Symmetries in First-Approximation S-Matrix Theory. II. High-Rank Scattering*

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A study is presented of scattering invariant under any of the groups $G=O(n)$, $S_p(n)$, or $SU(n)$, $n\gg1$, the scattered multiplets transforming as products of at most two n-quarks plus n-antiquarks. The low-energy dynamics is studied in "first approximation," and in addition it is assumed that total cross sections σ_{tot} at high energies are determined by Regge poles and/or cuts, and that the Regge parameters upon which σ_{tot} depends are determined mainly by the low-energy dynamics. It is then shown that as $n\rightarrow\infty$, σ_{tot} usually remains positive, as unitarity requires. This conclusion is not obvious; it requires detailed calculation to establish and is the main result of the paper. Unitarity is initially in doubt for these systems because in the limit $n\rightarrow\infty$ the crossing matrix elements governing the contributions from the vacuum and other singlet trajectories go to zero; and it is the singlet contributions which keep total cross sections positive for the observed case of small n. It is found that σ_{tot} remains positive only because the low-energy dynamics requires nonsinglet trajectories to be produced in pairs, such that each negative contribution to σ_{tot} from one trajectory is almost exactly canceled by a corresponding positive contribution to σ_{tot} from its twin trajectory. Occasionally this cancellation does not occur: If one of the initial multiplets is a representation of G which is not equivalent to its complex-conjugate representation, then in at least one of the direct channels, even after paired contributions are taken into account, σ_{tot} goes slightly negative, $\sigma_{\text{tot}}=O(n^{-1})<0$. This number is small enough that if the electromagnetic vertex for G transforms as a singlet rather than as one of the generators of G, then the contribution to σ_{tot} from photon exchange will drive σ_{tot} positive again. It is then assumed that at least one of the initial particles has spin. It is shown that so long as the leading nonsinglet trajectories obey $\epsilon\eta_p=+1$ ($\epsilon=$ signature, $\eta_p=$intrinsic parity), it is not possible to change the sign of σ_{tot} by flipping the helicity of one of the incoming particles from λ to $-\lambda$. A generalization (to arbitrary spin) of the Wagner–Sharp rules for "line reversal" is also derived: In elastic scattering, if one scattered particle is replaced by its antiparticle, then the contribution to σ_{tot} from a given exchanged multiplet changes by a factor of $\eta_{\text{ch}}=$charge-conjugation parity of the exchanged multiplet. A fairly complete discussion is given of the dependence of σ_{tot} on the C, P properties of the exchanged multiplets.

I. INTRODUCTION

In this paper we continue to investigate the stability of high-rank and high-dimensional symmetries within the framework of first-approximation S-matrix dynamics. We turn from high-isospin scattering (the subject of paper I)[1] to scattering of multiplets transforming as the lowest-dimensional representations of one of the high-rank simple Lie groups $G=S_p(n)$, $O(n)$, or $SU(n)$, $n\gg1$.

Crossing matrix elements $C(\mu_d,\mu_c)$ for such scattering can be calculated readily enough by well-known tensor techniques.[2] (μ_d,μ_c label the irreducible representations occurring in the direct and crossed channels.) It is by now clear that there are many elements of order unity left in the limit $n\rightarrow\infty$.[3–5] This is just to say that an investigation in the style of Pt. I, but with high rank replacing high isospin, has already been carried out, with similar negative results.

Paper II therefore considers an entirely different argument based on the assumption that high-energy (s or u), small-momentum-transfer (t) behavior of scattering amplitudes is determined by Regge poles and/or cuts. Hence one has for total cross sections

$$\begin{aligned}\lim_{s\to\infty}\sigma_{\text{tot}}(\mu_d=\mu_s)&=\lim(4\pi/ks^{1/2})\,\text{Im}A(\mu_d;t=0)\\&=\lim(4\pi/ks^{1/2})\,\text{Im}\sum_i C(\mu_d,\mu_{ti})A(\mu_{ti};t=0)\qquad(1.1\text{a})\\&=\sum_i C(\mu_d,\mu_{ti})\beta_i s^{\alpha_i(0)-1},\end{aligned}$$

and

$$\lim_{u\to\infty}\sigma_{\text{tot}}(\mu_d=\mu_u)=\sum_i C(\mu_d,\mu_{ti})\beta_i u^{\alpha_i(0)-1}\epsilon_i,\qquad(1.1\text{b})$$

in the simplest case of negligible Regge cuts. We have used the optical theorem; σ_{tot} is the cross section for $\mu_1\mu_2\rightarrow$ everything, while ImA is the imaginary part of the elastic scattering amplitude $\mu_1\mu_2\rightarrow\mu_1\mu_2$. β_i, ϵ_i, and $\alpha_i(t)$ are the residue, signature, and trajectory of a Regge pole in the crossed t channel μ_t. When the requisite $C(\mu_d,\mu_{ti})$ are calculated and inserted into the left-hand side of Eqs. (1.1) along with reasonable estimates of the parameters β_i and $\alpha_i(0)$, then for some choices of μ_u, μ_b, and μ_d the sum over i goes negative (hence $\sigma_{\text{tot}}<0$), in violation of unitarity.

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[1] Donald E. Neville, preceding paper, Phys. Rev. **163**, 1582 (1967), hereafter referred to as I.

[2] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee; Rev. Mod. Phys. **34**, 1 (1962); M. Hamermesh, *Group Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Chap. 10.

[3] Donald E. Neville, Phys. Rev. **132** 844 (1963); Phys. Rev. Letters **13**, 118 (1964).

[4] R. E. Cutkosky, J. Kalckar, and P. Tarjanne, in *Proceedings of the 1962 Annual International Conference on High-Energy Nuclear Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 653. These authors also noted a special case of the pair formation described in Sec. II.

[5] H. S. Mani, Gyan Mohan, Lalit K. Pande, and Virendra Singh, Ann. Phys. (N. Y.) **36**, 285 (1966).

The sums go negative because the large contributions from crossed singlet trajectories keep total cross sections positive in the observed case, broken $SU(n=3)$, while going to a high-rank group amounts to "turning off" this singlet contribution. That is, the singlet contribution $\mu_t=E$ (E for *ein*) contains a number which decreases steadily as the rank of G increases;

$$C(\mu_d,\mu_t=E)=+(N_1N_2)^{-1/2}, \quad N_x=\text{dimension of } \mu_x, \tag{1.2}$$

where the direct and cross channels are $\mu_1\mu_2\to\mu_d\to\mu_1\mu_2$ and $\mu_1\bar{\mu}_1\to\mu_t\to\mu_2\bar{\mu}_2$. Note that the singlet contribution can be positive for every choice of μ_d if the residue of the leading singlet trajectory is positive, since the sign in Eq. (1.2) is positive for all μ_d. The nonsinglet columns $C(\mu_d,\mu_t\neq E)$, on the other hand, must oscillate in sign in order to satisfy the orthogonality constraints implied by a very useful and powerful formula derived by Capps[6];

$$C(\mu_d,\mu_c)=(N_c/N_d)^{1/2}O(\mu_d,\mu_c), \quad O(\mu_d,\mu_c) \text{ orthogonal.} \tag{1.3}$$

Furthermore, the numbers $C(\mu_d,\mu_t\neq E)$, though often small also, are on the average much larger than $C(\mu_d,E)$ because of the factor $(N_c/N_d)^{1/2}$ in formula (1.3);

$$\bar{C}(\mu_d,\mu_t\neq E)/\bar{C}(\mu_d,E)=N_t^{1/2}=\text{order } r. \tag{1.4}$$

The bars mean rms averages over $O(\mu_d,E)$ and $O(\mu_d,\mu_t\neq E)$. r, the rank of G (number of simultaneously diagonalizable generators), is $n-1$ for $SU(n)$, and $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is integer, for $S_p(n)$ and $O(n)$. In the limit $r, n\to\infty$, therefore, the singlet contribution is negligible and the remaining terms oscillate in sign.

Our reasoning in the previous paragraph follows that of Foldy and Peierls.[7] These authors, who were considering isospin, proved that, whenever one crossed t-channel exchange dominates, it must have $I_t=0$, or total cross sections will go negative.

Of course, it must be verified that the various β_i, $\alpha_i(0)$ do not fall off in some way so as to make up for the falloff in $C(\mu_d,E)$. If it is assumed that those parameters are determined largely by the low-energy dynamics, then it is plausible that there will be no falloff in the β_i, $\alpha_i(0)$ (at least not by factors of r), because: (a) *every* multiplet in the low-energy region can exert a force on the annihilation channel; e.g., if μ_c is coupled to $\mu_a\mu_b$, then it can exert force on μ_t via μ_c exchange in $\mu_a\bar{\mu}_a\to\mu_b\bar{\mu}_b$ scattering (the annihilation channel is unique in this respect); and (b) direct calculation shows that the elements $C(\mu_t,\mu_c)$ through which these forces are exerted, i.e., the elements given the forces *into* t channels, do not fall off with r, but remain of order unity.

One must evaluate the sum (1.1) at a value of s just large enough that the background integral no longer contributes, and not, strictly speaking at $s=\infty$. Otherwise the singlet contribution will presumably dominate no matter what the value of $C(\mu_d,E)$, since the $C(\mu_t,\mu_c)$ which control low-energy dynamics in G suggest that a singlet will again have the largest $\alpha(0)$, just as in $SU(3)$.

From what has been said thus far, especially from Eqs. (1.2)–(1.4), there is no reason to suspect any dependence on the scattered multiplets μ_1 and μ_2; i.e., the argument might well be universal, applying to all high-rank scattering processes. On the contrary, for all but certain of the $SU(n)$ scattering processes studied, the dynamics forces the nonsinglet Regge trajectories to appear in pairs in the crossed t channel, such that every negative contribution to sum (1.1) from one member of a pair is almost exactly canceled by a positive contribution from the other member. The two members of a pair have almost identical residue and trajectory functions but opposite signatures, so that

$$\beta_{t1}s^{\alpha_{t1}(0)}\cong\beta_{t2}s^{\alpha_{t2}(0)}; \quad \epsilon_{t1}=-\epsilon_{t2}. \tag{1.5}$$

In this paper the notation $A\cong B$ means $A/B=1+O(r^{-1})$. Similarly, there is close agreement between the crossing matrix elements $C(\mu_d,\mu_{t1})$ and $C(\mu_d,\mu_{t2})$ occurring in Eq. (1.1) [cf. Eqs. (1.6a)–(1.6c) below for a detailed example] so that the cancellations result. The dynamical mechanism which produces the symmetry (1.5) is described in detail in Sec. II.

We have not checked every possible G-symmetric scattering process for pair cancellation, but only those processes involving multiplets of dimensionality $N\leq$ order r^2. In terms of tensors or quarks, the multiplets studied transform as tensors with ≤ 2 subscripts plus superscripts, or equivalently, as products of ≤ 2 quarks plus antiquarks. One could describe this paper as a study of tensors of low rank (≤ 2), but groups of high rank ($r\to\infty$). $O(n)$ and $S_p(n)$ have three such low-rank representations each (the n-dimensional representation plus two rank-two representations transforming as tensors T_i, $T_{ij}=+T_{ji}$, and $T_{ij}=-T_{ji}$). $SU(n)$ has the same three, plus three more distinct, in equivalent representations obtained from the first three by complex conjugation (usually written with upper indices, as $T_i{}^*=T^i$), plus the adjoint representation $T_i{}^j$. The number of processes studied, though not large, is presumably large enough for gauging the range of applicability of the argument under discussion.

Basically, the cancellation, when it occurs, always occurs because of symmetry (1.5); but the detailed manner in which the cancellation occurs varies from process to process. For all the $O(n)$ and $S_p(n)$ scattering processes investigated, as well as for the $SU(n)$ scattering process with $\mu_1=\mu_2=$adjoint representation, the

[6] Richard H. Capps, in *Proceedings of the Twelfth Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).

[7] Leslie L. Foldy and Ronald F. Peierls, Phys. Rev. **130**, 1585 (1963).

twins have different G quantum numbers ($\mu_{t1} \neq \mu_{t2}$), and in about $\frac{1}{3}$ of the direct s channels the crossing matrix elements satisfy

$$C(\mu_d,\mu_{t1}) \cong -C(\mu_d,\mu_{t2}). \tag{1.6a}$$

Therefore, the two contributions (1.5) nearly cancel, giving a resultant one power of r smaller than expected, and of the same order as the singlet contribution. In another $\frac{1}{3}$ of the direct s channels the two contributions do not cancel,

$$C(\mu_d,\mu_{t1}) \cong C(\mu_d,\mu_{t2}) > 0, \tag{1.6b}$$

but there is nothing to prevent the resultant from being positive always, if the residues β_{ti} are positive. In the remaining $\frac{1}{3}$ of the direct s channels the $C(\mu_d,\mu_{t2})$ fluctuate far below the rms value predicted by Eq. (1.4), and are in consequence too small to exceed the singlet contribution:

$$|C(\mu_d,\mu_{t1})|/C(\mu_d,E) \cong |C(\mu_d,\mu_{t2})|/C(\mu_d,E) = O(1). \tag{1.6c}$$

In the u channels $\mu_d = \mu_u$, the contribution from one twin changes sign because of the ϵ factor in Eq. (1.1b); but the crossing matrix factors also change sign,

$$C(\mu_u,\mu_{ti}) = \eta_{\text{CG}} C(\mu_s = \mu_u,\mu_{ti})\,; \quad \eta_{\text{CG}} = \pm 1, \tag{1.7}$$

that is, we obtain the crossing matrix elements for the u channel by relabeling $\mu_s \to \mu_u$ (the *same* irreducible representations occur in s and u channels, for the processes we are at present considering) and multiplying by a factor of η_{CG} which is $+1$ or -1 depending upon μ_{ti}. Twins have opposite η_{CG}. Therefore, if the Lorentz and G quantum numbers are correlated, such that

$$\eta_{\text{CG}} = \epsilon_i, \tag{1.8}$$

then there will be no negative cross sections in the u channel either. η_{CG} is the interchange parity of the Clebsch-Gordan coefficient at the $\mu_{ti} \to \bar{\mu}_2 \mu_2$ vertex:

$$\langle \bar{\mu}_2 \bar{m}_2 \mu_2 m_2 | \mu_{ti} m_{ti} \rangle = \eta_{\text{CG}} \langle \mu_2 m_2 \bar{\mu}_2 \bar{m}_2 | \mu_{ti} m_{ti} \rangle. \tag{1.9}$$

There seems to be no dynamical argument which could exclude correlation (1.8); indeed, if the external particles μ_2 and $\bar{\mu}_2$ are spinless and identical, Eq. (1.8) is the only possibility allowed by Bose statistics.

[We give a brief derivation of Eq. (1.7). The s channel is $\mu_1\mu_2 \to \mu_1\mu_2$, the u channel is $\mu_1\bar{\mu}_2 \to \mu_1\bar{\mu}_2$, and μ_2 is equivalent to $\bar{\mu}_2$ for the processes presently under consideration. Hence the same multiplets μ_d occur in both the s and the u channels. Crossing matrices $C(s,t)$ and $C(u,t)$ are, however, not identical because $C(s,t)$ interchanges the second and third multiplets in the t channel $\mu_1\bar{\mu}_1 \to \bar{\mu}_2\mu_2$ whereas $C(u,t)$ interchanges $\bar{\mu}_2 \leftrightarrow \mu_2$, *then* interchanges the second and third multiplets. The additional interchange $\bar{\mu}_2 \leftrightarrow \mu_2$ gives rise to the factor of η_{CG} difference in Eq. (1.7).[8]]

[8] There may be an additional factor of (-1) difference in Eq. (1.7), if the "antiparticle rule" factor for multiplet μ_2 is (-1).

TABLE I. Columns of $C(\mu_d,\mu_t)$ for exchange of a singlet (1) or the adjoint representation ($\mathbf{35}$, F- or D-coupled); the reaction is $\mathbf{35}+\mu \to \mu_d \to \mathbf{35}+\mu$, with crossed channel $\mathbf{35}+\overline{\mathbf{35}} \to \mu_t \to \bar{\mu}+\mu$. $\mu=\mathbf{15}$ or $\mathbf{21}$. $G=SU(n)$, $n \gg 1$. All rank ≤ 2 representstions of $SU(n)$ are labeled by their dimensionalities for $n=6$. $|x|=1$, $|y|=2$, $xy<0$.

	$\mu_t=1$	35_D	35_F
$\mu_d=15$	$2^{1/2}n^{-2}$	$2^{-1/2}$	$-2^{-1/2}$
21	$2^{1/2}n^{-2}$	$2^{-1/2}$	$-2^{-1/2}$
(Rank 4)	$2^{1/2}n^{-2}$	$2^{-1/2}n^{-1}x$	$2^{-1/2}n^{-1}x$
(Rank 4)	$2^{1/2}n^{-2}$	$23^{1/2}n^{-1}y$	$2^{-1/2}n^{-1}y$

The calculations needed to verify Eqs. (1.6a)–(1.6c) were carried out by means of an approximation procedure described in the Appendix. This procedure gives the leading term in an expansion of $C(\mu_d,\mu_c)$ in powers of r^{-1}.

Similar miraculous cancellations occur for the $SU(n)$ scattering processes such that both μ_1 and $\mu_2 \neq$ adjoint representation. For these processes $\mu_{t1} = \mu_{t2}$, so that contributions from both twins are multiplied by one and the same coefficient $C(\mu_d,\mu_t)$. Hence the twins cancel in the u channel because of $\epsilon_{t1} = -\epsilon_{t2}$. In the s channel, either $C(\mu_d,\mu_t)$ is always positive, as in Eq. (1.6b); or, when negative, it is too small, as in Eq. (1.6c).

No pair cancellation occurs for the $SU(n)$ scattering processes such that $\mu_2 \neq \mu_1 =$ adjoint representation, μ_2 not a singlet. μ_2 is then inequivalent to its complex conjugate $\bar{\mu}_2$. For these processes the multiplets occurring in the u and s channels transform as the complex conjugates of one another ($\mu_u = \bar{\mu}_s$), both twins have the same interchange parity η_{CG} at the $\mu_{ti} \to \bar{\mu}_2\mu_4$ vertex, and $C(\mu_u = \bar{\mu}_s,\mu_{ti}) = C(\mu_s,\mu_{ti})$. Hence, twins cancel in either the s or the u channel, but not in both.

Tables I and II list the $C(\mu_d,\mu_t)$ for the $SU(n)$ processes not affected by pair cancellation. It is convenient to label the $SU(n)$ representations by their dimensionalities for $n=6$: $T_{ij} = T_{ji}$ becomes the $\mathbf{21}$, $T_{ij} = -T_{ji}$ the $\mathbf{15}$, $T_i{}^j$ the $\mathbf{35}$, and T_i the $\mathbf{6}$. [The corresponding $SU(3)$ dimensionalities are $\mathbf{6}$, $\mathbf{3}$, $\mathbf{8}$, and $\mathbf{3}$. $SU(3)$ dimensionalities are not suitable as labels because for $n=3$ T_i is equivalent to $T_{ij} = -T_{ji}$.] The tables were calculated using Eq. (1.2) or the method of the Appendix, except that elements depending on x and y

TABLE II.

(Same as Table I, except $\mu = \mathbf{6}$.)

	$\mu_t=1$	35_D	35_F
$\mu_d=6$	$n^{-3/2}$	$(n/2)^{1/2}$	$(n/2)^{1/2}$
(Rank 3)	$n^{-3/2}$	$-(2n)^{-1/2}$	$(2n)^{-1/2}$
(Rank 3)	$n^{-3/2}$	$(2n)^{-1/2}$	$-(2n)^{-1/2}$

For a discussion of the antiparticle rule see Ref. 11. The discussion there is for $SU(2)$, but an analogous rule applied to self-conjugate representations in G. The antiparticle rule factor for all the low-dimensional multiplets we shall be dealing with is always $(+1)$.

were calculated using constraints (1.3). Twins occurring in the t channels of the processes covered by Tables I and II both transform as **35**, yet do not have identical $SU(n)$ quantum numbers because one twin is always F-coupled, the other always D-coupled to the external **35 35** pair. [For $SU(n)$ as for $SU(3)$, **35** occurs twice in **35**$\times$**35**; F-type (D-type), coupling is odd (even) under the interchange $\mathbf{35}\leftrightarrow\mathbf{35}$.] Table I is missing a fourth column, referring to the exchange of a rank-four multiplet μ_t. In passing over this multiplet we have anticipated the lemma of Sec. II, which implies that rank >2 channels will not contain any Regge poles. Note that for the processes covered by Table I, it is not even necessary to switch from s to u channel in order to obtain a negative σ_{tot}. No matter what the signs of the β_{ti}, they are bound to add to a negative result in at least one of the four direct channels.

Note that a system will not be free from pair cancellation unless it contains both a multiplet $\mu\neq\bar{\mu}$ *and* a multiplet transforming as the adjoint representation. However, this second requirement seems to be satisfied automatically since an $SU(n)$ system containing only rank $\leq 2\,\mu\neq\bar{\mu}$ multiplets, and perhaps some singlets, does not appear to be self-consistent. See the discussion of low-energy dynamics in Sec. II, especially the corollary to the lemma.

Unfortunately, it seems quite possible to construct a low-dimensional $SU(n)$ system which contains only multiplets transforming as the adjoint representation or as singlets.[3] *Some* $SU(n)$ systems are free from pair cancellation, but definitely not all. Presumably the $SU(3)$ octuplet model, if generalized to $SU(n)$, would exhibit pair cancellation, whereas a generalized Sakata model would not.

Even though the processes covered by Tables I and II are not affected by pair cancellation, it could happen that total cross sections for these processes would remain positive because of contributions to σ_{tot} from exchange of weakly interacting particles, e.g., a contribution from photon exchange. If the residues β_{ti} of the leading nonsinglet trajectories are positive and the 35_F twin has negative signature, then from Tables I and II the direct channels where $\sigma_{\text{tot}}<0$ all have very small cross sections $\sigma_{\text{tot}}=$ order $n^{-1}=$ order r^{-1}. In such a situation contributions ordinarily dismissed as "weaker" could swamp the "strong" contribution and drive σ_{tot} positive again. Section III shows that if the $\gamma\to\mu\bar{\mu}$ vertices transform as singlets with respect to rotations in G, then photon exchange contributes the same amount to each direct channel as does the vacuum trajectory, but does not fall off with r the way the vacuum trajectory does. [At first glance, the contribution to sum (1.1) from γ exchange appears to be infinite, since the Coulomb amplitude calculated from a potential $\propto 1/R$ blows up in the forward direction. It is physically more realistic, however, to assume a shielded Coulomb potential $\propto\exp(-\alpha R)/R$, in which case the γ contribution becomes finite and perfectly well behaved everywhere.] The $\gamma\to\mu\bar{\mu}$ vertex must transform as one of the generators of G or electromagnetism will drive total cross sections positive. These statements are discussed in full detail in Sec. III. Here we shall merely note that, *a priori*, the electromagnetic interaction could transform either as a generator or as a singlet; and present knowledge of both electromagnetism and the strong interaction is not complete enough to rule out the latter possibility.

Section IV investigates the dependence of σ_{tot} on the helicities λ_1 and λ_2 of the incident multiplets μ_1 and μ_2. The Lorentz quantum numbers of the leading exchanged trajectories must obey certain constraints [cf. Eq. (4.26)] or it will be possible to change the sign of σ_{tot} by varying the λ_i or changing a μ_i to $\bar{\mu}_i$. These constraints do not seem severe enough to rule out spinning particles in a G-symmetric universe.

II. MECHANISM OF PAIR FORMATION

It may help some readers to follow the discussion of the present section if it is noted that pairing of trajectories occurs even for the observed symmetry $SU(n=3)$. Pignotti[9] has pointed out that the exchange of the vector octuplet trajectory $V=(\rho,K^*,K^*,\omega)$ in low-energy pseudoscalar-meson scattering produces not only V but also a twin trajectory having the same Lorentz quantum numbers and approximately the same $\alpha(t)$ and β but opposite signature and inverse D/F ratio. Arnold[10] has proposed a symmetry in which *all* $SU(3)$ meson multiplets are signature-doubled in this fashion. The twins we shall be discussing are likewise signature doublets with, in general, different G quantum numbers (e.g., inverse D/F ratios); however, in G, for n large enough, the Lorentz quantum numbers of the multiplet are irrelevant as far as twin formation is concerned: Every low-dimensional nonsinglet representation of G has a twin, baryon multiplets included.

We begin our analysis of low-energy dynamics in G by proving the following lemma.

Lemma: Let μ_d occur in the direct channel of some scattering process $\mu_1\mu_2\to\mu_d\to\mu_3\mu_4$ and let μ_c be a crossed exchange. Then a necessary condition that the element $C(\mu_d,\mu_c)$ giving the force on μ_d due to μ_c exchange be $\geq$order unity is

$$N_i/N_d\gtrsim 1,\quad N_c/N_d\gtrsim 1.\tag{2.1}$$

$N_x=$dimension of μ_x, and $1\leq i\leq 4$.

The multiplets we are presently studying have di-

[9] A. Pignotti, Phys. Rev. **134**, B630 (1964). The Pignotti trajectory lies too high in the J plane to produce physical 0^+ mesons; but some meson resonances observed in the >1 BeV mass region fit nicely the 2^+ assignment appropriate to its first Regge recurrence. [See Arthur H. Rosenfeld *et al.*, University of California Radiation Laboratory Report No. UCRL 8030, 1965 (unpublished).]

[10] Richard C. Arnold, Phys. Rev. Letters **14**, 657 (1965).

mensions N=order r^k, where k=rank of multiplet ≤ 2. Hence for them, Eq. (2.1) becomes

$$k_i/k_d \geq 1\,, \quad k_c/k_d \geq 1\,. \qquad (2.2)$$

For example, singlets are always dynamically negligible (k_i or $k_c=0$), while the only multiplets effective in producing rank-two multiplets are other rank-two multiplets ($k_d=2$).

A version of this lemma was proved for isospin scattering in Sec. II of I, and the proof given there generalizes immediately to G-invariant scattering.[1] As in I, it is assumed that self-consistency imposes upper bounds upon coupling constants. The bound required for our present application to low-dimensional scattering, is very mild: It suffices to assume that there is no dynamical mechanism which could cause a coupling constant to diverge as order $r^{1/2}$ or faster as the rank of the group increases. The requirement $(N_c/N_d)>1$ of Eq. (2.1) follows immediately from the presence of the $(N_c/N_d)^{1/2}$ factor in Capps's formula (1.3); while if any of the external multiplets μ_i satisfies $N_i/N_d \ll 1$, then the strong coupling $g(\mu_i\mu_j \to \mu_d)$ can be vertex-crossed to give a divergent coupling $|g(\bar{\mu}_d\mu_j \to \bar{\mu}_i)| = (N_d/N_i)^{1/2}|g(\mu_i\mu_j \to \mu_d)| \gg 1$, in violation of the assumption of bounded couplings. [For a discussion of vertex crossing see Eqs. (I2.4)ff.; or the introduction to Ref. 11.]

Corollary: Every $SU(n)$-symmetric system containing only rank ≤ 2 multiplets must have at least one multiplet transforming as the adjoint representation. Proof: In a G-symmetric system with only rank ≤ 1 multiplets there is no two-body reaction which could produce the rank-one multiplets, since rank-one–rank-one scattering produces only rank-even multiplets, while rank-one singlet scattering is unsuitable for producing rank-one multiplets because of the lemma. Therefore, the system must contain at least one rank-two multiplet μ, and from the lemma it must be produced in some rank-two–rank-two scattering process. If μ is not an adjoint representation, then a straightforward enumeration of rank-two–rank-two scattering processes with μ in the direct channel reveals that the adjoint representation is always one of the scattered multiplets. Q.E.D.

We need one more piece of background information. [Readers familiar with the concent of "signature" may glance at Eq. (2.4) for the notation, then pass on to the next paragraph.] In a Regge calculation it is necessary to continue into the complex J plane not one, but two amplitudes per J value. As input to a non-Regge calculation one uses the discontinuity of the following function (calculated for spinless external particles):

$$\sum_{\mu_c}\int_{-1}^{1} P_l(z)dz\,\frac{1}{\pi}\left[\int^{u}\frac{C(S1,S2)[A(\mu_{S2})]_L}{S_2'-S_2}dS_2'+\int\frac{C(S1,S3)[A(\mu_{S3})]_L}{S_3'-S_3}dS_3'\right]$$
$$=\sum\int P_l(z)dz\,(2\pi q_d^2)^{-1}\int dz'\left\{\frac{C(S1,S2)[A(\mu_{S2})]_L}{z'-z}+\frac{C(S1,S3)[A(\mu_{S3})]_L}{z'+z}\right\}$$
$$=(2\pi q_d^2)^{-1}\sum\int dz'\,Q_l(z')\{C(S1,S2)[A(\mu_{S2})]_L+(-1)^l C(S1,S3)[A(\mu_{S3})]_L\}\,. \qquad (2.3)$$

$[A]_L \equiv$ left-hand discontinuity of A. The discontinuity of Eq. (2.3) cannot be used for a Regge calculation, because the factor $(-1)^l$ is not analytic in l. Instead, one constructs two potentials and calculates two amplitudes by substituting in turn

$$V_\pm^{S1} \equiv \sum_{S2,S3}\{C(S1,S2)[A(\mu_{S2})]_L \pm C(S1,S3)[A(\mu_{S3})]_L\} \qquad (2.4)$$

for the curly brackets in Eq. (2.3). The amplitudes $A_\pm(S1)$ are physical at alternating values of l. The form of Eqs. (2.3) and (2.4) is unchanged when the external particles have spin; the $V_\pm$ and $A(\mu_c)$ merely acquire helicity subscripts, while P_l and Q_l are replaced by functions which are essentially Jacobi and associated Jacobi polynomials.[12] The factor $\pm 1 \equiv \epsilon$ in Eq. (2.2) is the "signature" of the amplitude, or of any trajectory which contributes to the amplitude.

Since the proof that representation μ of G must have a twin is essentially the same for every μ and G, we shall go through the proof only for $G=SU(n)$, μ=adjoint representation. Note that the only nonsinglet rank ≤ 2 representation which can occur in the annihilation channels of an $SU(n)$ process is the adjoint representation, so that anyway μ=adjoint is the only representation of $SU(n)$ which is of interest in the present context. As at Tables I and II, we label the rank ≤ 2 representations of $SU(n)$ by their $SU(6)$ dimensionalities: **6**, **15**, **21**, and **35** correspond to tensors T_i, $T_{ij}=-T_{ji}$, $T_{ij}=+T_{ji}$, and T_i^j, respectively, with of course, $\bar{\mathbf{6}} \to T^i$, etc.

We next ask what scattering processes could produce a **35**. From the lemma, we can neglect $\bar{\mathbf{6}}\mathbf{6}$ channels even though they can link to a **35**, because the **6** is rank-one while **35** is rank-two. However, there are

[11] Donald E. Neville, Phys. Rev. **160**, 1375 (1967).

[12] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964); F. Calogero, J. M. Charap, and E. J. Squires; Ann. Phys. (N. Y.) **25**, 325 (1964).

plenty of other channels, since **35** occurs

$$\text{once in } \mathbf{15}\otimes\overline{\mathbf{15}},\ \mathbf{21}\otimes\overline{\mathbf{21}},\ \mathbf{15}\otimes\overline{\mathbf{21}},\ \mathbf{21}\otimes\mathbf{15}, \tag{2.5a}$$

$$\text{twice in } \mathbf{35}\otimes\mathbf{35}=\mathbf{35}_F+\mathbf{35}_D+\cdots. \tag{2.5b}$$

The $\mathbf{35}\otimes\mathbf{35}=35_F$ and $\mathbf{35}\otimes\mathbf{35}=35_D$ vertices have $\eta_{CG}=-1$ and $+1$, respectively. We now prove

$$V_+(\mu_d=\mathbf{35})\cong V_-(\mu_d=\mathbf{35}) \tag{2.6a}$$

for processes with both the initial and the final channels in list (2.5a);

$$V_\pm(\mu_d=\mathbf{35}_D)\cong V_\mp(\mu_d=\mathbf{35}_F) \tag{2.6b}$$

for processes with the initial or final channel, but not both, in (2.5a); and

$$\begin{aligned} V_\pm(\mu_d=35_{DD})&\cong V_\mp(\mu_d=35_{FF}),\\ V_\pm(\mu_d=35_{DF})&\cong V_\mp(\mu_d=35_{FD}) \end{aligned} \tag{2.6c}$$

for the process **35 35** $\rightarrow$ **35 35**. The notation in Eqs. (2.6b) and (2.6c), $\mathbf{35}_{DF}$, for instance, means that the intermediate 35 is D-coupled to the initial **35 35** pair and F-coupled to the final **35 35** pair. Evidently, Eqs. (2.6a)–(2.6c) imply just what is wanted: If the potential is such as to produce a **35** resonance with signature ϵ, it will also produce a **35** with signature $-\epsilon$, and all D/F ratios inverted. Proof of (2.6a): From the lemma, we need consider only crossed rank ≥ 2 channels, since **35** is rank two. As in Eq. (2.4), we label the crossed channels $S2$ and $S3$, such that $V_+(d)+V_-(d)\propto S2$-channel exchanges and $V_+(d)-V_-(d)\propto S3$-channel exchanges. For a process with the initial and final channels in (2.5a), there is one rank-two and one rank-four representation in channel $S2$, while channel $S3$ contains only rank-four representations. From the lemma, the rank-four channels can not be resonant, because we have input only rank ≤ 2 representations. Hence $[A(\mu_{S3})]_L$ is negligible in Eq. (2.4), and Eq. (2.6a) follows. (We have glossed over a fine point: Actually, $[A(\mu_{S3})]_L$ must be not only negligible, but negligible of order r^{-2}, since from Capps's formula (1.3) the $C(S1,S3)$ multiplying $[A(\mu_{S3})]$ is order r. The forces *into* the $S3$ channel are order r^{-1}, from Capps's formula again with $N_c=$order r^2 and $N_d=$order r^4. Hence $A(\mu_{S3})=$order r^{-1}, while its discontinuity$=\mathrm{Im}A=|A|^2$ will be of order r^{-2}, as required.) Proof of (2.6b): Again, from the lemma and the discussion just preceding, we need consider only the contributions to $V_\pm$ from crossed rank-two channels. There are two such in each of $S2$ and $S3$, so that $V_\pm$ will contain crossing matrix elements from the following 2×2 sub-blocks of $C(S1,S2)$ and $C(S1,S3)$ (these sub-blocks were calculated by the method of the Appendix):

$$C(S1,S2)\cong \begin{array}{c} \\ 35_D \\ \\ 35_F \end{array}\!\!\begin{array}{c} \begin{array}{cc} \mathbf{15} & \mathbf{21} \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ & \\ -1 & -1 \end{array}\right] \end{array} \eta_1\eta_2/2^{3/2}, \tag{2.7}$$

$$C(S1,S3)\cong \begin{array}{c} \\ 35_D \\ \\ 35_F \end{array}\!\!\begin{array}{c} \begin{array}{cc} \overline{\mathbf{15}} & \overline{\mathbf{21}} \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ & \\ 1 & 1 \end{array}\right] \end{array} \eta_1\eta_2/2^{3/2}. \tag{2.8}$$

The $S1$, $S2$, and $S3$ channels are defined by

$$\begin{aligned} &\mu_1\bar{\mu}_2\rightarrow\mu_{S1}\rightarrow\bar{\mu}_3\mu_4,\\ &\mu_1\mu_3\rightarrow\mu_{S2}\rightarrow\mu_2\mu_4,\\ &\mu_1\bar{\mu}_4\rightarrow\mu_{S3}\rightarrow\mu_2\bar{\mu}_3, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\mu_1,\mu_2=\mathbf{15}\text{ or }\mathbf{21};\quad \mu_3,\mu_4=\mathbf{35},\\ &\eta_i=+1\ (-1)\text{ for }\mu_i=\mathbf{21}\ (\mathbf{15}). \end{aligned} \tag{2.10}$$

[There are two more matrices, identical to (2.7) and (2.8), for processes with the **35 35** pair in the *initial* state of the direct channel.] No matter which of the channels μ_{S2}, and μ_{S3} are resonant, the desired result, Eq. (2.6b), follows immediately, merely from the pattern of signs in Eqs. (2.7), (2.8), and (2.4). Proof of Eq. (2.6c): Similarly, the desired result follows immediately from the pattern of signs in the sub-blocks of $C(S1,S2)$ and $C(S1,S3)$ such that μ_{S1}, μ_{S2}, μ_{S3} are rank two. With the channels as in Eq. (2.9), and $\mu_1=\mu_2=\mu_3=\mu_4=\mathbf{35}$, the relevant sub-blocks are

$$2C(S1,S2)\cong \begin{array}{c} \\ 35_{DD} \\ 35_{FF} \\ 35_{DF} \\ 35_{FD} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 35_{DD} & 35_{FF} & 35_{DF} & 35_{FD} \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & & \\ 1 & 1 & & \\ & & -1 & -1 \\ & & -1 & -1 \end{array}\right] \end{array}, \tag{2.11}$$

$$2C(S1,S3)\cong \left[\begin{array}{cccc} 1 & -1 & & \\ -1 & 1 & & \\ & & -1 & 1 \\ & & 1 & -1 \end{array}\right]. \tag{2.12}$$

Elements left blank in matrices (2.11) and (2.12) are identically zero.

One can continue in the same fashion, through the rest of the low-dimensional representations of $SU(n)$, $O(n)$, and $Sp(n)$. The results desired always follow either from the lemma alone, or from the lemma plus analysis of the pattern of signs in relevant sub-blocks of $C(d,c)$.

Incidentally, we have also proved in passing another statement made in the Introduction: There are plenty of order-unity elements $C(d,c)$ mediating the forces into the annihilation channel (labeled $S1$ above). Note that twins are not expected to cancel one another at low energies, the way they do at high energies. At low energy the amplitude for exchange of μ_c is very signature-dependent. This means (fortunately) that the forces producing the nonsinglet annihilation channel multiplets should remain finite even after the presence of twin forces is taken into account; it also means (unfortunately) that twin formation will not destroy the stability of the low-energy region.

III. COULOMB EFFECTS

This section amplifies the few remarks made about the electromagnetic interaction in the Introduction.

The contribution to $\sigma_{\rm tot}(\mu_d)$ from photon exchange can exceed the contribution from exchange of a strongly interacting particle if the $\mu_i\bar{\mu}_i \to \gamma$ vertex transforms as a singlet. The strength of the γ pole in the t channel of $\mu_1\mu_2 \to \mu_1\mu_2$ is measured by the quantity

$$e_1e_2\delta(m_1,\bar{m}_1)\delta(m_2,\bar{m}_2)\,, \quad (\gamma=\text{singlet}) \tag{3.1}$$

where m_i and $\bar{m}_i$ index the particles in multiplets μ_i and $\bar{\mu}_i$.[13] Equation (3.1) simply says that for γ = singlet the charge e_i is the same for each particle in multiplet μ_i. For exchange of a strongly interacting singlet, the e_i in Eq. (3.1) are replaced by coupling constants g_i of strong-interaction magnitude, while the $\delta(m_i,\bar{m}_i)$ are replaced by G Clebsch-Gordan (CG) coefficients for $\mu_1\otimes\bar{\mu}_i$ = singlet:

$$g_1g_2\delta(m_1,\bar{m}_1)N_1^{-1/2}\delta(m_2,\bar{m}_2)N_2^{-1/2}. \tag{3.2}$$

It is the factors of $N_i^{-1/2}$ which cause the force from exchange of a strongly interacting singlet to "turn off" as $r\to\infty$ [cf. Eq. (1.2)]; the expression for γ singlet exchange, Eq. (3.1), does not have these factors, hence stays order unity (or more precisely, order 137^{-1}) in the limit $r\to\infty$. Indeed, the group-theory factor multiplying the γ exchange amplitude for γ = singlet is Eq. (1.2) times $(N_1N_2)^{-1/2}$, i.e., unity. The contribution from nonsinglet Regge pole μ_t, on the other hand, goes as $C(\mu_d,\mu_t)\leq$ order $(N_t/N_d)^{1/2}$, and can vary from order unity, for N_d small, to order $r^{-1/2}\ll$ order 137^{-1} for N_d large.

If the $\gamma\to\mu_i\bar{\mu}_i$ vertex transforms as one of the generators of G, then on the average γ exchange is always order 137^{-1} weaker than nonsinglet Regge-pole exchange, even when the latter is order $r^{-1/2}$ or r^{-1}. The electromagnetic (EM) vertex for $\gamma\to\mu_i\bar{\mu}_i$ transforming as a generator M of G is

$$e^2\langle\bar{\mu}_1\bar{m}_1|M^{(1)}|\mu_1m_1\rangle\langle\bar{\mu}_2\bar{m}_2|M^{(2)}|\mu_2m_2\rangle; \tag{3.3}$$

$M^{(i)}$ is the $N_i\otimes N_i$ dimensional representation of generator M. We suppose $M^{(i)}$ is diagonal, and normalized that all its matrix elements are integers. $M^{(i)}$ is proportional to a CG coefficient for $\mu_i\otimes\bar{\mu}_i$ = adjoint representation, so that Eq. (3.3) can be rewritten

$$e^2\langle\mu_1m_1\bar{\mu}_1\bar{m}_1|\mu_{\rm adj}m\rangle[{\rm tr}M^{(1)2}]^{1/2}\langle\mu_2m_2\bar{\mu}_2\bar{m}_2|\mu_{\rm adj}m\rangle \times{\rm tr}M^{(2)2}]^{1/2}, \tag{3.4}$$

if M is the mth generator of G. In the form (3.4), γ exchange is easier to compare to μ_t exchange, which is

$$g_1'g_2'\sum_j\langle\mu_1m_1\bar{\mu}_1\bar{m}_1|\mu_tm\rangle\quad\langle\mu_2m_2\bar{\mu}_2\bar{m}_2|\mu_tm\rangle. \tag{3.5}$$

We now switch from charge space to states of definite μ_d, m_d by multiplying Eqs. (3.4) and (3.5) by appropriate CG coefficients

$$\langle\mu_1m_1\mu_2\bar{m}_2|\mu_dm_d\rangle\langle\mu_1\bar{m}_1\mu_2m_2|\mu_dm_{\rm d}\rangle\,, \tag{3.6}$$

and summing over $m_1\bar{m}_1m_2\bar{m}_2$ but not m_d. Equation (3.5) becomes simply

$$g_1'g_2'C(\mu_{\rm d},\mu_t)\,. \tag{3.7}$$

The same holds for every m_d value because of G invariance. Equation (3.4) becomes an expression which varies with m_d because there is no sum over m in Eq. (3.4). We average over m_d in order to estimate a typical value for this expression; i.e., we operate upon it with

$$N_d^{-1}\sum_{m_d}. \tag{3.8}$$

The averaging (3.8) yields a CG sum which is independent of m, for the same reason that $C(\mu_d,\mu^t)$ is independent of m_d. Hence we can average over m as well as over m_d, and Eq. (3.4) becomes

$$e^2[{\rm tr}M^{(1)2}\,{\rm tr}M^{(2)2}]^{1/2}[C(\mu_d,\mu_t=\mu_{\rm adj})/N_{\rm adj}]. \tag{3.9}$$

The result of applying operator (3.8) to Eq. (3.7) is just Eq. (3.7) again. Hence, on the average, γ and μ_t exchange differ by a factor

$$(e^2/g_1'g_2')[{\rm tr}M^{1)2}\,{\rm tr}M^{(2)2}]^{1/2} \times N_{\rm adj}^{-1}[C(\mu_d,\mu_{\rm adj})/C(\mu_d,\mu_t)]. \tag{3.10}$$

The only processes for which Coulomb effects are of interest are those $SU(n)$ processes not affected by pair cancellation; for such processes $\mu_t=\mu_{\rm adj}$ and the last bracket is ±1 (it is usually $+1$, but can be -1 if μ_t is D-coupled while $\mu_{\rm adj}$ is F-coupled). Further, unless the particles in multiplets μ_1 and μ_2 are highly charged,

$$[{\rm tr}M_j^{(1)2}\,{\rm tr}M_j^{(2)2}]^{1/2}=\text{order }(N_1N_2)^{1/2} \leq\text{order }r^2. \tag{3.11}$$

Since $N_{\rm adj}^{-1}$ = order r^{-2}, ratio (3.10) is of order $(e^2/g_1'g_2')$ independent of μ_d. Q.E.D.

Dashen and Frautschi[14,15] have shown that the transformation properties of the electromagnetic and other

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[13] As pointed out in Sec. II of Ref. 11, sometimes it is necessary to distinguish between states $|\mu^*m\rangle$ and states $|\mu,\bar{m}\rangle$: The former have the simple properties under crossing and are rotated by matrices $D^*\epsilon G$ if $|\mu,m\rangle$ is rotated by D; the latter have the correct phases for use with Clebsch-Gordan coefficients and differ only by a unitary transformation from the former. To be absolutely correct in notation, one should replace the labels $\bar{\mu}_i$, $\bar{m}_i$ in Eqs. (3.1)–(3.5) by labels μ_i^*, m_i, then carry out a unitary transformation to $\bar{\mu}_i$, $\bar{m}_i$ states just before Eq. (3.6).

[14] Roger F. Dashen and Stephen C. Frautschi, Phys. Rev. **143**, 1171 (1966); **145**, 1287 (1966).

[15] These equations have the same group-theoretical structure as Dashen and Frautschi's equations for computation of first-order mass splittings, and this structure is such as to favor mass splittings (or weak currents) transforming as the lowest-dimensional representations of G. [See Donald E. Neville, Nuovo Cimento **43A**, 995 (1966).] Therefore, even if the entity being exchanged does not couple to a conserved current, it is nevertheless likely to couple to hadrons via a vertex transforming as a rank-two multiplet or singlet. The discussion given in the present section for the electromagnetic current would then apply to nonconserved current as well.

weaker currents are constrained by the unitarity equations which such currents must obey. Thus even though the weaker interactions are not bootstrapped, it is plausible (if not provable in first approximation) that the weaker currents might be constrained to transfer purely as generators and not at all as singlets.

The difficulties with electromagnetism described in this section are significant for two reasons. Firstly, the hope of S-matrix theory is to solve the strong-interaction part of the elementary-particle problem without having to solve the entire problem, weak, electromagnetic, and strong; yet in the present instance we have not been able to do this. Secondly, the present calculation is very crude; anyone who has considered the matter expects that at some stage in the S-matrix program the effects due to the weaker interactions will have to be included; but it is unexpected to find such effects coming in already at the first-approximation stage.

IV. DEPENDENCE OF σ_{tot} ON SPIN

Let us assume that the external scattered particles have spin indices which are not averaged over in computing σ_{tot}, so that σ_{tot}, β, and ImA depend on helicity arguments:

$$\begin{aligned}\sigma_{\text{tot}}(\mu_d)&=\sigma_{\text{tot}}(\mu_d;\lambda_1\lambda_2)\,,\\ A&=A(\lambda_1\lambda_2\lambda_3\lambda_4)\,,\\ \beta&=\beta(\lambda_1\lambda_2\lambda_3\lambda_4)\,.\end{aligned}\tag{4.1}$$

λ_i is the helicity of scattered multiplet μ_i. In the optical theorem the non-spin-flip amplitude must be used:

$$\lambda_1=\lambda_3\,,\quad \lambda_2=\lambda_4\,.\tag{4.2}$$

We define a quantity $B(\mu_t)$ as the result of applying a helicity crossing matrix $H=H(\lambda_1\lambda_2\lambda_3\lambda_4;\lambda_1'\lambda_2'\lambda_3'\lambda_4';st)$ to the amplitude $A(\mu_t)$ for μ_t exchange:

$$\sigma_{\text{tot}}(\mu_d)\propto \operatorname{Im}A(\mu_d;\lambda_1\lambda_2\lambda_1\lambda_2)=\sum_{ti,\lambda_i}C(d,ti)\operatorname{Im}B(\mu_{ti};\lambda_1\lambda_2\lambda_1\lambda_2)=\sum C(d,ti)\operatorname{Im}[HA(\mu_{ti};\lambda_1'\lambda_2'\lambda_3'\lambda_4')]$$

$$\propto\sum C(d,t_i)\operatorname{Im}\{H\xi_{\lambda_1'\lambda_3'}\xi_{\lambda_2'\lambda_4'}[d_{\lambda_{\text{in}}',-\lambda_f'}{}^{\alpha_i}(\pi-\theta_t)(-1)^{\lambda_i'}+\epsilon_i d_{\lambda_{\text{in}}',\lambda_f'}(\theta_t)]\}\,,\tag{4.3}$$

$$\lambda_{\text{in}}'=\lambda_1'-\lambda_3'\,;\quad \lambda_f'=\lambda_2'-\lambda_4'\,.\tag{4.4}$$

The last line of Eq. (4.3) is the expression given by Muzinich[16] for the contribution from exchange of one Regge pole, except that we have dropped some irrelevant functions of t only. The residue $\beta(\lambda_1'\lambda_2'\lambda_3'\lambda_4')$ has been factorized into coupling constants $\xi_{\lambda_1'\lambda_3'}\xi_{\lambda_2'\lambda_4'}$, using unitarity. The d^α are rotation matrices, and at infinity,

$$\operatorname{Im}\lim_{s\to\infty}d_{\lambda\text{in}',-\lambda_f'}{}^{\alpha}(\pi-\theta_t)(-1)^{\lambda_i'}\quad\text{or}\quad\operatorname{Im}\lim_{u\to\infty}d_{\lambda\text{in}',\lambda_f'}{}^{\alpha}(\theta_t)=\text{real const}\times(s\text{ or }u)^\alpha\operatorname{Im}\exp[-i\pi\alpha+\tfrac{1}{2}i\pi(\lambda_{\text{in}}'-\lambda_f')]\,;$$

$$\operatorname{Im}\lim_{u\to\infty}d_{\lambda',\text{in}\lambda_f'}{}^{\alpha}(\pi-\theta_t)(-1)^{\lambda_i'}\quad\text{or}\quad\operatorname{Im}\lim d_{\lambda\text{in}',\lambda_f'}{}^{\alpha}(\theta_t)=\text{real const}\times(u\text{ or }s)^\alpha\operatorname{Im}\exp[\tfrac{1}{2}i\pi(\lambda_{\text{in}}'-\lambda_f')]\,.\tag{4.5}$$

This section considers three questions about the behavior of the $\operatorname{Im}B(\mu_{ti})$:

(a) What happens to $\operatorname{Im}B(\mu_{ti})$ when particles μ_2 and μ_4 are replaced by their antiparticles; i.e., what happens under "line reversal," as Wagner and Sharp[17] term this replacement;

(b) what happens when one of the helicity arguments, say, $\lambda_1=\lambda_3$, is left fixed but the other argument, $\lambda_2=\lambda_4$, is allowed to vary; and

(b′) [a less ambitious version of question (b)] what happens when the sign of $\lambda_2=\lambda_4$ is changed?

For the purposes of this paper we are interested only in the case that μ_{ti} is a Regge trajectory. Every equation derived in this section, however, is valid also for a non-Regge exchange, provided that the equation does not involve a factor of ϵ somewhere.

As to question (a), if only crossing symmetry is assumed, then

$$\lim_{s\to\infty}\operatorname{Im}B(\mu_c;\lambda_1\lambda_2\lambda_1\lambda_2)=\lim_{u\to\infty}\times\operatorname{Im}\bar{B}(\mu_c;\lambda_1-\lambda_2\lambda_1-\lambda_2)\epsilon\,,\quad(t=0)\,.\tag{4.6}$$

[16] I. J. Muzinich, J. Math. Phys. **5**, 1481 (1964).

[17] W. G. Wagner and David H. Sharp, Phys. Rev. **128**, 2899 (1962).

The amplitudes B and $\bar{B}$ are crossed to the $\mu_1\mu_2\to\mu_1\mu_2(s)$ and $\mu_1\bar{\mu}_2\to\mu_1\bar{\mu}_2(u)$ elastic scattering channels, respectively; ϵ is the signature of Regge trajectory μ_c. If charge-conjugation invariance is assumed, relation (4.6) is true without the helicity-flip;

$$B(\mu_c;\lambda_1\lambda_2\lambda_3\lambda_4)=\bar{B}(\mu_c;\lambda_1\lambda_2\lambda_3\lambda_4)\eta_{\text{ch}}\,.\tag{4.7}$$

Here η_{ch} is the charge-conjugation parity of exchanged trajectory μ_c.

Equations (4.6) and (4.7) suffice if we are working in charge space; otherwise, we must consider also the change in the factor $C(s,c)\to C(u,c)$. As in the Introduction, Eqs. (1.7)–(1.9), when the scattered multiplet is equivalent to its complex conjugate, $C(s,c)$ and $C(u,c)$ are simply related; either $C(s,c)=\eta_{\text{CG}}\;C(u=s,c)$ or $C(s,c)=\eta_{\text{CG}}C(u=\bar{s},c)$. In such cases it is not possible to change the sign of σ_{tot} if the leading trajectories satisfy the constraint

$$\eta_{\text{CG}}=\epsilon=\eta_{\text{ch}}\,,\tag{4.8}$$

where $\eta_{\text{CG}}=\pm1$ is the interchange parity at the $\mu_t\to\bar{\mu}_2\mu_2$ vertex, as in Eq. (1.9).

[Throughout this section we shall treat only the simple case in which $C(s,c)$ and $C(u,c)$ differ only by ±1; the discussion for the general case is the same, except that $C(s,c)$ and $C(u,c)$ are completely unrelated, it is necessary to calculate the latter from scratch, and contributions to σ_{tot} from μ_t may change in magnitude as well as in sign when $s \leftrightarrow u$.]

Conceivably, both members of a trajectory pair could satisfy Eq. (4.8) simultaneously, since twins have opposite η_{ch} as well as opposite ϵ. [Proof: if one trajectory gives rise to resonance in a particle-antiparticle channel with orbital angular momentum L, total spin S, then $\eta_{\text{ch}}=(-1)^{L+S}$. The twin trajectory can produce resonances only at $L\pm1$, hence has opposite η_{ch}.]

To prove Eq. (4.6), one simply calculates H, first for $t\to s$, then for $t\to u$ crossing, using the recipes for H derived by Trueman and Wick[18] or by Muzinich[16]; one then compares results. According to these authors, H is of the form

$$H=d_{\lambda_1'\lambda_1}{}^{s_1}(\omega_1)d_{\lambda_2'\lambda_2}{}^{s_2}(\omega_2)d_{\lambda_3'\lambda_3}{}^{s_3}(\omega_3)d_{\lambda_4'\lambda_4}{}^{s_4}(\omega_4)(-1)^{\eta}. \quad (4.9)$$

S_i is the intrinsic spin of μ_i. In the physical region of the direct or crossed channels, the $\omega_i=\omega_i(s,t)$ are real angles. In the special case of forward elastic scattering ($t=0$, external masses $m_1=m_3$, $m_2=m_4$), $\omega_i=\frac{1}{2}\pi$. The following choice of phase agrees with the Trueman-Wick choice up to unobservable factors of $(-1)^{2s_i}$:

$$\begin{aligned}\eta(t\to s)&=(\lambda_2'-\lambda_3')+(\lambda_4'-\lambda_3')+(\lambda_4'-\lambda_2'),\\ \eta(t\to u)&=(\lambda_4'-\lambda_3')+(\lambda_4'-\lambda_3')+(\lambda_2'-\lambda_4').\end{aligned} \quad (4.10)$$

The first set of parentheses in each case is the difference of the helicities of the particles which are crossed (μ_2 and μ_3 are crossed on going from t to s; μ_4 and μ_3 on going from t to u). The second set of parentheses is the difference of the helicities of the particles which are "particle 2" in the crossed t channel (i.e., coupling order is significant; the amplitudes for $\mu_1\bar{\mu}_3\to\bar{\mu}_2\mu_4$ and $\mu_1\bar{\mu}_3\to\mu_4\bar{\mu}_2$ differ in phase, and we have chosen the first of these as the amplitude for the crossed channel reaction.) The third set of parentheses is the difference of the helicities of the particles which are "particle 2" in the direct channel. Equation (4.10) implies

$$\eta(t\to u)=\eta(t\to s)+(\lambda_2'-\lambda_4'). \quad (4.11)$$

Hence, except for a factor $(-1)^{\lambda_2'-\lambda'_4}$, $H(t\to s)$ and $H(t\to u)$ are identical. This phase factor may be removed by applying the identity

$$d_{\lambda'\lambda}{}^{s}(\omega)(-1)^{\lambda'-s}=d_{\lambda}{}^{s}{}_{\prime,-\lambda}(\pi-\omega), \quad (4.12)$$

to both $d^{s_2}(\pi/2)$ and $d^{s_4}(\pi/2)$. Equation (4.6) then follows.

Equation (4.7) is proved by invoking $T=U_c^{-1}TU_c$, U_c is the unitary charge conjugation transformation which reverses charge labels but does nothing to spin and momentum, hence does nothing to helicity:

$$\begin{aligned}\langle\bar{\mu}\bar{m}_2\mu m_4;JM\lambda_2\lambda_4|T|\mu_t m_t;JM\rangle&=\langle\cdots|U_c^{-1}TU_c|\cdots\rangle\\ &=\eta_{\text{ch}}\langle\mu m_2\bar{\mu}\bar{m}_4;JM\lambda_2\lambda_4|T|\bar{\mu}_t\bar{m}_t;JM\rangle.\end{aligned} \quad (4.13)$$

$|\bar{\mu}\bar{m}_i\rangle$ and $|\bar{\mu}_t\bar{m}_t\rangle$ are the antiparticles to $|\mu m_i\rangle$ and $|\mu_t m_t\rangle$, $\mu=\mu_2=\mu_4$.[19] Since $\mu\bar{\mu}$ are particle-antiparticle, their charge-conjugation parties cancel each other; η_{ch} is the charge parity of μ_t. $|\bar{\mu}_t\bar{m}_t\rangle$ and $|\mu_t m_t\rangle$ belong to the same multiplet [or if they do not, as for **10** and $\mathbf{\overline{10}}$ mesons in $SU(3)$, then linear combinations $|10m\rangle\pm|\overline{10}\bar{m}\rangle$ should be used in the kets of Eq. (4.13)]. From G invariance, the T-matrix elements in Eq. (4.13) are proportional to CG coefficients, so that the last line of Eq. (4.13) can be rewritten

$$\eta_{\text{ch}}\frac{\langle\mu m_2\bar{\mu}\bar{m}_4|\bar{\mu}_t\bar{m}_t\rangle}{\langle\mu m_4\bar{\mu}\bar{m}_2|\mu_t m_t\rangle}\times\langle\mu m_4\bar{\mu}\bar{m}_2;JM\lambda_2\lambda_4|T|\mu_t m_t;JM\rangle. \quad (4.14)$$

The initial T-matrix element, first line, Eq. (4.13) is used in computing $B(\mu_c)$, while the final T-matrix element, Eq. (4.14) is used in computing $\bar{B}(\mu_c)$. Hence all that remains to be shown is that the CG ratio in Eq. (4.14) is $+1$ independent of the m_i. It is certainly independent of the m_i because the Wigner-Eckart theorem for G may be used to write the initial and final T-matrix elements as the same tensor times perhaps different reduced matrix elements, with all the m_i dependence contained in the tensor.

The ratio is certainly $+1$ whenever there exists a set of m_i, $m_i=m_{i0}$, such that $\langle\mu m_{20}\bar{\mu}m_{40}|\bar{\mu}_t\bar{m}_{t0}\rangle\neq0$, $|\bar{\mu}_t\bar{m}_{t0}\rangle=|\mu_t m_{t0}\rangle$, and $m_{20}=m_{40}$ (i.e., whenever μ_t contains a self-conjugate particle m_{t0} which can couple to a particle-antiparticle pair m_{20}, $\bar{m}_{40}$). In every case we know of such m_{i0} exist (or can be constructed from linear combinations of the m_i). Consequently, we have stated our result as at Eq. (4.7). If the ratio should turn out to be a constant $\neq1$ in a given case, then this constant would multiply the right-hand side of Eq. (4.7).

Wagner and Sharp[17] derive symmetry (4.7) for the special cases $s_i\leq\frac{1}{2}$, and our result agrees with theirs. The agreement is not obvious for the $s_i=\frac{1}{2}$ case since these authors use a Dirac γ-matrix formalism and express their result as $\pm\epsilon$ times the sign change of each Dirac covariant under charge conjugation. In order to establish the agreement, it is necessary to evaluate each Dirac covariant in the t-channel center-of-mass frame, determine thereby the values of L and S allowed at the vertex, and then relate η_{ch} to ϵ via $\eta_{\text{ch}}=(-1)^{L+S}$, $\epsilon=(-1)^J$ for bosons, $J=L$ or $L\pm1$ depending on the parity. Formula (4.4) is much easier to remember than is the formula given by Wagner and Sharp.

As for question (b) [how does $\text{Im}B(\mu_c;\lambda_1\lambda_2\lambda_1\lambda_2)$

[18] T. L. Trueman and G. C. Wick, Ann Phys. (N. Y.) **26**, 322 (1964).

[19] Again (compare Ref. 13) to be absolutely correct in the notation, the labels $\bar{\mu}_i$, $\bar{m}_i$ should be replaced by μ_i^*, m_i.

vary with λ_2], we suspect it is far too general to be answered in any model-independent way. The λ_2 dependence of $B(\mu_c)$ comes from the factor

$$d_{\lambda_2'\lambda_2}{}^{s_2}(\pi/2)d_{\lambda_4'\lambda_4}{}^{s_2}(-1)^{(\lambda_{\text{in}}'-\lambda_f')/2}\xi_{\lambda_2'\lambda_4'}. \quad (4.15)$$

Because of the constraint $\lambda_2=\lambda_4$, there are only $(2S_2+1)$ linear combinations (4.15) of the $\xi_{\lambda_2'\lambda_4'}$ which contribute to total cross sections. Yet there are $(2S_2+1)^2$ independent $\xi_{\lambda_2'\lambda_4'}$ [or roughly $(2S_2+1)^2/4$ independent $\xi_{\lambda_2'\lambda_4'}$, if parity and charge-conjugation invariance are taken into account]. Hence there should be no difficulty in finding a set of $\xi_{\lambda_2'\lambda_4'}$ such that every linear combination (4.15) will have the same sign.

Evidently, from the preceding discussion there is no point to relaxing any of the C, P, T constraints, since negative cross sections would then be harder, not easier, to obtain. If we invoke either C or P we can determine the behavior of $B(\mu_c)$ when $\lambda_2\to-\lambda_2$ [question (6′)]. Invoking C:

$$A(\mu_c;\lambda_1\lambda_2\lambda_1\lambda_2)=A(\mu_c;\lambda_1-\lambda_2\lambda_1-\lambda_2)(-1)^J\eta_{\text{ch}} \quad (t=0). \quad (4.16)$$

Invoking P:

$$A(\mu_c;\lambda_1\lambda_2\lambda_1\lambda_2)=A(\mu_c;\lambda_1-\lambda_2\lambda_1-\lambda_2)(-1)^J\eta_p \quad (t=0). \quad (4.17)$$

η_p is the intrinsic parity of μ_c. J is any spin value at which the amplitude of signature ϵ reduces to the physical amplitude in the t channel. For boson trajectories, it happens that $(-1)^J=\epsilon$; hence parity conservation implies a constraint

$$\epsilon=\eta_p, \quad (4.18)$$

just as charge-conjugation invariance implies constraint (4.8).

Evidently Eqs. (4.16) and (4.17) also require $\eta_{\text{ch}}=\eta_p$, i.e., CP parity$=+1$, or μ_c will not contribute to σ_{tot}. This selection rule following from CP invariance is doubtless familiar to many readers. The pion is a good example: It has $\eta_{\text{ch}}=-\eta_p=1$, and the non-spin-flip amplitude from pion exchange vanishes in the forward direction. Hence, given CP conservation, constraints (4.8) and (4.18) are not entirely independent.

Equations (4.16) and (4.17) are proved using the constraints

$$\xi_{\lambda_2'\lambda_4'}=\xi_{-\lambda_2',-\lambda_4'}\eta_p\eta_{p2}\eta_{p4}(-1)^{J-s_2-s_4} \quad (4.19)$$

$$=\xi_{-\lambda_2',-\lambda_4'}\eta_p(-1)^J, \quad (4.20)$$

and

$$\xi_{\lambda_2'\lambda_4'}=\xi_{\lambda_4'\lambda_2'}\eta_{\text{ch}}(-1)^J, \quad (4.21)$$

imposed by P and C conservation, respectively. Equations (4.20) and (4.21) are true in particle-antiparticle channels only. Equation (4.21) is presumably a known symmetry, although it is not listed by Jacob and Wick.[20]

[20] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

Probably the easiest way to derive it is to write

$$|JM\lambda_2'\lambda_4'\rangle=\sum_{LS}\langle S_2\lambda_2'S_4-\lambda_4'|S\lambda_f'\rangle\langle LOS\lambda_f'|J\lambda_f'\rangle \times|LSJM\rangle[(2L+1)/(2J+1)]^{1/2}, \quad (4.22)$$

and then use the known action of charge conjugation on $|LSJM\rangle$;

$$U_c|LSJM\rangle=(-1)^{L+S}|LSJM\rangle. \quad (4.23)$$

Repeated use of the following $SU(2)$ CG symmetries replaces the factor $(-1)^{L+S}$ by a factor of $(-1)^J$ and interchanges $\lambda_2'\leftrightarrow\lambda_4'$:

$$(-1)^{J_1+J_2-J_3}\langle J_1m_1j_2m_2|j_3m_3\rangle =\langle j_1-m_1j_2-m_2|j_3-m_3\rangle =\langle j_2m_2j_1m_1|j_3m_3\rangle, \quad (4.24)$$

so that one gets

$$U_c|JM\lambda_2'\lambda_4'\rangle=(-1)^J|JM\lambda_4'\lambda_2'\rangle. \quad (4.25)$$

The action of U_c on $\xi_{\lambda_2'\lambda_4'}=\langle JM\mu_t|T|JM\lambda_2'\lambda_4'\rangle$, Eq. (4.21), then follows.

In summary, the highest-lying trajectories must obey either $\eta_{\text{ch}}=-\eta_p$ (in which case they do not contribute to σ_{tot}) or else

$$\eta_{\text{ch}}=\eta_p=\epsilon=(-1)^J=\eta_{\text{CG}}. \quad (4.26)$$

It is noteworthy that these constraints do not depend explicitly on the spins of the external multiplets; however, if the external particles are spinless and identical, then the constraints are automatically satisfied because of Bose statistics.

In view of our belief that low-energy dynamics in $SU(3)$ and G will be much the same, it is worthwhile pointing out that both leading nonsinglet trajectories in $SU(3)$ satisfy constraints (4.26). The vector octuplet trajectory has $\eta_p=\epsilon=-1$, and the Pignotti octuplet trajectory has $\eta_p=\epsilon=+1$.

Since we have already seen one miracle, pair cancellation, it is perhaps not untoward to speculate that a second miracle will take place and force all $\eta_{\text{ch}}=\eta_p$ trajectories to obey Eq. (4.26). In other words, perhaps there is a general dynamical principle which would rule out leading trajectories with axial-vector resonances.

V. CONCLUSION

(Please note also the concluding paragraph to each of Secs. III and IV.)

We conclude not only that two particular arguments are invalid (one against high-isospin scattering in I; and one against high-rank low-dimensional scattering in II); but also that there exist *no further* arguments which can be evaluated at present. We reason to the latter conclusion as follows. The systems studied appear stable in both the low-energy region (cf. I; and the first two paragraphs of II) and the high-energy region near the forward direction (cf. II). [The high-isospin systems of I cannot be proven unstable by the Regge argument of II because the specific example discussed in Sec. IV of I has no high-isospin resonance in any t channel; hence the high-isospin analog of Eq. (II1.4), is not satisfied by that example.] These two energy regions constitute the full range of applicability of present-day first-approximation dynamics; *there are no other energy regions left to examine.* Hence there are no other arguments left to be evaluated.

The state of dynamical ignorance just described could, of course, change overnight.

The present treatment is far less complete group-theoretically than dynamically. If we have covered every energy region, we certainly have not covered every symmetry. In particular, some knowledge of symmetries which are both of high rank and high dimensional would be useful for putting the present results in perspective. Is it possible to establish the stability of a G-symmetric system, no matter how high-dimensional the multiplets it may contain? Or are the systems thus far studied merely a tiny minority, the stable limits, as group rank or multiplet rank $\rightarrow 1$, of a much larger class of systems all of which are unstable? The present results at one and at the same time stand by themselves, yet are, so to speak, only the outer edges, or the frame, to a picture which has not yet been completely drawn.

It is possible to write recurrence relations which determine $SU(2)$ crossing matrix elements $C(I_d,I_c)$. Presumably recurrence relations can be written for a higher symmetry $C(\mu_d,\mu_c)$ as well. It is not clear at present whether such equations will be tractable; but doubtless they will be so in enough special cases that the rest of this picture can be sketched in.[21]

The argument of II works for low-dimensional systems only under special circumstances (presence of non-self-conjugate representations, weaker currents transforming as generators); but for higher-dimensional systems the argument may very well not work at all. Either the electromagnetic vertex for these systems transforms as a singlet, in which case photon exchange will keep total cross sections positive as in Sec. III, or the electromagnetic vertex transforms as a generator, in which case many states will be so highly charged that it becomes meaningless to talk about G invariance any longer. [Compare the increasing irrelevance of $SU(2)$ invariance for nuclear dynamics as atomic number increases.] Therefore, the stability of high-dimensional systems should be tested against the ideas of I, rather than those of II. In particular, the discussion given for isospin in Sec. II of I should also be valid for high dimensional scattering in G, with an appropriate change in notation $I_i \rightarrow \mu_i$: In order for the low-dimensional region to be stable, there must exist an element $C(I_d,I_c)=(N_c/N_d)^{1/2}O(I_d,I_c)$ such that $O(I_d,I_c)$ is of order unity even though O is a large matrix.

[21] For the higher groups, as for isospin, it is possible to construct highly correlated "strong-coupling" models analogous to those discussed in Sec. II of paper I, especially Ref. 11 of I. Therefore, for the higher groups, as for isospin, it will not be obvious how to remove the first-approximation restriction to systems having a small number of multiplets.

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APPENDIX: CROSSING MATRIX CALCULATIONS

In order to illustrate an approximation technique useful in the limit $r\rightarrow\infty$, in this Appendix, we derive enough elements $C(d,c)$ to verify pair cancelation for scattering of any four rank-two self-conjugate (μ and $\bar{\mu}$ equivalent) representations of G. The various channels are coupled in the orders

$$\begin{aligned}&\mu_1\mu_2\rightarrow\mu_s\rightarrow\mu_3\mu_4\,,\\&\mu_1\bar{\mu}_3\rightarrow\mu_t\rightarrow\bar{\mu}_2\mu_4\,,\\&\mu_1\bar{\mu}_4\rightarrow\mu_u\rightarrow\mu_3\bar{\mu}_2.\end{aligned}\tag{A1}$$

The following derivation is sketchy; it will be assumed the reader is already somewhat familiar with the tensor technique for calculating crossing matrices. Ordinarily calculations using this technique are lengthy because tensors must be made traceless. To begin with, it is necessary to calculate a complicated set of trace-removing terms to add on to each tensor; and then their presence in turn complicates all further manipulations with the tensor, such as the calculation of its normalization constant. In the limit $r\rightarrow\infty$, however, the trace-removing terms (abbreviated TRT in the formulas below) fall off as r^{-1} and can be neglected. Therefore the discussion which follows is lengthy only because it is necessary to explain all the symbols and conventions. The actual mathematics involved is a matter of some six lines of straightforward manipulation.

The states in multiplets $\mu_1,\cdots,\mu_4$ transform under G like the components of tensors $A_j{}^i$, $B_j{}^i$, $C_j{}^{i*}$, and $D_j{}^{i*}$. The asterisks denote complex conjugation, necessary because μ_3 and μ_4 are final states in the s channel. We adopt this channel as the direct channel. Strictly speaking, the one-upper-, one-lower-index notation is necessary only for $SU(n)$. Tensors for the two rank-two representations of $Sp(n)$ and $O(n)$ can be written with two lowered indices, $T_{ij}=\pm T_{ji}$, because for these groups upper and lower indices are equivalent. However, since we want the same notations and derivation to apply to all three groups, we will not use the

T_{ij} forms. For $O(n)$ we simply write the second index as an upper index ($T_{ij}\equiv T_i{}^j=\pm T_j{}^i$); while for $Sp(n)$ we raise the second index using the $n\times n$ skew symmetric form $g^{jk}=-g^{kj}$ preserved by the group ($T_i{}^j\equiv g^{jk}T_{ik}$).

We now calculate the group-theory factor P_t which appears in the amplitude for exchange of μ_t.[22]

Only amplitudes $P_{t\alpha}$ such that $\mu_{t\alpha}=$rank two are of interest since only these will contain Regge trajectories; $\alpha=1, 2, 3, \cdots$ indexes the irreducible representations μ_t occurring in the t channel; the rank-two $P_{t\alpha}$ are of the form

$$N_{t\alpha}^{-1}P_{t\alpha}=\sum_{ijkm}(A_i{}^j\bar{C}_j{}^k+\eta_{t\alpha}A_j{}^k\bar{C}_i{}^j)\times(\bar{B}_i{}^mD_m{}^k+\eta_{t\alpha}'\bar{B}_m{}^kD_i{}^m)^*+(\mathrm{TRT}),\quad (\mathrm{A2})$$

i.e., all have one index on each A contracted with one index on a C; and similarly for B and D. $\eta_{t\alpha}$ and $\eta_{t\alpha}'$ are the interchange parities η_{CG} of the particles in the initial and final states of the t channel. We have introduced the notation

$$\begin{aligned}\bar{T}_j{}^k&\equiv T_k{}^{j*}\quad[SU(n),Sp(n)],\\ \bar{T}_j{}^k&\equiv\pm T_k{}^{j*}\quad[O(n)]\end{aligned}\quad(\mathrm{A3})$$

to emphasize that under symmetrization and contraction an upper index on an asterisked tensor is equivalent to a lower index on an unasterisked one, and vice versa. For $O(n)$, the $\pm$ sign applies according as $T_k{}^j=\pm T_k{}^j$. For a given G the normalization constants $N_{t\alpha}^{-1}$ are independent of α up to terms of order unity:

$$\begin{aligned}N_{t\alpha}^{-1}&\cong 2n\quad[SU(n)],\\ N_{t\alpha}^{-1}&\cong 4n\quad[O(n),Sp(n)].\end{aligned}\quad(\mathrm{A4})$$

$C(\mu_s,\mu_{t\alpha})$ is calculated from $P_{t\alpha}$ by expanding $P_{t\alpha}$ in direct channel amplitudes P_s and then reading off coefficients;

$$P_{t\alpha}=\sum_{s\beta}C(\mu_{s\beta},\mu_{t\alpha})P_{s\beta}.\quad(\mathrm{A5})$$

Equation (A2) must therefore be rearranged from $ACBD$ (t-channel) order to $ABCD$ (s-channel) order:

$$\begin{aligned}N_{t\alpha}^{-1}P_{t\alpha}\cong&(A_i{}^jB_m{}^i)(C_k{}^jD_m{}^k)^*+\eta_{t\alpha}\eta_{t\alpha}'(A_j{}^kB_k{}^m(C_j{}^iD_i{}^m)^*\\&+\eta_{t\alpha}(A_j{}^kB_m{}^i)(C_j{}^iD_m{}^k)^*\\&+\eta_{t\alpha}'(A_i{}^jB_k{}^m)(C_k{}^jD_i{}^m)^*.\end{aligned}\quad(\mathrm{A6})$$

The TRT have been dropped. The first two terms are amplitudes for rank-two transitions in the direct channel (because of the contraction between A and B, and C and D); the last two are amplitudes for rank-four transitions. The first two terms can be broken up into irreducible amplitudes by expanding each set of parentheses in tensors having definite parity under interchange; e.g., for the first set of parentheses,

$$\begin{aligned}A_i{}^jB_m{}^i&=\tfrac{1}{2}(A_m{}^iB_i{}^j+A_i{}^jB_m{}^i)-\tfrac{1}{2}(A_m{}^iB_i{}^j-A_i{}^jB_m{}^i)\\&\equiv\tfrac{1}{2}[T_+(AB)-T_-(AB)].\end{aligned}\quad(\mathrm{A7})$$

The last two terms of (A5) are reduced by breaking up each set of parentheses into tensors of definite parity under interchange of upper or lower indices; e.g.,

$$\begin{aligned}A_j{}^kB_m{}^i=&\tfrac{1}{4}(A_j{}^kB_m{}^i+A_j{}^iB_m{}^k-A_m{}^kB_j{}^i-A_m{}^iB_j{}^k)\\&+\tfrac{1}{4}(\text{odd, even})+\tfrac{1}{4}(\text{odd, odd})+\tfrac{1}{4}(\text{even, even})\\&\equiv\tfrac{1}{2}[T_O{}^E(AB)+T_E{}^O(AB)\\&\qquad+T_O{}^O(AB)+T_E{}^E(AB)].\end{aligned}\quad(\mathrm{A8})$$

The first set of parentheses, first line of Eq. (A8) is (even, odd), i.e., even under interchange of i and k, and odd under interchange of j and m. When Eqs. (A7) and (A8) are inserted in (A6) cross terms like $T_O{}^E(AB)T_E{}^O(CD)^*$ vanish because anitisymmetric indices cannot be contracted with symmetric ones; the remaining terms are

$$\begin{aligned}N_{t\alpha}^{-1}P_{t\alpha}\cong&\tfrac{1}{4}(1+\eta_{t\alpha}\eta_{t\alpha}')[T_+(AB)T_+(CD)^*\\&+T_-(AB)T_-(CD)^*]+\tfrac{1}{4}(-1+\eta_{t\alpha}\eta_{t\alpha}')\\&\times[T_+(AB)T_-(CD)^*+T_-(AB)T_+(CD)^*]\\&+\tfrac{1}{4}(\eta_{t\alpha}+\eta_{t\alpha}')[T_E{}^E(AB)T_E{}^E(CD)^*\\&-T_O{}^O(AB)T_O{}^O(CD)^*]+\tfrac{1}{4}(\eta_{t\alpha}-\eta_{t\alpha}')\\&\times[T_O{}^E(AB)T_O{}^E(CD)^*-T_E{}^O(AB)T_E{}^O(CD)^*].\end{aligned}\quad(\mathrm{A9})$$

The rank-two representations of G which occur in the crossed channel also occur in the direct channel; thus the amplitudes in the first two brackets of (A9) are form-identical to the amplitudes $N_{t\alpha}^{-1}P_{t\alpha}$ of Eq. (A2). Hence we set

$$T_+(AB)T_+(CD)^*=P_{s\beta}N_{s\beta}^{-1},\quad\beta=1,$$

and similarly for $\beta=2, 3, 4$. Using $N_{s\beta}^{-1}\cong N_{t\beta}^{-1}$ as well as Eq. (A5), we get

$$C(\mu_{s\beta},\mu_{t\alpha})\cong\tfrac{1}{4}(\pm1+\eta_{t\alpha}\eta_{t\alpha}'),\quad(\mu_{s\beta},\mu_{t\alpha}\text{ rank two}).\quad(\mathrm{A10})$$

The last two sets of brackets of Eq. (A9) contain amplitudes $N_{s\beta}^{-1}P_{s\beta}$, $\mu_{s\beta}=$rank four, $N_{s\beta}^{-1}$ being normalization constant of order unity. Hence

$$C(\mu_{s\beta},\mu_{t\alpha})\cong\tfrac{1}{4}(\eta_{t\alpha}\pm\eta_{t\alpha}')N_{t\alpha}N_{s\beta}^{-1},\quad(\mu_{s\beta}\text{ rank four, }\mu_{t\alpha}\text{ rank two}).\quad(\mathrm{A11})$$

Whenever $(\eta_{t\alpha}\pm\eta_{t\alpha}')\neq0$, elements (A11) are of order r^{-1} because $N_{t\alpha}N_{s\beta}^{-1}$ is of order $n^{-1}=$ order r^{-1}.

Formulas (A9)–(A11) simplify considerably for $G=Sp(n)$ or $O(n)$, because $T_\pm(AB)$ is always orthogonal to $T_\pm(CD)$, whether the s channel is inelastic or not. Hence there are only two rank-two multiplets in each channel ($\alpha,\beta=1, 2$); the second bracket in Eq. (A9) always vanishes; $\eta_{t\alpha}=\eta_{t\alpha}'=+1(-1)$ according as $\alpha=1$

[22] The P_t should be called projection operators or amplitudes, according as the tensors $A_j{}^i, \cdots, D_i{}^{j*}$ are taken to transform as annihilation operators for states or as states. The latter convention is adopted here; either convention gives the same final answer for $C(d,c)$. The P_t used in Ref. 3 should also be called amplitudes.

(2); Eq. (A10) is always $+\frac{1}{2}$; and Eq. (A11) simplifies to either $\pm\frac{1}{2}N_{t\alpha}N_{s\beta}^{-1}$, or a number of order r^{-2}.

For $SU(n)$ [necessarily elastic; $SU(n)$ has only one rank-two self-conjugate multiplet; $\mu_1=\mu_2=\mu_3=\mu_4=\mathbf{35}$] the discussion is similar except that there are four rank-two amplitudes in each channel ($\alpha, \beta=1, \cdots 4$); they are the generalizations to $SU(n)$ of the $D\to D$, $F\to F$, $D\to F$, and $F\to D$ transitions of $SU(3)$. The sign $\eta_{t\alpha}$ (respectively $\eta_{t\alpha}'$) is $+1$ or -1 according as the initial (final) state in the crossed t channel is D- or F-coupled.

Equations (A11) and (A10) lead to crossing matrix elements of type (1.6a) and (1.6b), respectively, when the η_{CG} are such that the parentheses in Eqs. (A10) and (A11) do not vanish; when the parentheses vanish, Eqs. (A10) and (A11) yield crossing matrix elements of type (1.6c). Elements of type (1.6c) depend on the TRT neglected at Eq. (A2).

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Regge-Pole Analysis of $pn\to np$ and $p\bar{p}\to n\bar{n}$ Scattering*

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The differential cross sections for the reactions $pn\to np$ and $p\bar{p}\to n\bar{n}$ have been investigated. It is found that besides the ρ and $R(A_2)$ trajectories, the π and B trajectories must be included. A variety of schemes suggested by four-dimensional symmetry have been investigated. The existence of various daughter trajectories does not suffice to explain the data, though the data can be fitted with a parity doublet, of which the pion may or may not be a member. In the former case some structure must be introduced into the pion residue function.

INTRODUCTION

We have investigated the differential cross sections for the two charge-exchange processes (I) $pn\to np$, and (II) $p\bar{p}\to n\bar{n}$ within the framework of Regge-pole phenomenology.[1] In the absence of cuts these reactions are presumed controlled by the exchange of $I=1$, $B=0$, $Y=0$ trajectories. The main features of the data which must be explained are (a) the exceptionally sharp peak in the differential cross section of process I with a width of about 0.01 GeV^2, (b) the fact that this sharp peak persists to very low energies and the width is almost energy independent, (c) the large difference in the magnitudes of the cross sections for processes I and II at the same value of energy and momentum transfer (for $|t|>0.02$ GeV^2), and (d) the energy dependence of $p\bar{p}\to n\bar{n}$ data. Feature (c) can be explained only by the existence of both positive and negative G-parity trajectories which interfere with opposite signs in the two processes.

It has been known for some time that the data cannot be satisfactorily explained with only ρ and $R(A_2)$ trajectories. Even if rapidly varying residue functions are chosen so that the sharp peak of process I is fitted (and this can be done), the difference of magnitude of the two cross sections I and II cannot be explained, since the ρ and R trajectories are roughly equal over the region of interest, and having opposite signature, yield little interference. Moreover, small residues for ρ and R amplitudes are suggested by the total cross-section differences[2] ($\sigma_{\bar{p}p}-\sigma_{\bar{p}n}$ and $\sigma_{pp}-\sigma_{pn}$) which (while possessing large experimental errors) are consistent with zero in the high-energy region under consideration. Since only t-channel sense-sense triplet amplitudes which do not vanish at $t=0$ can contribute to s-channel total cross sections, in this analysis only the ρ and R contribute to these differences. It is therefore to be expected that lower-lying $I=1$ trajectories which have not been considered in the usual analysis of data up to the present time will play a prominent role here.

Qualitatively one might expect the pion trajectory to be an important factor in determining the sharp peak of the $pn\to np$ cross section, because of the proximity of the pion pole to the forward direction. Extrapolation of the pion residue to the known pion-nucleon coupling constant indicates in fact that the pion contribution must be large near the forward direction (whether or not the pion amplitude vanishes at $t=0$), and thus should be included in the analysis. Until recently, it was assumed that the amplitude to which the pion contributes must vanish at $t=0$, and thus it was difficult to see how the pion could give rise to a sharp peak. The recent developments in the understanding of

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