PHYSICAL REVIEW VOLUME 163, NUMBER 5 25 NOVEMBER 1967

Symmetries in First-Approximation S-Matrix Theory.*† I. High-Isospin Scattering

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(Received 14 October 1966; revised manuscript received 27 April 1967)

We construct a simple example of a high-isospin system and verify (in "first approximation"; that is, we drop all contributions to the unitarity relations except those from the nearest two-body channels, assume that the number of such channels is small, and approximate crossed singularities by poles) that it satisfies the requirements suggested by an S-matrix theory of strong interactions. The forces producing each particle act through isospin crossing matrix elements of reasonable size, and no crossing matrix element is so large that the force into a direct channel will cause pathologies such as negative-energy bound states or (in Regge theory) trajectories violating the Froissart limit. Strictly speaking, the stability of our example is not completely established, since we do not carry out a complete dynamical calculation (with some specific choice of spins and cutoff functions), but only study the sizes of the relevant isospin crossing matrix elements. It is also shown that a mechanism suggested by the "strong-coupling" model is not suitable for producing high-isospin particles, at least in the low-energy region, because if the model were treated in other than static approximation it would develop negative-energy bound states in the annihilation channel.

I. INTRODUCTION

THERE are three curious facts about the abstract space symmetry obeyed by the strong interactions: (a) It is low dimensional, at least at low energies (e.g., there are no isospin-10 pions), (b) it is of low rank (there are only *three* linearly conserved commuting quantities: hypercharge, baryon number, and I_3), and (c) it is not a simple group. It is a direct product of $SU(2)$, baryon number, and hypercharge symmetries; and it is a special kind of direct product at that, namely, one obtained by breaking a higher symmetry.

Chew and Frautschi[1] have suggested that the above facts do not have to be put into a dynamical calculation but may be part of the output, provided that the dynamical assumptions exclude arbitrary couplings and masses. They have advanced this suggestion within the framework of analytic S-matrix theory, but even if one believes that fields are more fundamental than the S matrix, one may yet consider strong-interaction symmetries as derived rather than basic; the self-consistency requirements of S-matrix theory may be taken as subsidiary constraints which ensure the uniqueness of the field-theoretic solution.

In their original paper, Chew and Frautschi put forth no detailed program for testing their idea, nor did they estimate to what accuracy S-matrix elements would have to be calculated before it became obvious that the wrong choice of symmetry was leading to a violation of an S-matrix axiom. The present paper and its successor (paper II) propose such a detailed program, one which can be evaluated given only present-day, rather inaccurate dynamics. Two specific arguments are considered, one to prove that scattering systems involving high-isospin multiplets are unstable, and a second (in paper II) to prove that scattering systems involving low-dimensional representations of high-rank simple Lie groups are unstable.

Both arguments are fallacious (though it must be affirmed that in each case some work is required in order to establish this fact). Indeed at the end of paper II the opinion will be ventured that these two are the only arguments that can be evaluated at the present time, so that investigation of the symmetries considered herein should be dropped until further advances in the dynamics are made. Needless to say, some will disagree with this conclusion and consider it too strong. We would be the first to hope that they are correct. Other people will say, "we knew it all along."[2]

In an S-matrix theory the symmetry enters the dynamics entirely through the crossing matrix elements $C(\mu_d,\mu_c)$ appearing in the crossing relations

$$A(\mu_d)=\sum_{\mu_c} C(\mu_d,\mu_c)A(\mu_c)\,, \tag{1.1}$$

where $A(\mu_d)$ and $A(\mu_c)$ are the scattering amplitudes in direct and crossed channels, and μ denotes some set of parameters for labeling the irreducible representations of the symmetry; $\mu\equiv I$ for $SU(2)$. For simplicity, we have suppressed the Lorentz crossing matrix factor in Eq. (1.1).[3] To test a symmetry experimentally, one needs only the Clebsch-Gordan coefficients, which give branching ratios, allowed vertices, and multiplet dimensionalities. For a "theoretical test," however, one needs

* Work performed under the auspices of the U. S. Atomic Energy Commission.
† Part of this work was done while the author was a U. S. Air Force Office of Scientific Research Fellow at CERN, Geneva, Switzerland.
[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 395 (1961).

[2] "And now we know why we knew we knew it all along."
[3] For helicity crossing matrices: I. J. Muzinich, J. Math. Phys. **5**, 1481 (1964); T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964). For crossing of representations of the homogeneous Lorentz group: A. O. Barut, I. J. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963); Steven Weinberg, *ibid.* **133**, 1318 (1964); **134**, 882 (1964).

rather the $C(\mu_d,\mu_c)$, quantities determined entirely by group theory.[4,5]

As mentioned previously, the discussion proceeds entirely within the framework of present-day S-matrix theory. We make use of the usual "nearest singularity" dynamics (neglecting all but nearby two-body channels, keeping only the nearest crossed singularities).[6] In addition, we shall limit ourselves to systems containing only a small number of high-isospin or high-rank multiplets, in order to avoid a multichannel problem of intractable size. Our dynamical assumptions are more restrictive than the usual nearest-singularity one, therefore, and to describe them we shall use the phrase "first approximation," rather than "nearest-singularity approximation." All our results are a test of the bootstrap hypothesis "to a first approximation only."

We now sketch the first argument (for high isospins). It is well known for isospin symmetry (and it has been proved by Capps[7] for a general compact Lie group) that the elements of $C(\mu_d,\mu_c)$ are proportional to the elements of an orthogonal matrix $O(\mu_d,\mu_c)$;

$$C(\mu_d,\mu_c)=(N_c/N_d)^{1/2}O(\mu_d,\mu_c)\,,\quad N_i=\text{dimension of }\mu_i. \tag{1.2}$$

In order to obtain a first rough estimate of the magnitude of $C(\mu_d,\mu_c)$, one might replace $O(\mu_d,\mu_c)$ by the rms average over all elements of $O(\mu_d,\mu_c)$; in virtue of orthonormality this rms average would be $m^{-1/2}$, if $O(\mu_d,\mu_c)$ is an $m\times m$ matrix. (That is, m is the number of direct or crossed channels; $m=\max I_d-\min I_d+1=\max I_c-\min I_c+1$ for isospin.)

$$\bar{C}(\mu_d,\mu_c)=(N_c/N_d)^{1/2}m^{-1/2}. \tag{1.3}$$

When the external isospins are large, typically there will be a large number of direct and crossed channels and m will be a large number. If Eq. (1.3) is a good estimate, then perhaps

$$\lim_{I_d\to\infty} C(I_d,I_c)\ll 1 \tag{1.4}$$

for enough elements $C(I_d,I_c)$ that a high-isospin system is impossible. In other words, if Eq. (1.3) is close to the truth, then the force due to I_c exchange is being distributed rather equally over the large number of direct channels available, and in no one channel would there be enough concentration of force to produce a resonance or bound state.

The fallacy here is simply that for a few elements estimate (1.3) is not good; sizeable fluctuations away from the rms average do occur. In Sec. III we give a systematic procedure for locating such elements, and in Sec. IV we construct a simple example of a high-isospin system bootstrapped with their aid.[8]

Of course, one might try to refute the above argument on dynamical rather than group-theoretical grounds, arguing that for some reason Eq. (1.4) is not a sufficient condition for instability. There are many ways to attack the above argument. Our belief about the dynamical objections we have seen, however, is that all would have been rather easy to refute provided that criterion (1.4) had been *fully* satisfied, i.e., provided that enough elements had been *vanishingly* small, not just order 0.1 or so. For example, if one were to try to compensate for small elements $\ll 0.1$ by assuming small masses or large couplings, then one would have merely replaced the original problem with a tougher problem: Given that the system requires extreme masses and couplings to produce resonances, then how are the extreme masses and couplings themselves to be produced? In order to save space, at the risk of being sketchy on some points, we have chosen to examine the chain of reasoning only at the one link where it seems weakest.

It will perhaps be instructive to mention briefly one other link which we might have chosen to examine. Suppose one were to argue that high-isospin systems may contain a large number of multiplets; hence that one of the first-approximation restrictions listed above is unreasonable. If in particular there were resonances in a large number ($\geq$order $m^{1/2}$) of crossed amplitudes $A(I_c)$, then the sum in Eq. (1.1) might come out to be order unity, even though each term in the sum were only order $m^{-1/2}$. Such a result would imply strong correlations between the magnitudes and signs of the various $A(I_c)$, because the sum (1.1) would have to come out of order unity in a large number ($\simeq$order $m^{1/2}$) of the direct channels in order that the large number of crossed multiplets be produced in the first place. It is conceivable that a system possessing such a high degree of correlation could be constructed; for a suggestive result in this connection, see the discussion of the static model given later in this Introduction, particularly Ref. 12. Therefore, even had we been able to establish the argument suggested by Eq. (1.4), we might have found it impossible or very difficult to extend the result to systems containing a large number of multiplets.

The discussion of Sec. III should be useful to those who work frequently with isospin-crossing matrix elements. Even when the scattered isospins are small, the construction described at the beginning of that section

[4] $C(\mu_d,\mu_c)$ is equal to a product of four or more Clebsch-Gordan coefficients, summed over all the magnetic quantum numbers. For a detailed discussion, at least for the isospin case, see Ref. 5.

[5] Donald E. Neville, Phys. Rev. **160**, 1375 (1967).

[6] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962). F. Zachariasen, Pacific International Summer School in Physics, Honolulu, Hawaii, 1965 (unpublished).

[7] Richard H. Capps, in *Proceedings of the Twelfth Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).

[8] At an earlier time we were convinced that Eq. (1.4) was correct and that we had a valid proof for it [Donald E. Neville, Phys. Rev. Letters **13**, 118 (1964)]; the counterexample of Sec. III forces us to abandon this position.

is quite useful for obtaining order-of-magnitude estimates of $C(I_d,I_c)$.

This is the first of a two-part paper. The second part will discuss high-rank scattering as well as the restrictions imposed upon symmetries when the scattering amplitude is assumed to be analytic in the complex J plane.

II. INTERACTIONS BETWEEN LOW- AND HIGH-DIMENSIONAL MULTIPLETS: THE STRONG-COUPLING MODEL

This section derives a condition governing the interactions between isospins varying widely in dimensionality [Eq. (2.1) below], and, in the course of the derivation, discusses the shortcomings of the "strong-coupling" model as a method for producing high isospins.[9] The usual strong-coupling model involves production of high-dimensional baryons by scattering of a low-dimensional meson, typically a pion with $I=1$, and hence is subject to (and in fact violates) condition (2.1). Later on in the section, another argument against the stability of high-isospin systems, one not based on Eq. (1.4), will be described and refuted.

The present section should be useful in understanding some of the remarks made at the end of Sec. IV; otherwise, Sec. II is not needed for later sections. In particular, in the example constructed at the beginning of Sec. IV, all multiplets are of about the same dimensionality; hence that example satisfies condition (2.1) and is free of the difficulties discussed below in connection with the strong-coupling model.

The condition will be stated in a strong and a weak form. Suppose it is desired to produce a multiplet of dimensionality N_d by scattering multiplets of dimensionality N_i ($i=1, \cdots, 4$) and exchanging a multiplet of dimensionality N_c. Suppose further that one of the inequalities

$$N_c/N_d \gtrsim O(1)\,, \quad N_i/N_c \gtrsim O(1) \tag{2.1}$$

is not satisfied. Then (weak form) either the group-theoretic factor associated with I_c exchange will be quite small [falling off as $(N_c/N_d)^{1/2}$ or $(N_i/N_c)^{1/2}$ typically], or the group-theoretic factors associated with I_c exchange in a related scattering process, $I_i\bar{I}_i \rightarrow I_i'\bar{I}_i'$, will be very large {diverging as $(N_c/N_i)^{1/2} \times[1+(N_i/N_c)^{1/2}]^{-1/2}$}. I_i' is the other external particle at the $I_i\rightarrow I_c$ vertex; e.g., if in the original process I_c is in crossed channel $I_1I_3\rightarrow I_c\rightarrow I_2I_4$ and N_1 violates the criterion $N_1/N_c \ll O(1)$, then $I_i=I_1$, $I_i'=I_3$, and the related process having the divergence is $I_1\bar{I}_1\rightarrow I_3\bar{I}_3$.

In its weak form, the criterion is only a mathematical statement that whenever inequalities (2.1) are not satisfied, certain crossing matrix elements are very large or very small. In order to obtain a stronger form with dynamical content, one must assume that relevant coupling constants $g(I_iI_i'\rightarrow I_c)$ and masses m_i, m_c are of normal strong-interaction magnitude; otherwise the non-group-theoretic factor in the I_c exchange term may be large or small enough to make up for a small or large group-theoretic factor. If the couplings and masses are normal, then the strong form of the criterion follows: It is possible to produce low-dimensional isospins by scattering and exchanging higher-dimensional isospins, but difficult or dangerous to do the reverse (produce high-dimensional ones by scattering and/or exchanging low-dimensional ones as in the strong-coupling model).

The criterion in its strong form is clearly a qualitative guide, not a quantitative statement: The imprecise "$O(1)$" in inequalities (2.1) cannot be replaced by a more precise limit unless one knows quite a bit about the masses and couplings of the system. The strong-coupling model happens to violate the second of inequalities (2.1) rather severely, so that presumably a precise value of the "$O(1)$" need not be determined for this case.

We now give a proof (a proof of the weak form, strictly speaking; and a proof of the strong form should the dynamics allow).

The first inequality in Eq. (2.1) was established by Capps: An upper bound $|C(d,c)|\leq(N_c/N_d)^{1/2}$ follows immediately from Eq. (1.2), and in turn the necessity for $N_c/N_d\gtrsim O(1)$ follows immediately from this upper bound.[7]

To establish the second inequality, let us suppose first it is violated, i.e., that the system contains a coupling $g(I_1I_2\rightarrow I_c)$ with $N_1/N_c\ll O(1)$. We then calculate the crossing matrix elements $C(d,c)$ for I_c exchange in the crossed channel of $I_1\bar{I}_1\rightarrow I_2\bar{I}_2$ scattering, for the case $I_d=0$. It is appropriate to label the direct channel the t channel. For $I_t\ll\min[I_1^{1/2},I_2^{1/2}]$ there is an approximate formula due to Racah and Edmonds[10]:

$$(-1)^{I_t}C(I_t,I_s)=C(I_t,I_u)\cong N_c(N_1N_2)^{-1/2}P_t(\cos\theta)\,. \tag{2.2}$$

I_c is in either the s ($I_1\bar{I}_2\rightarrow I_1\bar{I}_2$) or u ($I_1I_2\rightarrow I_1I_2$) channel. $P_t(\cos\theta)$ is the ordinary Legendre polynomial of order I_t and argument

$$\cos\theta=[I_c(I_c+1)-I_1(I_1+1)-I_2(I_2+1)] \times[4I_1(I_1+1)I_2(I_2+1)]^{-1/2}. \tag{2.3}$$

The P_t factor is of order unity, at least for $I_t\cong 0$. The $N_c(N_1N_2)^{-1/2}$ factor, however, is $\gtrsim$ order

$$N_c[N_1(N_c+N_1)]^{-1/2}=(N_c/N_1)^{1/2}[1+(N_1/N_c)]^{-1/2},$$

[9] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940); **14**, 633 (1941). For the strong-coupling model expressed in bootstrap language, see E. S. Abers, L. A. P. Balázs, and Y. Hara, Phys. Rev. **136**, B1382 (1964). For the strong-coupling model discussed in group-theoretical language, see T. Cook, C. G. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

[10] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1960), 2nd ed.

hence diverges as $(N_c/N_1)^{1/2}$ for large enough (N_c/N_1). In the usual static model this divergence is particularly serious, since $I_1=1$. With a divergence such as this, it is difficult to see how one can avoid getting a ghost. Even with crossing matrix elements of order unity, one often has trouble avoiding ghosts in the annihilation channel.

Usually this divergence goes unnoticed because the strong-coupling model neglects baryon recoil, hence is not suitable for studying the t channel. (Further, the divergence will not occur if the baryon masses m_c increase rapidly enough with isospin, so that the criterion is applicable only in its weak form. In the context of the present paper we cannot invoke high mass as a way out, because we are interested only in producing *low*-mass high isospins—our first approximation dynamics may well not work in the high-energy region anyway, and we choose to say nothing about that region.) Clearly this divergence would persist even if the theory were made fully relativistic.[11]

It is entirely possible that the s- and u-channel exchanges, though separately divergent, might add to give a finite result in some of the direct channels. If this happens in a channel with orbital angular momentum L_t, it should be possible to switch to $L_t\pm 1$ and get a finite force, since the u-channel contribution changes sign with L_t.

At this point we have established both criteria (2.1), but seem to have arrived at an apparent paradox: High-isospin systems will produce low isospins easily [or, at least every high-isospin coupling $g(I_1I_2\rightarrow I_c)$ can be made to exert strong forces into low-isospin channels; to prove this, invoke Eq. (2.2) again, but this time with all three of N_1, N_2, and N_c large, $N_1\cong N_2\cong N_0$, and $I_t\cong 0$; then $C(I_t,I_c)=O(1)$]; but if low isospins are coupled to high ones, divergences will result in t channels; therefore, will not all high-isospin systems become unstable because of their by-product low isospins? The answer is "no" (even if it be granted that high isospins produce low isospins easily), because those couplings which produce the divergences are of the form $g(I_HI_L\rightarrow I_H')$, whereas the couplings produced by scattering high isospins are of the form $g(I_H\bar{I}_H'\rightarrow\bar{I}_L)$. [We use H for high and L for low isospin. In our notation for couplings we place to the left of the arrow the particle which is internal and to the right the particle which is external. Thus $g(I_HI_L\rightarrow I_H')^2$ is the residue at the I_H' pole in $I_HI_L\rightarrow I_HI_L$ scattering, while $g(I_H\bar{I}_H'\rightarrow\bar{I}_L)^2$ is the residue at the $\bar{I}_L$ pole in $I_H\bar{I}_H'\rightarrow I_H\bar{I}_H'$ scattering.] The couplings $g(I_HI_L\rightarrow I_H')$ and $g(I_H\bar{I}_H'\rightarrow I_L)$ differ, not only by an analytic continuation in the four-momenta of I_H' and I_L, but also by a group-theory factor which makes $g(I_HI_L\rightarrow I_H')$ much too weak to cause any divergence[5]:

$$g(I_HI_L\rightarrow I_H')=(-1)^{I_H+I_L-I_H'}(N_L/N_H')^{1/2}g(I_H\bar{I}_H'\rightarrow\bar{I}_L). \quad (2.4)$$

In the strong-coupling model, one assumes that the coupling on the *left* is strong, and one gets a divergence; in a high-isospin system which happens to produce some by-product low isospins, one knows that the coupling on the *right* is strong; hence the coupling on the left is not, and there is no divergence. Formula (2.4) is of interest if only for the seeming paradox which it implies: A coupling can be bootstrapped, yet be superweak.

We do not want to emphasize the paradoxical aspects of Eq. (2.4) unduly: The implications of the formula are perhaps paradoxical, but its existence certainly is not, and might perhaps have been anticipated. A coupling is only a kind of amplitude, for one particle in and two out instead of two particles in and two out; consequently, one should expect a group-theory factor to appear when a coupling is crossed, just as the numbers $C(I_d,I_c)$ appear when a scattering amplitude is crossed. Indeed, Eq. (1.2) holds for a scattering diagram with any number of legs, in particular for a vertex function with three legs. Hence, if one sets $O(\mu_d,\mu_c)=\pm 1$ in Eq. (1.2) (there is only one direct or crossed channel) and substitutes g's for A's in Eq. (1.1), one gets the vertex-crossing formula (2.4) immediately, except for the sign.

Conceivably, the vertex crossing could work both ways, of course: It could greatly *enhance* a coupling as well as suppress it. There is a unitarity argument against enhancement of a coupling to a bound state, however, while an enhanced resonance coupling will not be so dynamically effective as its large size might indicate. Let us denote the enhanced coupling by $g=g(I_HI_H'\rightarrow I_L)$; we suppose the coupling $\bar{g}=\bar{g}(\bar{I}_LI_H'\rightarrow\bar{I}_H)$ has been produced in $\bar{I}_LI_H'$ scattering, then vertex-crossed to give a superlarge coupling: $\bar{g}=O(1)$, $g=O(N_H/N_L)^{1/2}\gg 1$. If I_L is a bound state, unitarity could be violated because a large value of the coupling constant implies a large value for the magnitude of the asymptotic wave function; and in simple models a bound on the former follows immediately from the unitarity bound on the latter. Geshkenbein and Ioffe[12] derive such bounds utilizing model-independent

[11] Or if one altered the group-theory structure. It is possible to construct a strong-coupling model in which the $I=1$ pion is replaced by a pion having arbitrary isospin, in particular, by one having a very large isospin $I_\pi\gg 1$ [Cook *et al.*, Ref. 9; as well as V. Singh, Phys. Rev. **144**, 1275 (1966); S. K. Bose, *ibid.* **145**, 1247 (1966)]. Also it is possible to adjust the minimum isospin in the chain of baryon resonances to any desired value, in particular to min $I_B\cong I_\pi$. Such a model appears to suffer from the same disease as the ordinary one, since there is no upper limit to I_B, and eventually the divergence in Eq. (2.2) will go as the ratio $(N_B/N_\pi)^{1/2}$.

This model is of interest from another point of view, however. It is an instance of a high-isospin system with a very *large* number of multiplets ($\gg\bar{m}^{1/2}$, $\bar{m}$ being the dimensionality of a typical crossing matrix in the system, $\bar{m}\simeq 2I_\pi$). Of especial interest is the large amount of correlation between the crossed exchanges; in fact *every* crossed (u) channel contains a baryon resonance, yet the force comes out attractive in all direct (s) channels.

[12] B. V. Geshkenbein and B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. **44**, 1211 (1963); **47**, 1832 (1964) [English transls.: Soviet Phys.—JETP **17**, 820 (1963); **20**, 1235 (1965)].

assumptions. If I_L is a resonance rather than a bound state, then in the limit $g\rightarrow\infty$ there is no infinite force; rather, the pole in the complex plane corresponding to the resonance recedes an infinite distance from the real axis. For I_L, either a bound state or a resonance, there is also a self-consistency problem: It is not enough to produce the large coupling *indirectly*, in $\bar{I}_L I_H'$ scattering followed by vertex crossing; the coupling must also come out large when produced *directly*, in $I_H I_H'$ scattering. Experience with simple models indicates that superlarge couplings are difficult to produce directly.

The discussion just given may, of course, also be used to justify some of the coupling-constant assumptions required in moving from the weak to the strong form of criterion (2.1). Vertex-crossing considerations also lend further support to the second of inequalities (2.1), $N_i/N_c \gtrsim O(1)$. As this criterion is increasingly violated, i.e., as the external multiplets get smaller, $N_i/N_c\rightarrow 0$, vertex crossing suggests that the coupling of I_i to I_c should fall off as $(N_i/N_c)^{1/2}$, in order that crossing $I_i\leftrightarrow I_c$ should not lead to a superlarge coupling plus a problem with self-consistency or unitarity.

In summary, there is no contradiction in *some* low-to-high couplings [those of the form $g(I_H\bar{I}_H'\rightarrow I_L)$] being strong. Even though strongly coupled, a low-isospin I_L is ineffective in producing high isospins; it is ineffective both as an exchanged multiplet [because of the $(N_c/N_d)^{1/2}$ factor in Eq. (1.2)] and as an external multiplet [because of the $(N_L/N_H')^{1/2}$ factor in Eq. (2.4)]. A strong coupling of the form $g(I_H\bar{I}_L\rightarrow I_H')$ is dangerous, rather than ineffective, since it can produce divergences in the t channels as at Eqs. (2.2) ff.; or it can be crossed to a superlarge coupling as at Eq. (2.4). Finally, vertex crossing effects are more likely to suppress a low-to-high coupling to $\ll O(1)$, rather than enhance it to $\gg O(1)$.

All the conclusions of the previous few paragraphs (except those regarding t-channel divergences) follow from the existence of the $(N_i/N_j)^{1/2}$ factors in Eqs. (1.2) and (2.4). Since the same ratios recur in the crossing relations for higher groups, the same conclusions follow for interactions between low- and high-dimensional representations of the higher groups. [In addition, there is strong evidence that the analogs of Eqs. (2.2) and (2.3) exist for the higher symmetries, hence, that the t-channel conclusions also follow: Quite generally, $|C(\mu_t=\text{singlet},\mu_c)|=N_c(N_1N_2)^{-1/2}$.[5]] Indeed, Capps[7] first applied the $(N_c/N_d)\gtrsim O(1)$ criterion not to $SU(2)$, but rather to $SU(3)$, pointing out that the $(N_c/N_d)^{1/2}$ factor in Eq. (1.2) inhibits the formation of 27-plet resonances. Presumably the weakness of low-dimensional representations would be apparent for singlets in $SU(3)$, were it not for symmetry-breaking effects. Thus in the exact $SU(3)$ limit the ω meson would be pure $SU(3)$ singlet, and the forces due to ω scattering or exchange would be damped by group-theory factors typically of order $\frac{1}{8}^{1/2}$. The observed ω, however, is a singlet only in the $SU(2)$ sense; from the $SU(3)$ point of view ω is part singlet and part octuplet; and the octuplet part is not damped.

The discussion of vertex-crossing matters given above is, of course, quite relevant to the stated purpose of this paper: It is essential to give a complete discussion of the interactions between low and high isospins, lest the mistaken impression arise that low isospins make a high-isospin system unstable. We wish now to discuss very briefly a topic which may well be irrelevant to our own purpose or to anyone else's. Since a high-isospin system cannot be ruled out theoretically at the present time, we investigate the possibility of such a system coexisting alongside the familiar low-isospin one.

Firstly, the familiar isospins cannot be produced as byproducts in the scattering of some as yet undetected set of high isospins: Low isospins produced in such a manner are always coupled *weakly* to one another; they are coupled strongly only to their "parent" high isospins. The reason for this may be seen by considering the following simple model. We suppose isospin $I_L=O(1)$ is coupled strongly to its parent $I_H\gg O(1)$ via coupling $g(I_H\bar{I}_H\rightarrow I_L)=O(1)$; then because of I_H exchange in the off-diagonal process $I_L\bar{I}_L\rightarrow I_H\bar{I}_H$, perhaps $I_L\bar{I}_L$ is coupled to some additional low-isospin I_L' as well: $I_L\bar{I}_L\rightarrow I_L'\rightarrow I_H\bar{I}_H$. An upper bound on the crossing matrix element for I_H exchange in $I_L\bar{I}_L\rightarrow I_H\bar{I}_H$ scattering follows from Eq. (1.2): $|C(I_L,I_H)|\leq(N_H/N_L)^{1/2}$. Since the crossed coupling squared is $g(I_HI_L\rightarrow I_H)^2$ rather than $g(I_HI_H\rightarrow I_L)^2$, the exchange diagram contains a factor of (N_L/N_H) from vertex crossing, Eq. (2.4); hence the net |force| producing $g(I_L\bar{I}_L\rightarrow I_L)^2$ is $\leq(N_L/N_H)^{1/2}g(I_HI_H\rightarrow I_L)^2\ll 1$. Similarly, for any other mechanism one might think of to produce a coupling of the type $g(I_LI_L'\rightarrow I_L'')$: The mechanism always involves external low isospins coupled to internal high isospins, and hence the crossed exchanges are always damped by vertex crossing factors of order $(N_L/N_H)^{1/2}$ from Eq. (2.4). Therefore, if the observed low isospins were produced by high ones, the observed isospins would be weakly coupled to one another, which is absurd.

The observed low isospins cannot play an "active" role either, if we are to avoid the divergences described at Eqs. (2.2)–(2.3) ff. The conclusion, then, is that the couplings between the observed system and any high-isospin system would have to be very weak; e.g., (from an analysis of vertex and crossing matrix factors similar to that given in the previous paragraph) the amplitude for $I_L\bar{I}_L\rightarrow I_H\bar{I}_H$, $I_L\ll I_H$ (i.e., associated production of very high isospins) would have to be of order $\leq(N_L/N_H)^{1/2}$.

III. LOCATING ELEMENTS OF ORDER UNITY

In this section we outline a simple procedure for locating elements $O(I_d,I_c)$ of order unity, $O(I_d,I_c)$ being the orthogonal matrix introduced in Eq. (1.2). At least some of these elements will have $(N_c/N_d)^{1/2}\gtrsim 1$, and hence will give rise to elements $C(I_d,I_c)$ of order $\gtrsim$ unity. The first step is to note that $O(I_d,I_c)$ for two-body scattering is proportional to a 6-J symbol (or symmetrized Racah coefficient)[13]:

$$O(I_d,I_c)=(-1)^{I_d+I_c+I_\beta-I_\gamma}(N_cN_d)^{1/2}\begin{Bmatrix} I_\alpha & I_\beta & I_d \\ I_\delta & I_\gamma & I_c \end{Bmatrix}. \quad (3.1)$$

The direct and crossed channels are

$$I_\alpha\bar{I}_\beta\rightarrow I_d\rightarrow\bar{I}_\gamma I_\delta\,, \quad (3.2a)$$

$$I_\alpha I_\gamma\rightarrow I_c\rightarrow I_\beta I_\delta\,. \quad (3.2b)$$

The phase given in Eq. (3.1) needs modification occasionally, and detailed rules for doing this [in fact a complete derivation of (3.1)] are (is) given elsewhere.[5] It suffices for present purposes simply to note that the curly bracket in Eq. (3.1) is a neat shorthand for a lengthy sum over four $SU(2)$ Clebsch-Gordan (CG) coefficients:

$$\{\ \}=\sum_{(\text{all } m\text{'s})}\cdots\langle m_\alpha,\,-m_\gamma|m_c\rangle\langle -m_\beta,\,m_\delta|m_c\rangle \times\langle m_\alpha m_\beta|m_d\rangle\langle m_\gamma m_\delta|m_d\rangle\,, \quad (3.3)$$

where three center dots indicate some known, but irrelevant factors such as $N_c^{-1/2}$, and the CG coefficients are abbreviated $\langle I_\alpha m_\alpha I_\beta m_\beta|I_d m_d\rangle=\langle m_\alpha,m_\beta|m_d\rangle$, etc. The first two coefficients are those which occur in the crossed amplitude (some of the m's have a minus sign because m values reverse sign on crossing); and the last two coefficients are needed to project out into the desired direct channel.

The next step is to utilize a construction due to Wigner and shown in Fig. 1.[14] The curly bracket, like all quantum-mechanical amplitudes, has a classical limit for large values of its arguments. In particular, if five angular momentum or isospin vectors $\mathbf{I}_\alpha\mathbf{I}_\beta\mathbf{I}_\gamma\mathbf{I}_\delta\mathbf{I}_d$ are added as shown in Fig. 1, then the only length which can vary (once the five lengths $|\mathbf{I}_\alpha|,\cdots,|\mathbf{I}_d|$ are fixed) is the one shown dashed; and $|(N_cN_d)^{1/2}\{\ \}|^2=|O(I_d,I_c)|^2$ gives the probability that this length shall be $|\mathbf{I}_c|$. [Note that each face of the tetrahedron in Fig. 1 corresponds to one of the CG coefficients in Eq. (3.3).] Classically, this probability equals the fraction $d\phi/\pi$, where $d\phi$ is the change in ϕ, the angle between faces $I_\alpha I_\beta I_d$ and $I_\gamma I_\delta I_d$, as the dashed length changes from (I_c) to $|I_c|+1$. Wigner computes this fraction from the geometry and gets, for the classical limit of the 6-J symbol,

$$|O(I_d,I_c)|=(N_dN_c)^{1/2}(24\pi V)^{-1/2}, \quad (3.4)$$

where V is the volume of the tetrahedron. From formula (3.4) it is clear *that the largest elements are to be found where the tetrahedron degenerates into a figure of lower dimension*, either area or line. [In fact, formula (3.4) blows up at $V\rightarrow 0$. The blowup is not real, of course; it results from the approximations made in treating the geometry.] Another interesting result is that *wherever the tetrahedron cannot be formed, the 6-J symbol classically is zero*. For instance, if one continues to lengthen $|I_c|$ beyond the point $\phi=\pi$ (where Fig. 1 has collapsed to an area) the figure ruptures, and classically, $O(I_d,I_c)=0$. Quantum-mechanically, the probability for such $|I_c|$ will not be zero; but it will decrease exponentially as $|I_c|$ is lengthened beyond the point of rupture, like a wave function tunneling through a potential barrier. Even when the external isospins are small, so that the situation is highly nonclassical, Wigner's tetrahedron is still a fairly reliable guide, and elements in classically forbidden regions are found to be down by factors typically of order $1/e$.

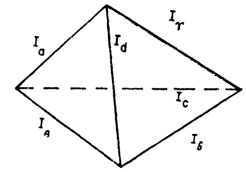

FIG. 1. Wigner's tetrahedron.

Intuitively, one might guess that $O(I_d,I_c)$ is larger at straight-line degeneracies than at area degeneracies, and we shall study the former first. We label the external scattered multiplets I_α, I_β, I_γ, and I_δ as in Eqs. (3.2a) and (3.2b) and Fig. 1. In order for that figure to become (or approximate) a straight line, one of the following constraints must be satisfied (or approximately satisfied):

$$\begin{aligned} &\text{Either} && I_\alpha+I_\beta\cong I_d\cong I_\gamma+I_\delta\,, && (\text{l.l.}) \\ &\text{or} && I_\alpha+I_\gamma\cong I_c\cong I_\beta+I_\delta\,, && (\text{u.r.}) \\ &\text{or} && I_\alpha-I_\gamma\cong\pm I_c\cong I_\beta-I_\delta && (\text{u.l.})\,. \end{aligned} \quad (3.5)$$

If $O(I_d,I_c)$ is pictured as a square array in the (I_d,I_c) plane, then elements $O(I_d,I_c)$ satisfying (3.5) lie at the corners of this array; the abbreviations (l.l., etc.) indicate at which corner of $O(I_d,I_c)$ (lower left, upper right, etc.) the constraint is satisfied. Racah has derived a finite sum for the 6-J bracket, a sum which is usually very lengthy but reduces to a single term at corners (in fact at edges)[10,14]:

$$\begin{Bmatrix} I_\alpha & I_\beta & I_d \\ I_\delta & I_\gamma & I_c \end{Bmatrix}=\prod_{p,q}\Delta_p\sum_r(-1)^r(r+1)! \times\{[r-I(p)]![K(q)-r]!\}^{-1}. \quad (3.6)$$

In this formula there is one index p for each face of the tetrahedron in Fig. 1. If $p=1$ denotes the face $(I_\alpha I_\beta I_d)$, then

$$I(1)\equiv I_\alpha+I_\beta+I_d\,,$$

$$\Delta(1)\equiv\{(I_\alpha+I_\beta-I_d)!\,(I_\alpha-I_\beta+I_d)!\,(-I_\alpha+I_\beta+I_d)!/[I(1)+1]!\}^{1/2}. \quad (3.7)$$

[13] F. J. Dyson, Phys. Rev. **100**, 344 (1955).
[14] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

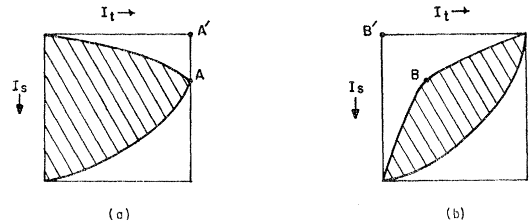

FIG. 2. Classically allowed regions (shown shaded) of $C(d,c)$ for (a) t-to-s and (b) u-to-s crossing of the elastic process $I_1\bar{I}_2\rightarrow I_s\rightarrow \bar{I}_1\bar{I}_2$. The figures are drawn for the case $I_2/I_1=2$. As $I_2/I_1\rightarrow 1$, the points A and B approach the points A′ and B′. The boundary curves for Figs. 2(a) and 2(b) were computed by setting $V=0$ in Eqs. (3.10) and (3.11), respectively.

Similarly for $p=2$, 3, 4. There is one index q for each quadrilateral which can be formed from Fig. 1 by deleting two nonintersecting edges, i.e., $(I_\alpha I_\beta I_\delta I_\gamma)$, $(I_\alpha I_d I_\delta I_c)$, and $(I_\beta I_d I_\gamma I_c)$. If $q=1$ indexes $(I_\alpha I_\beta I_\delta I_\gamma)$, then

$$K(1)\equiv I_\alpha+I_\beta+I_\delta+I_\gamma\,, \tag{3.8}$$

and similarly for $q=2$, 3. The sum is over $\max I(p) \leq r\leq \min K(q)$; and at edges it is always $\max I(p) =\min K(q)$. The final step in investigating points of straight-line degeneracy is therefore to verify, via Racah's sum, that at a given point of straight-line degeneracy $C(I_d,I_c)$ is indeed of order unity.

This final check is necessary because a degeneracy (3.5) is only a necessary, not a sufficient, condition for $O(I_d,I_c)$ to be of order unity [and of course we are interested in $C(d,c)$ rather than $O(d,c)$, so that the factor $(N_c/N_d)^{1/2}$ must be examined]. Usually, however, the degeneracy points where $O(I_d,I_c)$ is indeed of order unity are easily distinguished from those where it merely attains a relative maximum, because at the former points the classically allowed region is narrow in both the column and row directions. Figure 2, for instance, gives the classically allowed regions for u-to-s and t-to-s crossing of an elastic scattering process $I_1\bar{I}_2\rightarrow I_s\rightarrow I_1\bar{I}_2, I_1I_2\rightarrow I_u\rightarrow I_1I_2$, and $I_1\bar{I}_1\rightarrow I_t\rightarrow I_2\bar{I}_2$, drawn for the special case $I_2/I_1=2$. The boundary curves are solutions of the equation $V=0$, where

$$\begin{aligned}(12V)^2=&I_d^2I_c^2(I_\alpha^2+I_\beta^2+I_\gamma^2+I_\delta^2-I_d^2-I_c^2)\\&+I_\alpha^2I_\delta^2(I_d^2+I_c^2+I_\beta^2+I_\gamma^2-I_\alpha^2-I_\delta^2)\\&+I_\beta^2I_\gamma^2(I_d^2+I_c^2+I_\alpha^2+I_\delta^2-I_\beta^2-I_\gamma^2)\\&-I_d^2I_\alpha^2I_\beta^2-I_d^2I_\gamma^2I_\delta^2-I_c^2I_\alpha^2I_\gamma^2-I_c^2I_\beta^2I_\delta^2\end{aligned} \tag{3.9}$$

in the general case, and

$$(12V)^2=I_t^2\{I_s^2[2(I_1^2+I_2^2)-I_s^2-I_t^2]-(I_1^2-I_2^2)^2\} \tag{3.10}$$

for $t\leftrightarrow s$ crossing of an elastic scattering reaction, i.e., $I_\alpha=I_\gamma\equiv I_1$, $I_\beta=I_\delta\equiv I_2$, $(I_d,I_c)=(I_s,I_t)$; and

$$(12V)^2=[2(I_1^2+I_2^2)-(I_s^2+I_u^2)]\times[I_s^2I_u^2-(I_1^2-I_2^2)^2] \tag{3.11}$$

for $s\leftrightarrow u$ crossing of an elastic scattering reaction, i.e., $I_\alpha=I_\delta\equiv I_1$, $I_\beta=I_\gamma\equiv I_2$, and $(I_d,I_c)=(I_s,I_u)$. Both left-hand corners in Fig. 2(a) are points of straight-line degeneracy; nevertheless, one would not expect $|O(I_d,I_c)|=1$ because the entire left edge is classically allowed and the sum of all $|$edge elements$|^2$ must add up to unity by orthonormality. In Fig. 2(b), on the other hand, the two skew corner points are the only allowed points in their respective rows and columns; hence, one expects (and indeed finds) $O(I_s,I_u)=$order unity at or near the skew corners.

At degeneracy points away from edges one must compute $O(I_d,I_c)$ by other methods because Racah's sum becomes too lengthy. It is possible to obtain $O(I_d,I_c)$ at such points by first deriving a three-term recurrence relation for the 6-J bracket from the Biedenharn-Elliott identity, then solving this recurrence relation approximately.[15] By this method one finds that elements $O(d,c)$, located at degeneracy points not too near corners, are of order $m^{-1/3}\ll$order unity. This result is in agreement with one's intuition that at area degeneracies $O(d,c)$ is smaller than at straight-line degeneracies. Since the noncorner degeneracies do turn out to be so small, we shall not digress here to write up the recurrence relation and describe its solutions in detail.[16] In any case, even if one is given only the order-unity elements already found (at corners), it appears possible to produce a high-isospin system (see Sec. IV).

IV. HIGH-ISOSPIN SCATTERING: A SPECIFIC EXAMPLE

Even though we now have found *some* elements of order unity, in Sec. III, it is not yet clear that we have found *enough* such elements. Perhaps in every high-isospin system there is always at least one isospin which is needed to produce other isospins, yet cannot itself be produced because none of the crossed exchanges assigned to produce it have an order-unity element through which to act. In order to clear up this and

[15] L. C. Biedenharn, J. Math. Phys. **31**, 287 (1953); J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953). There are many *two*-term recurrence relations derived from the identity and quoted in the literature; see, for example, Ref. 10, p. 98. Such relations are somewhat inconvenient for studying the crossing matrix, however, since invariably they connect two 6-J symbols belonging to entirely different crossing matrices.

[16] However, a few notes on the method of solution may not be amiss, because the writeup will likely be a long time appearing. We let $f=f(I_d)$ denote the 6-J bracket of Eq. (3.1) and suppose that f obeys a three-term recurrence relation $A_+F(I_d+1)+A_0f(I_d)+A_-f(I_d-1)=0$, with all arguments except I_d kept fixed. Any three-term relation can be rewritten as a second-order difference equation $B_2\delta^{(2)}f+B_1\delta^{(1)}f+B_0f=0$, with $\delta^{(1)}\equiv f(I_s+1)-f(I_s)$; $\delta^{(2)}f\equiv f(I_s+1)-2f(I_s)+f(I_s-1)$. Difference equations suggest differential equations, and classical limits suggest WKBJ solutions. WKBJ solutions suggest the ansatz $f=R\exp iS$ for f, with R and S expandable as a power series in a small parameter (here the parameter, or parameters, are the $1/I_i$ rather than $\hbar$). The WKBJ solution goes through as in the usual quantum-mechanical case, except that differencing replaces differentiating; and, at degeneracy points, one obtains the *difference* equation for the Bessel function, rather than the differential equation.

related uncertainties, we consider a simple example of a high-isospin system satisfying the following requirements:

(a) (A minimal version of the self-consistency requirement): Every particle used as a crossed or external particle in one reaction is produced in the direct channel of that reaction or of another reaction of the system.

(b) If I_d is produced by exchange of I_c, then the relevant crossing factor $C(I_d,I_c)$ must be of order unity.[17] [Of course, possibly this requirement is too strict; that is, possibly an element $|C(I_d,I_c)|=0.01$, say, could produce a resonance in channel I_d if the couplings of I_c were huge. It is of interest, however, to see how strict the requirements can be made and still be satisfied. This is especially true with regard to the present requirement, since in Sec. II we had occasion to argue against the existence of superlarge couplings.]

(c) If channels I_d, I_c are resonant and channel $I_d'\neq I_d$ is not, then $|C(I_d,I_c)|>|C(I_d',I_c)|$.

(d) No crossed forces diverge: $C(I_d,I_c)\leq O(1)$ for all I_d and all resonant I_c.

Example: Consider three high-isospin particles 1, 2, and 3 with isospins $I_1I_2I_3$, $I_1\gg 1$, and $I_k=kI_1$. They are assumed to have distinct antiparticles $I_k\neq\bar{I}_k$ and non-isospin quantum numbers such that (at least) the following vertices are allowed and strong:

$$g(I_1I_2\to\bar{I}_1)\,,\quad g(I_1\bar{I}_2\to I_3)\,. \tag{4.1}$$

There is no lack of reactions for satisfying requirement (a). In fact, the couplings (4.1) imply that each of the I_i must appear in at least the following channels:

$$\begin{aligned} I_1&:\ \bar{I}_1\bar{I}_2,\, I_2I_3,\\ I_2&:\ \bar{I}_1\bar{I}_1,\, I_1\bar{I}_3,\\ I_3&:\ I_1\bar{I}_2. \end{aligned} \tag{4.2}$$

So as to stay within the framework of "first-approximation" dynamics, we shall assume the I_i have such mass ratios that they are stable, or at least the I_i have such long lifetimes that the dynamically important many-body states containing only stable particles are well approximated by the two-body channels (4.2).

In order to check requirements (b)–(d), one must have explicit expressions for the relevant crossing matrix elements. If $g(I_iI_j\to I_k)$ is a coupling (4.1) or one such as $g(I_i\bar{I}_k\to\bar{I}_j)$ obtained from (4.1) by crossing two lines, then because we have deliberately chosen $I_k=kI_1$ it follows that

$$|I_i\pm I_j|=I_k\,, \tag{4.3}$$

i.e., the crossing matrix elements for I_k or $\bar{I}_j$ exchange lie at the edges and corners of $C(I_d,I_c)$, where the order-unity elements are located, according to Sec. II.

Thus to check that the forces upon I_i are strong enough to produce I_i [requirement (b)], one must study corner elements. Similarly, to check for the absence of multiplets other than I_1, I_2, and I_3 and to rule out divergences [requirement (c) and (d)] one must study the edge elements $C(I_d,I_c=\min I_c)$ and $C(I_d,I_c=\max I_c)$. Checking requirements (b)–(d) for our example or any other example constructed from corner forces is therefore a straightforward matter of substituting into the Racah sum, Eq. (3.6).

Since the check is so straightforward, we shall not describe all of it in detail. However, it is useful to describe one part of the check, say, that of the I_1 bootstrap, to give an idea of the order of magnitude of the crossing matrix elements involved. From Eq. (4.2) the T matrix for this bootstrap is 2×2 in channel space, the two channels being $\bar{I}_1\bar{I}_2$ and I_2I_3. Let us consider the diagonal elements first. From couplings (4.1), the only crossed force in these elements is I_3 exchange in the u channel of $\bar{I}_1\bar{I}_2\to\bar{I}_1\bar{I}_2$. (As is conventionally done, we label $I_1=I_d$ the s channel and the crossed annihilation and elastic scattering channels the t and u channels, respectively.) For s-to-u crossing of an elastic scattering amplitude, the maxima of $|C(I_d,\text{ext}I_c)|$ lie in the skew corners of $C(I_d,I_c)$. ($\text{ext}I_c=$ extremum of $I_c=\max I_c$ or $\min I_c$.) For the $I_1I_2\to I_2I_1$ amplitude, $C(I_1,I_3)=N_3/N_2\cong\frac{3}{2}$, so that requirement (b) is satisfied; and the falloff on moving away from this element up the edge is $C(I_s+1,I_3)/C(I_s,I_3)\approx N_1/N_2=\frac{1}{2}$, so that requirements (c) and (d) are satisfied.

Since there is no crossed force in the other diagonal amplitude $I_2I_3\to I_2I_3$, I_1 will not be linked strongly to channel I_2I_3 [contrary to requirement (b) and Eq. (3.1)] unless forces in the off-diagonal elements $\bar{I}_1\bar{I}_2\leftrightarrow I_2I_3$ are strong in channel $I_d=I_1$. In the off-diagonal elements the only crossed forces are from $\bar{I}_1$ exchange. We get $|C(I_1,\bar{I}_1)|=N_1/N_2\approx\frac{1}{2}$; hence condition (b) is indeed satisfied. The falloff away from this element is $|C(I_s+1,I_1)|/|C(I_s,I_1)|\approx(N_1N_3/N_2^2)^{1/2}\approx\frac{3}{4}^{1/2}$. This last falloff satisfies requirement (c), although just barely; the somewhat more rapid falloff of the diagonal element (by a factor of $N_1/N_2\approx\frac{1}{2}$) may perhaps make up for this gentle off-diagonal behavior.

Of course, maybe this diagonal falloff is not sufficient, and maybe some multiplets in addition to I_1, I_2, and I_3 will be excited. In general, the crossing matrix elements for I_1, I_2, and I_3 exchange in this example never diverge, although they are usually order unity for channels with $I_d\simeq kI_1$ ($k=0$, 1, 2, 3, 4, or 5). If any of these additional multiples of I_1 were excited, perhaps their reaction back upon the original multiplets would destroy the self-consistency of the system. Unfortunately, our present techniques are not powerful enough for us to say anything meaningful about this possibility; the problem rapidly gets out of hand as the number of multiplets in the system increases.

[17] There may be vertex crossing factors multiplying $C(I_d,I_c)$, from Eq. (1.4), if the crossed coupling was produced originally in a reaction with I_c external. We ignore such factors in this section since all the isospins of our example have about the same dimensionality.

Another possible source of difficulty might be a blowup in the antiparticle-particle channels with $I_t \simeq 0$, since every coupling exerts a force on these channels [e.g., $g(I_1\bar{I}_2 \to I_3)$ exerts force on $I_t \simeq 0$ via I_3 exchange in $I_1\bar{I}_1 \to I_2\bar{I}_2$ scattering]; and the relevant crossing matrix elements are order unity; see Eq. (2.2). If these $I_t \simeq 0$ resunances all lie on Regge trajectories, then Eq. (2.2) suggests the highest lying of these will have $I_t = 0$, as in the observed case. Therefore, if we want divergences we should look at this channel first. We find that even the forces into $I_t = 0$ in our example are not obviously so strong as to drive the singlet trajectory over the Froissart limit.[18] The largest element of the form $C(I_t = 0, I_c)$ is $C(0, I_c) = \frac{3}{2}\sqrt{2}$, for I_3 exchange in $I_1\bar{I}_1 \to I_2\bar{I}_2$ scattering; in the other crossed channel of the same reaction, $\bar{I}_1$ exchange acts through an element $C(0, I_1) = \frac{1}{2}\sqrt{2}$. (Of course, there may be vertex crossing factors multiplying these numbers. These factors will be of order unity, not such as to produce obvious divergences.) Note that for comparison purposes, the |elements| for ρ exchange in the s and u channels of $\pi\pi$ scattering are both $+1$.

It should be stressed that our example was *designed* to produce the relatively small forces on the $I_t = 0$ channel just described. Thus we have been careful to keep the ratio $(\max I_i / \min I_i)$ of order unity in order to avoid the type of t-channel divergence discussed in connection with the strong-coupling model at Eq. (2.2). In addition, we have chosen non-self-conjugate multiplets $I_i \neq \bar{I}_i$: Had we allowed $I_2 = \bar{I}_2$, for instance, then I_3 could be exchanged in both crossed channels of $I_1\bar{I}_1 \to I_2\bar{I}_2$ scattering, and the net force on $I_t = 0$ would be doubled. It is possible to design a system so that the forces into the $I_t = 0$ channels are small, but the constraints on such a system are stringent.

So long as the high-isospin system contains only a small number of multiplets, at least one of the dynamically important crossing matrix elements must satisfy $|O(d,c)| \cong 1$ as well as $|C(d,c)| \cong 1$. The only way to obtain the contrary result, $|O(d,c)| \ll 1$ but $|C(d,c)| \cong 1$, would be to arrange matters so that $(N_c/N_d)^{1/2} \gg 1$; but if all the dynamically important crossing matrix elements in the system satisfy $N_c \gg N_d$, then there is no force to produce the largest isospin in the system, unless we assume the existence of an infinite number of multiplets of everincreasing isospin. [The specific example given in this section is rather special in that *all* dynamically important elements satisfy $|O(d,c)| \cong 1$.]

Moreover, there must be at least one element which satisfies not only $|O(d,c)| \cong 1$, but also $m \gg 1$, where m is the dimensionality of $O(d,c)$ as in Eq. (1.3). Though we did not mention the fact in Sec. III, there is a trivial way to construct elements satisfying $|O(d,c)| \cong 1$, a way which does not require any study of Wigner's tetrahedron. One can choose the external isospins such that $m \cong 1$, whence obviously $|O(d,c)| \cong 1$. For example, one can choose an external isospin to be $\simeq 0$; this is an obvious way to get $m \cong 1$, but it is also dangerous, as we have seen in connection with the strong-coupling model in Sec. II. The safe way is to choose all the external isospins large but make the process highly inelastic. For instance, in our example the inelastic process $\bar{I}_1\bar{I}_1 \to I_1\bar{I}_3$, $I_k = kI_1$ has $m = 1$ even though $I_d = I_2 \gg 1$. Inelastic reactions of this type always have one external or exchanged multiplet larger than the multiplet produced, however; hence, as in the $N_c \gg N_d$ example discussed in the previous paragraph, one cannot rely exclusively on $m \cong 1$ elements unless the system contains an infinite number of multiplets. If one wishes to avoid infinite regress, there is no shortcut: One must invoke Wigner's tetrahedron and locate an element satisfying both $|O(d,c)| \cong 1$ and $m \gg 1$.

When the direct and/or crossed channel is elastic, elements satisfying $|O(d,c)| \cong 1$, $m \gg 1$ are hard to find, even though requirements (3.5) are always satisfied at at least two corners. One cannot take I_d or $I_c = I_t$ (annihilation channel), because resonances in the t channel do not exert strong forces on the high isospins in the s and u channels [recall the discussion in connection with Fig. 2(a)]; and conversely, the forces into the t channel from s and u channels are strong only near $I_t = 0$ [see Eq. (2.2)]. One must take $(I_d, I_c) = (I_s, I_u)$ or (I_u, I_s). [Even for these cases the corner elements are not of order unity unless one is away from equal-isospin scattering $I_\alpha = I_\beta = I_\gamma = I_\delta$. In s-to-u crossing of $I_\alpha I_\beta \to I_\alpha I_\beta$, the largest elements, those in the skew corners of Fig. 2(b), are $|O(d,c)| \cong |N^2 - N^2|^{1/2} / \max(N_\alpha, N_\beta)$.] Whence the predominance of s-to-u bootstraps and inelastic processes in our example.

Finally, it is no accident that the falloffs $C(I_d \pm 1, I_c)/C(I_d, I_c)$ in our example are large. The fastest rates of falloff [and incidentally, the largest values of $|O(I_d, I_c)|$] are obtained when a constraint (3.5) is satisfied exactly. If (3.5) is not satisfied exactly, the maxima of $|O(I_d, I_c)|$ do not lie in the corner, but at two points displaced slightly from the corner, one point lying on each of the edges meeting at the corner. The falloffs from these points must be gentler if the maxima are smaller, in order that the sum rule $\sum_{d \text{ or } c} |O(I_d, I_c)|^2 = 1$ be preserved. The system should therefore tend to have a smaller number of multiplets, the more closely the corner constraints (4.3) are satisfied.

If a given high-isospin system contains at most two high isospins I_1, $I_2 \gg 1$, and $I_1/I_2 \equiv r$, then no matter what the value of r [subject to $r = O(1)$, of course] not enough of the relevant elements $C(I_1, I_1)$, $C(I_2, I_1)$, etc., will lie in corners for the system to be self-consistent. Therefore, a high-isospin system must contain at least three high isospins. Our example was essentially the simplest that could be constructed, therefore. Despite its simplicity, the example is quite representative of the properties of high-isospin systems containing a relatively small number of multiplets.

[18] M. Froissart, Phys. Rev. **123**, 1053 (1961).