\mathbf{P}$.

$$C_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q})=-C_{LK,ji}{}^{ba}(\mathbf{P},-\mathbf{Q})=C_{KL,ij}{}^{ab}(\mathbf{P})+\sum_{n=1}^{m}C_{KL,ij,k_1\cdots k_n}{}^{ab}(\mathbf{P})Q_{k_1}\cdots Q_{k_n}. \tag{60a}$$

The $C_{KL,ij,k_1\cdots k_n}{}^{ab}(\mathbf{P})$ are symmetric in the k indices and satisfy

$$C_{KL,ij}{}^{ab}(\mathbf{P})=-C_{LK,ji}{}^{ba}(\mathbf{P}), \quad C_{KL,ij,k_1\cdots k_n}{}^{ab}(\mathbf{P})=(-1)^{n+1}C_{LK,ij,k_1\cdots k_n}{}^{ba}(\mathbf{P}). \tag{60b}$$

The successive powers of $\mathbf{Q}$ in momentum space correspond to successive derivatives of δ functions in position space. Thus the form (60) allows for an arbitrary, but finite number of derivative of δ functions. It is seen that one integrated ETCR, say $\mathbf{p}=\mathbf{0}$ is given by

$$\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{0},\mathbf{q})=C_{KL,i}{}^{cb}(\mathbf{P},-\mathbf{P})=C_{KL,ij}{}^{ab}(\mathbf{P})+\sum_{n=1}^{m}(-1)^nC_{KL,ij,k_1\cdots k_n}{}^{ab}(\mathbf{P})P_{k_1}\cdots P_{k_n}, \tag{61a}$$

and the twice integrated ETCR $\mathbf{p}=\mathbf{0}=\mathbf{q}$, is

$$\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{0},\mathbf{0})=C_{KL,ij}{}^{ab}(\mathbf{0},\mathbf{0})=C_{KL,ij}{}^{ab}(\mathbf{0}). \tag{61b}$$

We shall find that our previous techniques, without further assumptions, set *no* conditions on $C_{KL,ij}{}^{ab}(\mathbf{P})$, and only a very mild restriction on the $C_{KL,ijk_1\cdots k_n}{}^{ab}(\mathbf{P})$. With further assumptions, we can show that the coefficients of powers of Q higher than the first vanish, i.e., at most one derivative of a δ function is present in the ST. However, nothing can be said about $C_{KL,ij}{}^{ab}(\mathbf{P})$, reflecting the fact that models exist which give different results for the once or twice integrated ETCR of space components of current densities. (We have in mind the quark model, the gauge-field model, etc. These results can be contrasted with those of Sec. III. There we also found that the ETCR of the time-space components of current densities involved at most one derivative of a δ function. However, we were able to go further and evaluate completely the once integrated ETCR and set symmetry restrictions on the ST present in the local ETCR.)

[7] S. Weinberg, Phys. Rev. Letters, 18, 507 (1967).

We begin by assuming only the results of Secs. I to III. First, we note that from general principles it follows that

$$[H,K_j{}^a(\mathbf{p})]=-i\partial_0 K_j{}^a(\mathbf{p})\,, \qquad (62a)$$

$$[M^{0i},K_j{}^a(\mathbf{p})]=-ip_iK_j{}^a(\mathbf{p})-\partial^0\frac{\partial}{\partial p_i}K_j{}^a(\mathbf{p})+i\delta^{ij}K_0{}^a(\mathbf{p})\,. \qquad (62b)$$

Then we use the Jacobi identity, first with H, $K_0{}^a(\mathbf{p})$, $L_j{}^b(\mathbf{q})$. This yields

$$-p_i[K_i{}^a(\mathbf{p}),L_j{}^b(\mathbf{q})]-[D^a(\mathbf{p}),L_j{}^b(\mathbf{q})]-\epsilon_{abc}\partial_0(K\cdot L)_j{}^c(\mathbf{p}+\mathbf{q})+p_i[R_{KL,ij}{}^{ab}(\mathbf{p}+\mathbf{q}),H]-i[K_0{}^a(\mathbf{p}),\partial_0L_j{}^b(\mathbf{q})]=0\,. \qquad (63a)$$

We have used in (63a) Eqs. (12), (33), and (62). D^a here is the divergence of $K_\mu{}^a$. In order to evaluate the last ETCR we proceed as follows:

$$\begin{aligned}[K_0{}^a(\mathbf{p}),\partial_0L_j{}^b(\mathbf{q})]&=\partial_0[K_0{}^a(\mathbf{p}),L_j{}^b(\mathbf{q})]-[\partial_0K_0{}^a(\mathbf{p}),L_j{}^b(\mathbf{q})]\\&=i\epsilon_{abc}\partial_0(K\cdot L)_j{}^c(\mathbf{p}+\mathbf{q})+\partial_0S_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q})\\&\quad+i[D^a(\mathbf{p}),L_j{}^b(\mathbf{q})]+ip_i[K_i{}^a(\mathbf{p}),L_j{}^b(\mathbf{q})]\,. \end{aligned}\qquad (63b)$$

Inserting (63b) in (63a) gives the not very interesting fact that

$$[H,R_{KL,ij}{}^{ab}(\mathbf{p})]=-i\partial_0R_{KL,ij}{}^{ab}(\mathbf{p})\,. \qquad (63c)$$

Slightly more interesting is the Jacobi identity for M^{0i}, $K_0{}^a(\mathbf{p})$, $L_j{}^b(\mathbf{q})$. The identity gives

$$[M^{0i},S_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q})]=i(p_i+q_i)S_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q})+\frac{\partial}{\partial q_i}\partial_0\mathcal{S}_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q})+p_k\left[\frac{\partial}{\partial p_i}-\frac{\partial}{\partial q_i}\right]\mathcal{C}_{KL,kj}{}^{ab}(\mathbf{p},\mathbf{q})\,. \qquad (64a)$$

Using (39a) and differentiating (64a) with respect to p_l and setting $\mathbf{p}=\mathbf{0}$, then gives

$$[M^{0i},R_{KL,lj}{}^{ab}(\mathbf{q})]=iq_iR_{KL,lj}{}^{ab}(\mathbf{q})+\partial_0\frac{\partial}{\partial q_i}R_{KL,lj}{}^{ab}(\mathbf{q})+\left[\frac{\partial}{\partial p_i}-\frac{\partial}{\partial q_i}\right]\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p},\mathbf{q})\,|_{\mathbf{p}=\mathbf{0}}\,. \qquad (64b)$$

Substituting (64b) into (64a) yields a condition on the commutator $\mathcal{C}_{KL,kj}{}^{ab}(\mathbf{p},\mathbf{q})$

$$p_K\left[\frac{\partial}{\partial p_i}-\frac{\partial}{\partial q_i}\right]\mathcal{C}_{KL,kj}{}^{ab}(\mathbf{p},\mathbf{q})=p_K\left[\frac{\partial}{\partial r_i}-\frac{\partial}{\partial q_i}\right]\mathcal{C}_{KL,kj}{}^{ab}(\mathbf{r},\,\mathbf{p}+\mathbf{q})\,|_{\mathbf{r}=\mathbf{0}}\,, \qquad (65a)$$

or in the notation of (60)

$$(P+Q)_k\frac{\partial}{\partial Q_i}C_{KL,kj}{}^{ab}(\mathbf{P},\mathbf{Q})=(P+Q)_k\times\left(\frac{\partial}{\partial Q_i}C_{KL,kj}{}^{ab}(\mathbf{P},\mathbf{Q})\right)\bigg|_{\mathbf{Q}=-\mathbf{P}}\,. \qquad (65b)$$

This is the only condition on the ETCR of the spatial components of current densities that we are able to derive using *only* Poincaré invariance (and the assumptions of Secs. I to III).

Note that because (65b) involves differentiation with respect to Q_i, (65b) places *no* restrictions on those parts of $C_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q})$ which are independent of $\mathbf{Q}$, or are proportional to $\mathbf{Q}$. Specifically, we cannot use Poincaré invariance to obtain the once or twice integrated ETCR.

We wish also to call attention to the fact that Eq. (64b) exhibits the result that $C_{KL,lj}{}^{ab}(\mathbf{P},\mathbf{Q})$ is related to the Lorentz transformation properties of $R_{KL,ij}{}^{ab}(\mathbf{P})$ in the following fashion. The commutator of $M^{\mu\nu}$ with any local operator is always of the form (in position space)

$$[M^{\mu\nu},O(x)]=i(x^\mu\partial^\nu-x^\nu\partial^\mu)O(x)+\mathcal{L}^{\mu\nu}[O]\,, \qquad (66)$$

viz., the commutator has a term involving differentiation and an additional term here called $\mathcal{L}^{\mu\nu}[O]$, whose form is determined by the finite dimensional representations of the Lorentz group according to which O transforms. Thus when O is a scalar, $\mathcal{L}^{\mu\nu}[O]$ vanishes; if it is a vector O^α, $\mathcal{L}^{\mu\nu}[O^\alpha]$ is $ig^{\mu\nu}O^\nu-ig^{\nu\alpha}O^\mu$, etc. Examining the right-hand side of (64b) it is seen that, of the three terms there the first two represent in momentum space the differential operator, while the third term reflects the Lorentz transformation properties of $R_{KL,lj}{}^{ab}$. Thus we have

$$\mathcal{L}^{0i}[R_{KL,lj}{}^{ab}(\mathbf{P})]=\frac{\partial}{\partial Q_i}C_{KL,lj}{}^{ab}(\mathbf{P},\mathbf{Q})\,|_{\mathbf{Q}=-\mathbf{P}}\,. \qquad (67)$$

This is as far as we can get in the analysis of the space-space ETCR without making further assumptions. We now assume the local version of (62):

$$[H_{00}(\mathbf{p}),K_j{}^a(\mathbf{q})]=-i\partial^0K_j{}^a(\mathbf{p}+\mathbf{q})+ip_jK_0{}^a(\mathbf{p}+\mathbf{q})\,. \qquad (68)$$

This local ETCR implies (62); however, further ST could be present in (68) which integrate to zero so that (62) remains valid. Unlike the analogous ETCR with $K_0{}^a$ (12), we are unable to derive (68) from an action principle. Thus (68) must be considered to be an outright assumption.[8]

[8] Our assumption that the $[H_{00},K_i{}^a]$ ETCR has the form (68) implies (at least for systems with spin ≤ 1) that $[H^{0i}(\mathbf{p}),K_0{}^a(\mathbf{q})]=iq_iK_0{}^a(\mathbf{p}+\mathbf{q})$. To see this, we consider the Jacobi identity for $H_{00}(\mathbf{p})$, $H_{00}(\mathbf{q})$, and $K_0{}^a(\mathbf{k})$, and use (12), (68), and the ETCR $[H_{00}(\mathbf{p}),H_{00}(\mathbf{q})]=(p_k-q_k)H^{0k}(\mathbf{p}+\mathbf{q})$, which holds for systems with spin ≤ 1. The result is $(p_k-q_k)[H^{0k}(\mathbf{p}+\mathbf{q}),K_0{}^a(\mathbf{k})]=i(p_k-q_k)\times k_kK_0{}^a(\mathbf{p}+\mathbf{q}+\mathbf{k})$. We have assumed that the ETCR of the divergence of $K_\mu{}^a$ with H_{00} has no ST. The desired result follows from the above equation by differentiating with respect to p_i-q_i and holding p_i+q_i constant.

Using (68), we consider the Jacobi identity of $H_{00}(\mathbf{k})$, $K_0{}^a(\mathbf{p})$, $L_j{}^b(\mathbf{q})$. This leads to

$$[H_{00}(\mathbf{k}),S_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q})]=-i\partial_0 S_{KL,j}{}^{ab}(\mathbf{p},\mathbf{q}) +ip_l[\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p}+\mathbf{k},\,\mathbf{q})-\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p},\,\mathbf{q}+\mathbf{k})]. \quad (69a)$$

The usual differentiation by p_i and then $\mathbf{p}=\mathbf{0}$, gives

$$[H_{00}(k),R_{KL,ij}{}^{ab}(\mathbf{q})]=-i\partial_0 R_{KL,ij}{}^{ab}(\mathbf{q}) +i[\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{k},\mathbf{q})-\mathcal{C}_{KL,ij}{}^{ab}(\mathbf{0},\,\mathbf{q}+\mathbf{k})]. \quad (69b)$$

Finally, substituting (69b) into (69a) yields a condition on $\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p},\mathbf{q})$

$$p_l[\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p}+\mathbf{k},\mathbf{q})-\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{p},\mathbf{q}+\mathbf{k})] =p_l[\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{k},\mathbf{p}+\mathbf{q})-\mathcal{C}_{KL,lj}{}^{ab}(\mathbf{0},\,\mathbf{p}+\mathbf{q}+\mathbf{k})]. \quad (70a)$$

(It is seen that the previous condition on $\mathcal{C}_{KL,lj}{}^{ab}(p,q)$, Eq. (65a) can be obtained from the present by differentiating (70a) by k_i and setting $\mathbf{k}=\mathbf{0}$.) It is more convenient to write (70a) in a different fashion. We use the notation (60), define $\mathbf{p}'=\mathbf{p}+\frac{1}{2}\mathbf{k}$, $\mathbf{Q}'=\mathbf{Q}-\frac{1}{2}\mathbf{k}$, and suppress the prime. Then (70a) becomes

$$(P+Q)_l[C_{KL,lj}{}^{ab}(\mathbf{P},-\mathbf{P})-C_{KL,lj}{}^{ab}(\mathbf{P},\mathbf{Q})] =(P+Q)_l[C_{KL,lj}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k})-C_{KL,lj}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k})]. \quad (70b)$$

We now show that condition (70b) is sufficient to establish the fact that the space-space ETCR involves at most one derivative of a δ function. First, differentiate (70b) by k_n to give

$$0=(P+Q)_l\left[\frac{\partial}{\partial k_n}C_{KL,lj}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k}) -\frac{\partial}{\partial k_n}C_{KL,lj}{}^{ab}(\mathbf{P},\,+\mathbf{Q}+\mathbf{k})\right]. \quad (71a)$$

Next differentiate (71a) by Q_i

$$\frac{\partial}{\partial k_n}C_{KL,ij}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k})-\frac{\partial}{\partial k_n}C_{KL,ij}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k}) =(P+Q)_l\frac{\partial^2}{\partial k_n\partial Q_i}C_{KL,lj}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k}). \quad (71b)$$

Finally, differentiate (71a) by k_i

$$(P+Q)_l\frac{\partial^2}{\partial k_n\partial k_i}C_{KL,lj}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k}) =(P+Q)_l\frac{\partial^2}{\partial k_n\partial k_i}C_{KL,lj}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k}) =(P+Q)_l\frac{\partial^2}{\partial k_n\partial Q_i}C_{KL,lj}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k}). \quad (71c)$$

Substituting (71b) for the right-hand member of (71c) yields

$$\frac{\partial}{\partial k_n}C_{KL,ij}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k})=\frac{\partial}{\partial Q_n}C_{KL,ij}{}^{ab}(\mathbf{P},\,\mathbf{Q}+\mathbf{k}) =\frac{\partial}{\partial k_n}C_{KL,ij}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k}) -(P+Q)_l\frac{\partial^2}{\partial k_n\partial k_i}C_{KL,lj}{}^{ab}(\mathbf{P},\,-\mathbf{P}+\mathbf{k}). \quad (71d)$$

Setting $\mathbf{k}$ to zero in (71d) shows that $(\partial/\partial Q_n)C_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q})$ is linear in $\mathbf{Q}$. Hence in the notation (60a), we have

$$C_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q})=C_{KL,ij}{}^{ab}(\mathbf{P})+C_{KL,ij,k_1}{}^{ab}(\mathbf{P})Q_{k_1} +C_{KL,ij,k_1k_2}{}^{ab}(\mathbf{P})Q_{k_1}Q_{k_2}. \quad (72)$$

Next we substitute the form (62) into (70b) to yield

$$(P+Q)_i(P+Q)_{i_1}k_{i_2}C_{KL,ij,i_1i_2}{}^{ab}(\mathbf{P})=0. \quad (73a)$$

This equation states that $C_{KL,ij,i_1i_2}{}^{ab}(\mathbf{P})$ is antisymmetric in i and i_1. We know already that it also is symmetric in i_1 and i_2. Hence, the following equations show that it vanishes. (We exhibit only the indices ij, i_1i_2.)

$$ij,i_1i_2=-i_1j,ii_2=-i_1j,i_2i\,, \quad ij,i_1i_2=ij,i_2i_1=-i_2j,ii_1=-i_2j,i_1i=i_1j,i_2i. \quad (73b)$$

We have thus established that the most general form of the ETCR of the space components of currents, which is implied by (68), is

$$C_{KL,ij}{}^{ab}(\mathbf{P},\mathbf{Q})=C_{KL,ij}{}^{ab}(\mathbf{P})+C_{KL,ij,k}{}^{ab}(\mathbf{P})Q_k, \quad (74)$$

viz., at most one derivative of δ function is present. Note that even the local ETCR (68) is not sufficient to determine the once or twice integrated commutator. Finally, note that (64b) and (74) now read

$$[M^{0i},R_{KL,lj}{}^{ab}(\mathbf{P})]=tP_iR_{KL,lj}{}^{ab}(\mathbf{P}) +\partial_0\partial/\partial p_iR_{KL,lj}{}^{ab}(\mathbf{P})+C_{KL,lj,i}{}^{ab}(\mathbf{P}), \quad (75a)$$

i.e.,

$$\mathcal{L}^{0i}[R_{KL,lj}{}^{ab}(\mathbf{P})]=C_{KL,lj,i}{}^{ab}(\mathbf{P}). \quad (75b)$$

This implies that $C_{KL,lj,i}{}^{ab}$ is determined by $R_{KL,lj}{}^{ab}$ and its Lorentz-transformation properties. In particular, the space-space ETCR contain no Schwinger terms if and only if $R_{KL,lj}{}^{ab}$ is a position-space scalar, i.e., of the form $R_{KL,ij}{}^{ab}=\delta_{ij}R_{KL}{}^{ab}$ (as in the σ model).