# Electron-Electron Bremsstrahlung from a Quantum Plasma ($Z=1$)*

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The nonrelativistic limit of the Born cross section for electron-electron bremsstrahlung is integrated over a Maxwell–Boltzmann electron distribution. Results for the spectrum are compared with the electron-ion ($Z=1$) bremsstrahlung spectrum found in the literature. The long-wavelength limit is obtained and compared with recent classical calculations. Corrections to the spectrum for low temperatures and high frequencies are discussed. It is concluded that at electron temperatures $kT_e \geq 10$ keV, electron-electron bremsstrahlung should be included in calculations of the emission spectrum, especially in the short-wavelength region.

## I. INTRODUCTION

THE importance of electron-ion (e-i) bremsstrahlung[1] in a fully ionized gas, both as an energy-loss mechanism,[2] and as a tool for measuring plasma parameters (density, temperature),[3,4] has been recognized for some time. Since electron-electron (e-e) bremsstrahlung is quadrupole in nature, and e-i bremsstrahlung is of a dipole nature, it is expected that e-e is down from e-i by a factor $\sim kT_e/mc^2$ ($k$ is the Boltzmann constant, $T_e$ is the electron temperature, and $mc^2$ is the electron rest energy).

The importance of e-e bremsstrahlung in high-temperature plasmas has been pointed out, in the literature, in a calculation of the total energy radiated by a quantum plasma as a function of $kT_e/mc^2$.[5] Recently, a classical calculation of the spectrum due to e-e bremsstrahlung[6] estimates that e-e is down from e-i by an approximate factor of $\frac{2}{5}(kT_e/mc^2)$ in the low-frequency region ($h\nu \ll kT_e$).

The purpose of this paper is to calculate e-e bremsstrahlung from a Maxwell-Boltzmann (MB) hydrogenic plasma, in the Born approximation and in the nonrelativistic limit. In Sec. II, the cross section is discussed and the validity of the nonrelativistic Born approximation is pointed out.

In Sec. III, the Born cross section is integrated over a nonrelativistic equilibrium (MB) electron distribution and the bremsstrahlung spectrum is obtained. The long-wavelength limit is taken.

In Sec. IV, the results are presented and a comparison is made with the dipole e-i spectrum. The ratio of quadrupole e-e to dipole e-i emission at a particular frequency is found to be $\sim(6/5\sqrt{2})kT_e/mc^2$ for long wavelengths, increasing to $\sim 3$–$5kT_e/mc^2$ in the short-wavelength region.

In Sec. V the total energy emitted by bremsstrahlung is discussed and compared with the results for e-i emission.

A correction to the Born approximation for low temperatures and short wavelengths is given in Sec. VI. After integrating the corrected cross section over the distribution, it is found that the correction is small when the temperature is high enough so that e-e bremsstrahlung is important.

In Sec. VII, the results of the work are summarized and applications to "lab plasma" physics and astrophysics are discussed.

## II. BORN CROSS SECTION

The Born cross section for e-e bremsstrahlung, in the nonrelativistic limit, is just the matrix element of the quadrupole operator between initial and final two-electron states. Integrating over angles, the cross section for emitting a photon with frequency between $\nu$ and $\nu+d\nu$ in a collision of two electrons, where $\epsilon_1$ is the total kinetic energy in the center-of-mass (c.m.) system, is given by[7]

$$d\sigma_\nu(\epsilon_1) = (4/15)\alpha r_0^2 \Bigg\{ [17-3x^2/(2-x)^2](1-x)^{1/2} + \left[\frac{12(2-x)^4 - 7(2-x)^2x^2 - 3x^4}{(2-x)^3}\right] \times \ln\left[\frac{1}{x^{1/2}} + \left(\frac{1}{x} - 1\right)^{1/2}\right] \Bigg\} \frac{dx}{x}, \quad (1)$$

where

$$x = h\nu/\epsilon_1, \quad (2)$$

$\alpha = e^2/\hbar c$ is the fine structure constant, and $r_0 = e^2/mc^2$ is the classical radius of the electron. The factor

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[1] We define bremsstrahlung, in general, to be the radiation emitted by an electron as it decelerates in the field of other charged particles.

[2] R. F. Post, Rev. Mod. Phys. **28**, 338 (1956).

[3] J. E. Drummond, *Plasma Physics* (McGraw-Hill Book Company, Inc., New York, 1961), p. 319.

[4] D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion* (John Wiley & Sons, Inc., New York, 1961); G. Belefi, *Radiation Processes in Plasmas* (John Wiley & Sons, Inc., New York, 1966).

[5] J. Stickforth, Z. Physik **164**, 1 (1961); J. Garrison (private communication).

[6] T. J. Birmingham, J. M. Dawson, and R. M. Kulsrud, Phys. Fluids **9**, 2014 (1966); T. J. Birmingham, Princeton University Report No. MATT-386 (1966) (unpublished).

[7] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience Publishers, Inc., New York, 1965), Chap. V.

$dx/x = d\nu/\nu$ reflects the familiar infrared divergence. The final energy of the electrons in the c.m. system, $\epsilon_2$, has been eliminated by the energy conserving $\delta$ function, requiring

$$\epsilon_1 = \epsilon_2 + h\nu . \tag{3}$$

The requirement for the validity of the Born approximation for the scattering of two electrons is[8]

$$e^2/\hbar v_{1,2} \ll 1 , \tag{4}$$

where

$$v_{1,2} = (2\epsilon_{1,2}/\mu)^{1/2} \tag{5}$$

and

$$\mu = \tfrac{1}{2}m \text{ (reduced mass).} \tag{6}$$

If (4) is satisfied, the scattering of the electrons due to the Coulomb repulsion can be neglected.

We have also used the nonrelativistic approximation. This requires that

$$\frac{v_{1,2}}{c} \ll 1 \tag{7}$$

or, equivalently,

$$e^2/\hbar v_{1,2} \gg \alpha = e^2/\hbar c . \tag{8}$$

Therefore, the cross section (1) is valid in the region

$$\alpha \ll e^2/\hbar v_{1,2} \ll 1 . \tag{9}$$

We will find later that if the temperature of the gas is high enough so that quadrupole emission becomes an important correction, $kT_e \geq 10$ keV, and the right-hand side of (9) is automatically satisfied. We can satisfy the left-hand side of (9) if $kT_e \leq 25$ keV. Therefore, the nonrelativistic Born method can be used in the region

$$10 \text{ keV} \leq kT_e \leq 25 \text{ keV} . \tag{10}$$

## III. THE MAXWELL-BOLTZMANN AVERAGED CROSS SECTION

Consider a gas with a uniform density of electrons. If the density is not too high nor the temperature too low, collective effects will be unimportant. The criterion is

$$h\nu_p \ll kT_e , \tag{11}$$

where

$$\omega_p{}^2 = (2\pi\nu_p)^2 = 4\pi N_e e^2/m \tag{12}$$

and $N_e$ is the electron density. If (11) is satisfied, the "dressing" of the photon due to its interaction with the plasma as a whole[9] will have a negligible effect in the frequency range which contributes to the total energy radiated. For most lab plasmas,

$$h\nu_p \approx 0.01 kT_e , \tag{13}$$

so that (11) is satisfied. Therefore, we can simply compute the radiation emitted from the plasma by summing the contribution of a single pair of electrons over all possible pairs of electrons in the gas at density $N_e$ = number of electrons/cm$^3$.

The number of photons emitted per unit time per unit volume with frequency between $\nu$ and $\nu + d\nu$ will then be

$$P^{\text{e-e}}(\nu, kT_e)d\nu = \frac{N_e{}^2}{2} \int\int (d\mathbf{p}_1)(d\mathbf{p}_2) N_e{}^{m\text{M-B}}(\mathbf{p}_1) \times N_e{}^{m\text{M-B}}(\mathbf{p}_2) \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m} d\sigma(|\mathbf{p}_1 - \mathbf{p}_2|, \nu) , \tag{14}$$

where

$$N_e{}^{m\text{M-B}}(\mathbf{p}) = \left(\frac{1}{2\pi mkT_e}\right)^{3/2} e^{-|p|^2/2mkT_e} \tag{15}$$

and $d\sigma$ is the cross section, integrated over angles, for emitting a photon with frequency $\nu$ in the collision of two electrons of momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. In the Appendix, we reduce the integral over the two distributions to an integral over a single Maxwell-Boltzmann distribution of reduced mass particles. From Eqs. (A21), (A7), and (1) we have

$$P^{\text{e-e}}(\nu, kT_e)d\nu = \frac{8}{15} \frac{\alpha r_0{}^2 N_e{}^2 h\nu d(h\nu) I(\lambda)}{(\pi m)^{1/2}(kT_e)^{3/2}} , \tag{16}$$

where

$$\lambda \equiv h\nu/kT_e , \tag{17}$$

$$I(\lambda) = \int_0^1 dy\, A(\lambda, y) , \tag{18a}$$

and

$$A(\lambda, y) = e^{-\lambda/y}/y^3 \Bigg\{ [17 - 3y^2/(2-y)^2](1-y)^{1/2} + \left[\frac{12(2-y)^4 - 7(2-y)^2y^2 - 3y^4}{(2-y)^3}\right] \times \ln\left[\frac{1}{y^{1/2}} + \left(\frac{1}{y} - 1\right)^{1/2}\right] \Bigg\} . \tag{18b}$$

The long-wavelength limit of the quadrupole e-e spectrum is most easily obtained by going back to the Born cross section $d\sigma_\nu$ given by Eq. (1). We have

$$\lim_{x \to 0} d\sigma_\nu \equiv d\sigma_0 = \frac{16}{5} \alpha r_0{}^2 \left[\frac{17}{12} + \ln\left(\frac{4}{x}\right)\right] \frac{dx}{x} . \tag{19}$$

Using (A21) and (A7),

$$\lim_{h\nu \to 0} P^{\text{e-e}}(\nu, kT_e)d\nu \equiv P^{\text{e-e}}(0, kT_e)d\nu = \frac{(4/\pi m)^{1/2} N_e{}^2}{(kT_e)^{3/2}} \int_{h\nu}^{\infty} d\epsilon_1 \epsilon_1\, e^{-\epsilon_1/kT_e} d\sigma_0 . \tag{20}$$

[8] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., Chap. V.

[9] J. D. Stack and A. M. Sessler, Phys. Fluids 6, 1193 (1963).

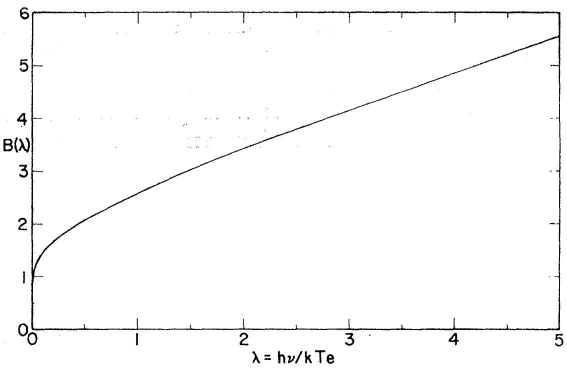


FIG. 1. Ratio of quadrupole to dipole emission.

Substituting (19) into (20), we obtain

$$P^{e\text{-}e}(0,kT_e)d\nu=\frac{32}{5}\alpha r_0^2N_e^2\left(\frac{kT_e}{\pi m}\right)^{1/2}\frac{d\nu}{\nu}$$
$$+\left[\ln\left(\frac{4kT_e}{h\nu}\right)+\frac{17}{12}-0.577+1\right]. \quad (21)$$

We have used the relations[10]

$$\int_0^\infty dx\, e^{-x}\ln x=-\gamma=-0.577 \quad (22a)$$

and

$$\int_0^\infty dx\, e^{-x}x\ln x=-\gamma+1. \quad (22b)$$

## IV. NUMERICAL RESULTS AND COMPARISON WITH THE DIPOLE SPECTRUM

The dipole e-i spectrum from a hydrogenic Maxwell-Boltzmann plasma is given by the expression[3]

$$P^{e\text{-}i}(\nu,kT_e)d\nu=(16/3)(2/\pi mkT_e)^{1/2}\alpha r_0^2N_e^2mc^2e^{-\lambda/2}\times K_0(\tfrac{1}{2}\lambda)d\nu/\nu, \quad (23)$$

where $K_0(x)$ is the modified Bessel function of the second kind.[10–12] It diverges as $\ln(2/x)$ for small $x$ and decreases as $e^{-x}/x^{1/2}$ for large $x$. The long-wavelength limit of (23) is

$$\lim_{h\nu\to 0}P^{e\text{-}i}(\nu,kT_e)d\nu\equiv P^{e\text{-}i}(0,kT_e)d\nu=\frac{16}{3}\left(\frac{2}{\pi mkT_e}\right)^{1/2}$$
$$\times\alpha r_0^2N_e^2mc^2\frac{d\nu}{\nu}\left[\ln\left(\frac{4kT_e}{h\nu}\right)-0.577\right]. \quad (24)$$

Equation (24) is obtained from (23) by using the relation[10–12]

$$\lim_{x\to 0}K_0(x)=\ln(2/x)-\gamma. \quad (25)$$

Forming the ratio $P^{e\text{-}e}(0,kT_e)/P^{e\text{-}i}(0,kT_e)$, we find from (21) and (24)

$$\frac{P^{e\text{-}e}(0,kT_e)}{P^{e\text{-}i}(0,kT_e)}=\frac{6}{5\sqrt{2}}\left(\frac{kT_e}{mc^2}\right). \quad (26)$$

Dr. T. Birmingham has recently calculated this ratio exactly, for a Maxwellian distribution, using the classical methods of Ref. 6. His result is in agreement with Eq. (26).

Next we define

$$B(\lambda)\equiv\frac{P^{e\text{-}e}(\nu,kT_e)}{P^{e\text{-}i}(\nu,kT_e)}\left(\frac{mc^2}{kT_e}\right) \quad (27)$$

as the ratio of quadrupole to dipole emission at a particular frequency. From (16), (23), and (27),

$$B(\lambda)=\left(\frac{1}{10\sqrt{2}}\right)\frac{\lambda^2I(\lambda)}{e^{-\lambda/2}K_0(\frac{1}{2}\lambda)}, \quad (28)$$

where $I(\lambda)$ is defined by Eq. (18). The numerical evaluation of $I(\lambda)$ was carried out on a CDC 6600. In Fig. 1, we plot $B(\lambda)$ versus $\lambda$. From (27) and (26),

$$\lim_{\lambda\to 0}B(\lambda)=6/5\sqrt{2}, \quad (29)$$

in agreement with the numerical results for small $\lambda$. $B(\lambda)$ is seen to rise from the value given in (29) to $\sim 5.5$ for $\lambda=5$. Thus there is a substantial increase in the ratio of quadrupole to dipole emission as the short-wavelength portion of the bremsstrahlung spectrum is reached.

To compare the quadrupole and dipole spectra directly, let us define

$$P^{(e\text{-}i,e\text{-}e)}(\nu,kT_e)d\nu\equiv C^{(e\text{-}i,e\text{-}e)}(N_e,kT_e)\times S^{(e\text{-}i,e\text{-}e)}(\lambda)d\lambda, \quad (30)$$

where, from (16) and (23),

$$S^{e\text{-}i}(\lambda)=e^{-\lambda/2}K_0(\tfrac{1}{2}\lambda)/\lambda, \quad (31)$$

$$S^{e\text{-}e}(\lambda)=\lambda I(\lambda), \quad (32)$$

$$C^{e\text{-}i}(N_e,kT_e)=(16/3)(2/\pi mkT_e)^{1/2}\alpha r_0^2N_e^2mc^2, \quad (33)$$

and

$$C^{e\text{-}e}(N_e,kT_e)=(8/15)(kT_e/\pi m)^{1/2}\alpha r_0^2N_e^2. \quad (34)$$

We write the quadrupole and dipole spectra in units of $C^{e\text{-}i}(N_e,kT_e)$ as given in Eq. (33). The dipole spectrum is

$$\frac{P^{(e\text{-}i)}(\nu,kT_e)}{C^{e\text{-}i}(N_e,kT_e)}=S^{e\text{-}i}(\lambda)=\frac{e^{-\lambda/2}K_0(\frac{1}{2}\lambda)}{\lambda} \quad (35)$$

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[10] I. M. Ryshik and I. S. Gradstein, *Tables of Series, Products, and Integrals* (Veb Deutscher Verlag Der Wissenschaften, Berlin, 1957).

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1958), 4th ed. p. 373.

[12] Tables of $e^xK_0(x)$ can be found in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and J. A. Stegun, National Bureau of Standards, Applied Mathematics Series, No. 55 (U. S. Government Printing Office, Washington, D. C., 1964), p. 417.

and the quadrupole spectrum is

$$\frac{P^{\text{e-e}}(\nu,kT_e)}{C^{\text{e-i}}(N_e,kT_e)}=\frac{C^{\text{e-e}}(N_e,kT_e)}{C^{\text{e-i}}(N_e,kT_e)}S^{\text{e-e}}(\lambda)$$

$$=\frac{1}{10\sqrt{2}}\left(\frac{kT_e}{mc^2}\right)\lambda I(\lambda)\,. \quad (36)$$

In terms of $B(\lambda)$, Eq. (36) can be written

$$\frac{P^{\text{e-e}}(\nu,kT_e)}{C^{\text{e-i}}(N_e,kT_e)}=\left(\frac{kT_e}{mc^2}\right)B(\lambda)S^{\text{e-i}}(\lambda)\,. \quad (37)$$

For

$$kT_e=(10,20)\text{ keV}\,, \quad (38)$$

$$\frac{P^{\text{e-e}}(\nu,kT_e)}{C^{\text{e-i}}(N_e,kT_e)}=(0.02,0.04)B(\lambda)S^{\text{e-i}}(\lambda)\,. \quad (39)$$

In Fig. 2, we plot $S^{\text{e-i}}(\lambda)$, the dipole spectrum, and $0.04B(\lambda)\ S^{\text{e-i}}(\lambda)$, the quadrupole spectrum, normalized in the same manner, for $kT_e=20$ keV. Experimentally, one would measure the sum of these two spectra plus the quadrupole e-i contribution.[13] We see from Fig. 2 that the spectra "cutoff" in the region $1<\lambda<10$. For the quadrupole e-e spectrum, a rough estimate of the contribution of the short-wavelength region to the total energy emitted shows that greater than 90% of the energy is emitted in the region $\lambda\lesssim 5$, with about 60% of this energy in the region

$$0.5\leq\lambda\lesssim 5\,. \quad (40a)$$

For the e-i spectrum, about 50% of the energy is in the region

$$0.5\leq\lambda\leq 4\,. \quad (40b)$$

We find that the e-e contribution will be 4–10% (8–20%) of the e-i contribution in the frequency range, $0.5\leq\lambda\leq 5$, for at emperature $kT_e=10$ (20) keV. The quadrupole e-i contribution in this region is 2–4% (2–7%) for $kT_e=10$ (20) keV.[13] The e-e contribution to the experimentally measured spectrum, in the short-wavelength region, would be approximately

$$\frac{P^{\text{e-e}}(\nu)}{P^{\text{e-i}}(\nu)[1+0.05]}\simeq\frac{(0.95)P^{\text{e-e}}(\nu)}{P^{\text{e-i}}(\nu)}=0.95B(\lambda)\left(\frac{kT_e}{mc^2}\right) \quad (41)$$

for 20 keV.

Therefore, the "average" contribution of quadrupole e-e to the short-wavelength portion of the bremsstrahlung spectrum of a hydrogenic plasma at 20 keV is 14% and proportional to $kT_e$. Since $\gtrsim 50\%$ of the total energy is emitted in this region, we estimate that $\gtrsim 7\%$ of the total energy in the bremsstrahlung spectrum (e-e+e-i) comes from e-e collisions.

[13] R. L. Gluckstern, M. H. Hull, Jr., and G. Breit, U. S. Atomic Energy Commission Report No. AECD-4246, [Yale LA3, 1953 (unpublished)]; C. Quigg, Lawrence Radiation Laboratory Report No. UCRL 50227 (1967) (unpublished).

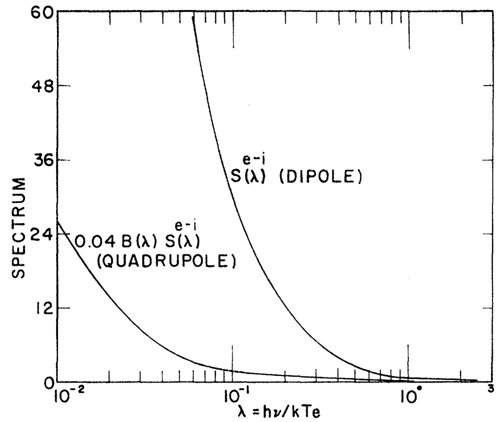


FIG. 2. Dipole and quadrupole spectra at 20 keV.

## V. TOTAL ENERGY EMITTED

Following the notation of Ref. 5, we define $W_{\text{e-e}}$ as the total energy emitted per unit time per unit volume due to e-e bremsstrahlung. Then

$$W_{\text{e-e}}(\tau)=\int_0^\infty h\nu P^{\text{e-e}}(\nu,kT_e)d\nu\,, \quad (42)$$

where

$$\tau=kT_e/mc^2\,, \quad (43)$$

and $P^{\text{e-e}}(\nu,kT_e)$ is given by Eqs. (A21) and (1). The integral (42) is easily evaluated by integrating first over frequency and then over the electron energy, since[7]

$$\int_0^\infty d\sigma_\nu(\epsilon_1)h\nu\simeq 8\alpha r_0^2\epsilon_1\,. \quad (44)$$

Using (44), we obtain

$$W_{\text{e-e}}(\tau)\simeq 32\alpha r_0^2N_e^2(kT_e)^{3/2}/(\pi m)^{1/2}\,, \quad (45)$$

in agreement with Ref. 5.

Using the well-known result[2,14]

$$W_{\text{e-i}}(\tau)=\frac{32}{3}\left(\frac{2kT_e}{\pi m}\right)^{1/2}\alpha r_0^2N_e^2mc^2\,, \quad (46)$$

we obtain

$$W_{\text{e-e}}(\tau)/W_{\text{e-i}}(\tau)=(3/\sqrt{2})\tau\,. \quad (47)$$

If $kT_e=20$ keV, $\tau\simeq 0.04$, and $W_{\text{e-e}}(\tau)/W_{\text{e-i}}(\tau)\simeq 8\%$, in good agreement with our estimate of Sec. IV.

## VI. CORRECTION TO THE BORN APPROXIMATION FOR LOW TEMPERATURES

It is well known that the Born cross section for e-i bremsstrahlung can be corrected by a simple factor,[15] if the incoming electron energy is high enough, in order

[14] C. F. Wandel, T. Hesselberg Jensen, and O. Kofoed-Hansen, Nucl. Instr. Methods 4, 246 (1959).

[15] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., Chap. V.

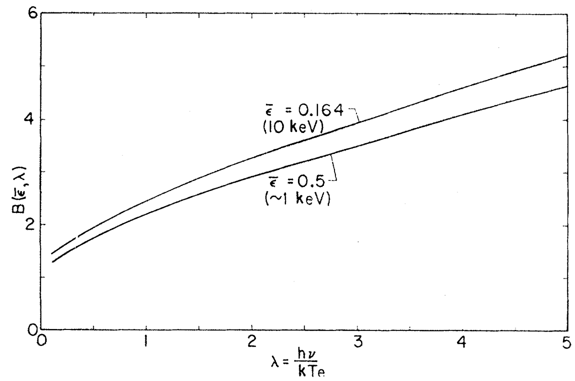


FIG. 3. Corrected quadrupole to dipole ratio.

to obtain agreement with exact results. In the exact calculation, the matrix element of the dipole operator is taken between wave functions which are solutions of the Schrödinger equation for an electron in the Coulomb field of the ion. This simple factor has been derived by Elwert,[16] and its validity has been investigated in subsequent numerical calculations.[17] In Berger's work, it is found that the Elwert corrected result is within 2% of the exact result if the energy of the incoming electron is greater than 100 eV. The simple correction factor, which we denote by $f_{\mathrm{BE}}$ (Born-Elwert), is given by

$$f_{\mathrm{BE}}=(\eta/\eta_0)(1-e^{-2\pi\eta_0})/(1-e^{-2\pi\eta})\,, \tag{48}$$

where

$$\eta=Ze^2/\hbar v,\quad \eta_0=Ze^2/\hbar v_0\,, \tag{49}$$

and $v_0(v)$ is the velocity of the incoming (outgoing) electron. This factor is proportional to the probability for finding the final electron at the position of the ion. Since

$$f_{\mathrm{BE}}\geq 1\,, \tag{50}$$

the corrected cross section

$$d\sigma_c=f_{\mathrm{BE}}d\sigma_{\mathrm{Born}}\geq d\sigma_{\mathrm{Born}}\,, \tag{51}$$

so that there will be more bremsstrahlung. This must be true since the attractive potential causes the final electron to spend more time in the field of the nucleus.

For the e-e problem, the correction factor analogous to (48) is simply found by computing the probability of finding the two final electrons at the same position, using the exact solutions of the Schrödinger equations for scattering in a Coulomb field.[8] This factor is

$$f_{\text{e-e}}=(\eta_2/\eta_1)(e^{2\pi\eta_1}-1)/(e^{2\pi\eta_2}-1)\,, \tag{52}$$

where

$$\eta_{1,2}=e^2/\hbar v_{1,2}\,, \tag{53a}$$

and

$$v_{1,2}=(2\epsilon_{1,2}/\mu)^{1/2}\,, \tag{53b}$$

consistent with our notation in Secs. II and III. Since

$$f_{\text{e-e}}\leq 1\,, \tag{54}$$

the corrected cross section,

$$d\sigma_c^{\text{e-e}}=f_{\text{e-e}}d\sigma_\nu(\epsilon_1)\,, \tag{55}$$

where $d\sigma_\nu(\epsilon_1)$ is given by (1), will satisfy the inequality

$$d\sigma_c^{\text{e-e}}\leq d\sigma_\nu(\epsilon_1)\,. \tag{56}$$

This reflects the fact that the Coulomb repulsion will cause the electrons to spend less time close to each other, resulting in less bremsstrahlung, especially in the short-wavelength region. This effect disappears if the initial energy $\epsilon_1$ is high enough, since

$$f_{\text{e-e}}\to 1\,. \tag{57}$$

Using (55), (52), and (A21), we find that the corrected probability per unit time per unit volume for e-e bremsstrahlung is

$$P(\nu,kT_e,\bar{\epsilon})d\nu=\frac{8}{15}\frac{\alpha r_0^2N_e^2h\nu d(h\nu)}{(\pi m)^{1/2}(kT_e)^{3/2}}I(\bar{\epsilon},\lambda)\,, \tag{58}$$

where

$$I(\bar{\epsilon},\lambda)=\int_0^1\frac{dy}{(1-y)^{1/2}}\left\{\exp\left[\frac{\bar{\epsilon}}{[\lambda y(1-y)]^{1/2}}\right]-1\right\}^{-1}\times[\exp(\bar{\epsilon}/\lambda y)-1]A(\lambda,y)\,, \tag{59}$$

$$\bar{\epsilon}=(2\pi/\sqrt{2})(Ry/kT_e)^{1/2}\quad(\mathrm{Ry}=13.6\text{ eV})\,, \tag{60}$$

and $A(\lambda,y)$ is given by Eq. (18b).

The corrected ratio of quadrupole (e-e) to dipole (e-i) emission will be

$$B(\bar{\epsilon},\lambda)=\left(\frac{1}{10\sqrt{2}}\right)\frac{\lambda^2I(\bar{\epsilon},\lambda)}{e^{-\lambda/2}K_0(\frac{1}{2}\lambda)}\,, \tag{61}$$

where

$$\frac{P^{\text{e-e}}(\nu,kT_e,\bar{\epsilon})}{P^{\text{e-i}}(\nu,kT_e)}\equiv B(\bar{\epsilon},\lambda)\left(\frac{kT_e}{mc^2}\right). \tag{62}$$

The integral $I(\bar{\epsilon},\lambda)$ was numerically evaluated on a CDC 6600. The results for $B(\bar{\epsilon},\lambda)$ versus $\lambda$ are plotted in Fig. 3 for values of $\bar{\epsilon}=0.164$ and 0.5, corresponding to temperatures of 10 and ~1 keV. By comparing the results given in Figs. 1 and 3, we see that the correction due to the "Elwert factor" at 10 keV is $\leq 6\%$ for all frequencies. At ~1 keV, the correction becomes as large as 18% in the short-wavelength region. The correction becomes larger as the frequency increases, since the Coulomb repulsion manifests itself most strongly in close collisions.

At 20 keV, the correction ranges from 3% at $\lambda=0.5$ to 4% at $\lambda=5$. Since the region $0.5\leq\lambda\lesssim 5$ contributes $\geq 60\%$ of the total energy emitted, we estimate about a 2% correction to the total e-e bremsstrahlung. For

[16] G. Elwert, Ann. Physik **34**, 178 (1939).
[17] P. Kirkpatrick and L. Wiedmann, Phys. Rev. 67, 321 (1945); J. M. Berger, *ibid.* **105**, 35 (1957).

10 keV, the correction should be $\sim 3\%$. In Ref. 5, this correction is estimated by classical methods and found to be $\sim 1.3\%$ for $kT_e = 10$ keV. Therefore, the classical approximation seems to underestimate the Coulomb repulsion effect in e-e collisions by a factor $\sim 2$. On the other hand, this method seems to work well for the e-i case, as pointed out in Ref. 5.

To summarize, the results of Sec. IV on the quadrupole emission should be decreased by $\sim$3–4% at 20 keV and $\sim$4–6% at 10 keV, in the short-wavelength region $0.5 \leq \lambda \leq 5$.

## VII. CONCLUSIONS

The main result of this work is that e-e bremsstrahlung collisions contribute to the short-wavelength emission spectrum from a hydrogenic plasma when $kT_e \geq 10$ keV. In particular, the ratio of quadrupole (e-e) to dipole (e-i) emission is 5–10% for $\lambda = 0.5$–5 at 10 keV and proportional to the electron temperature. These results are good to $kT_e = 25$ keV because of the nonrelativistic approximations.

The dipole (e-i) spectrum has $\sim 50\%$ of its total energy in the region $0.5 \leq \lambda \leq 4$, and the quadrupole (e-e) has $\sim 60\%$ of its total energy in the region $0.5 \leq \lambda \leq 5$. Therefore, in the temperature region $kT_e = 10$ (20) keV, one can expect that $\sim 4\%$ (8%) of the total energy emitted by bremsstrahlung (e-e+e-i), in a hydrogenic plasma, comes from e-e collisions.

We also correct the Born cross section for low temperatures and high frequencies since the wave function describing the final state will deviate from a plane wave. It is found that the quadrupole e-e emission is decreased by $\sim 5\%$ (4%) at $kT_e = 10$ (20) keV, so that the correction is small.

It is obvious that a measurement of the emission spectrum of a plasma can be used to determine the electron temperature. This method has been used both in "lab plasma" physics[18] and in astrophysics.[19] From our work, we conclude that in the region $10\ \text{keV} \leq kT_e \leq 25$ keV, one should add the dipole (e-i) and the quadrupole (e-e) spectra in order to correctly describe the emission.

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## APPENDIX

First we will transform Eq. (14) into an integral over the total and relative momenta of the two electrons. Since $d\sigma$ only depends on the relative momentum, the integral over the total momentum can be done trivially. We then evaluate the remaining integral in the c.m. system of the two electrons. We finally obtain $P(\nu,kT_e) \times d\nu$ as an integral of the cross section (1) over a Maxwell-Boltzmann distribution of reduced-mass particles.

First, we define new variables.

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2\,, \quad \mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2\,, \tag{A1}$$

so that

$$\mathbf{p}_1 = \tfrac{1}{2}(\mathbf{P} + \mathbf{p}), \quad \mathbf{p}_2 = \tfrac{1}{2}(\mathbf{P} - \mathbf{p})\,. \tag{A2}$$

Now

$$(d\mathbf{p}_1)(d\mathbf{p}_2) = (d\mathbf{p})(d\mathbf{P})\,\partial(\mathbf{p}_1,\mathbf{p}_2)/\partial(\mathbf{p},\mathbf{P}) = \tfrac{1}{8}(d\mathbf{p})(d\mathbf{P})\,. \tag{A3}$$

Since the integration in (14) is over all $\mathbf{p}_1$ and $\mathbf{p}_2$, the transformed integral will go over all values of $\mathbf{p}$ and $\mathbf{P}$.

Using (15),

$$N_e^{m\text{M-B}}(\mathbf{p}_1) N_e^{m\text{M-B}}(\mathbf{p}_2) = (1/2\pi mkT_e)^3\, e^{-(|p_1|^2+|p_2|^2)/2mkT_e}\,. \tag{A4}$$

Reducing the two-particle scattering problem to the scattering of a reduced particle in an external field, the total energy can be written[20]

$$E = \frac{|p_1|^2}{2m} + \frac{|p_2|^2}{2m} = \frac{|P|^2}{2M} + \frac{|p|^2}{2\mu}\,, \tag{A5}$$

where

$$M = 2m \tag{A6}$$

$$\mu = \tfrac{1}{2}m = \text{reduced mass.} \tag{A7}$$

Therefore, using (A4), (A5), (A6), and (A7),

$$N_e^{m\text{M-B}}(\mathbf{p}_1) N_e^{m\text{M-B}}(\mathbf{p}_2) = (1/2\pi mkT_e)^3\, e^{-|p|^2/2\mu kT_e}\, e^{-|P|^2/2MkT_e}\,. \tag{A8}$$

Now

$$(1/2\pi mkT_e)^{3/2} \int (d\mathbf{p})\, e^{-|p|^2/2mkT_e} = 1 \tag{A9}$$

so that

$$N_e^{m\text{M-B}}(\mathbf{p}_1) N_e^{m\text{M-B}}(\mathbf{p}_2) = N_e^{\mu\text{M-B}}(\mathbf{p}) N_e^{M\text{M-B}}(\mathbf{p})\,. \tag{A10}$$

Using (A3), (A4), (A9), and (A10), (14) can be written

$$P^{\text{e-e}}(\nu,kT_e)d\nu = \frac{N_e^2}{16} \times \int (d\mathbf{p}) N_e^{\mu\text{M-B}}(\mathbf{p}) \frac{|\mathbf{p}|}{m} d\sigma(|\mathbf{p}|,\nu))\,. \tag{A11}$$

[18] K. Boyer, E. M. Little, W. E. Quinn, G. A. Sawyer, and T. F. Stratton, Phys. Rev. Letters **2**, 279 (1959); H. R. Griem, A. C. Kolb, and W. R. Faust, *ibid.* **2**, 281 (1959).

[19] See, e.g., I. S. Shklovsky, Astrophys. J. Letters **148**, No. 1, Part 2 (1967). We wish to thank Dr. W. H. Grasberger for pointing out this reference to us.

[20] See, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Corporation, Reading, Massachusetts, 1959), Chap. 3.

Now we go to the c.m. system where

$$\mathbf{P}=0, \tag{A12}$$

$$\mathbf{p}=2\mathbf{p}_1, \tag{A13}$$

$$\epsilon=|\mathbf{p}_1|^2/m, \tag{A14}$$

and

$$\text{relative velocity}=|\mathbf{p}|/m=2|\mathbf{p}_1|/m. \tag{A15}$$

This is equivalent to a reduced-mass particle with momentum $\mathbf{p}_1$, so that

$$\epsilon=|\mathbf{p}_1|^2/2\mu=|\mathbf{p}_1|^2/m \tag{A16}$$

$$\text{velocity}=|\mathbf{p}_1|/\mu=2|\mathbf{p}_1|/m. \tag{A17}$$

This situation can be represented symbolically as

$$\underset{m}{\overset{\mathbf{p}_1}{\longrightarrow}}\underset{m}{\overset{-\mathbf{p}_1}{\longleftarrow}}=\underset{\mu}{\overset{\mathbf{p}_1}{\longrightarrow}}.$$

Since we want to integrate over a "reduced particle" distribution, let us transform the integration variable from $\mathbf{p}$ to $\mathbf{p}_1$.

Using (A13),

$$(d\mathbf{p})=8(d\mathbf{p}_1). \tag{A18}$$

Finally,

$$P^{\mathrm{e-e}}(\nu,kT_e)d\nu=\tfrac{1}{2}N_e^2\int(d\mathbf{p}_1)N_e^{\mu\mathrm{M-B}}(\mathbf{p}_1)\frac{|\mathbf{p}_1|}{\mu}d\sigma_\nu, \tag{A19}$$

where $d\sigma_\nu$ is given by Eq. (1), where

$$\epsilon_1=|\mathbf{p}_1|^2/2\mu. \tag{A20}$$

Integrating over angles, we obtain

$$P^{\mathrm{e-e}}(\nu,kT_e)d\nu=\frac{(2/\pi\mu)^{1/2}N_e^2}{(kT_e)^{3/2}}\times\int_{h\nu}^{\infty}d\epsilon_1\epsilon_1\,e^{-\epsilon_1/kT}d\sigma_\nu(\epsilon_1). \tag{A21}$$

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# Kinetic Theory for the Interpretation of Measurements on Fluctuations in Radiation Distributions in Finite, Inhomogeneous Systems*

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A kinetic (transport) theory is presented for the first- and second-order (and, if necessary, higher) statistical moments of the number densities of the various particles and/or photons that describe the observable fluctuations in the radiation distribution from an emitting system. This treatment is particularly suitable for the analysis of finite, inhomogeneous systems that may be composed of detectors located outside of a radiating source. Because we are largely concerned with the utility of kinetic theory as a physical theory, considerable emphasis is placed upon an appropriate theoretical description of the actual observables of given experimental situations. The quantum Liouville equation is used to generate the coupled set of transport equations, and basic criteria for the applicability of transport and wave theories are discussed. Quantum-statistical effects are also quite naturally accounted for in cases where they are relevant. It is seen that fluctuation measurements are useful for inferring information relevant to the dynamic interactions within a given system. Such measurements often enjoy the feature of being passive with respect to the interacting system of interest. To illustrate the use of this spatially dependent form of kinetic theory on a system emitting optical radiation, we consider an example that interprets a fluctuation measurement on the radiation emergent from a finite nondispersive blackbody. We conclude by discussing the problems of statistical coupling between the radiation field and detector atom distributions.

## I. INTRODUCTION

THE primary objective of this work is to present a transport theory of the multiplet densities of radiation distributions to facilitate the analysis of measurements of fluctuations in radiation fields in which spatial inhomogeneities play a significant role. Our main concern will be for the development of the transport equations which describe the phase-space and time variation of physically interpretable singlet and doublet densities for the radiation system. A secondary objective is to apply these equations (as well as others needed for the description of detected particle densities) to an analysis of selected examples of fluctuation measurements on finite systems in which a consideration of spatially dependent effects is pertinent to a

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