than the $\mathbf{k}^2=0$ rule at all energies, and in particular near $W=1512$, it shows that the experimental points are not inconsistent with the partial waves believed to resonate there. The S_{11} and P_{11} partial waves are large and may explain the observed $\mathbf{k}^2$ dependence (Fig. 4).

The results are roughly of the same magnitude as the experimental data at 1610 MeV (between resonances), so it is possible that the model does give an estimate of the nonresonant part of the pion-electroproduction cross section.

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Simplified Procedure for Performing Absorption Corrections*

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An approximate procedure for carrying out absorption corrections is presented. It is simpler to apply than the usual procedure, in that the impact-parameter integrals are replaced by simple factors. One can perform the absorption corrections directly on the helicity amplitudes, without performing a partial-wave expansion. The procedure allows one to reproduce approximately all of the results of the usual method, including the polarizations of the produced particles, with only the parameters present in the usual approach.

OVER the past few years we have learned that a great deal of experimental data for inelastic two-body processes can be understood within the peripheral model with absorption corrections.[1–3] As is well known, the results for the production angular distributions, the decay angular distribution of final-state resonances, and the interferences between different possible exchanges are the main successes of the model. On the other hand, various failures of the model are equally well known, particularly those concerned with energy dependence.

In spite of its weaknesses, the peripheral model with absorption corrections is clearly useful in analyzing data, and we would like to report here an approximate procedure for carrying out the absorption corrections which is considerably simpler to apply than those currently in use. This procedure allows one to reproduce approximately all of the results of the usual method, including the polarizations of the particles, with only the parameters present in the usual approach. One can perform the absorption corrections directly on the helicity amplitudes, without the necessity for a partial-wave expansion.

Several uses for such a simplified procedure come to mind. First, one can apply the absorptive corrections more easily than previously in conventional situations to obtain information on the spins of new resonances, to extract information on π-π scattering from production of a pion pair (or, of course, on scattering of any pair of particles that can be produced peripherally), to take account of momentum-transfer dependence introduced by absorption when extrapolating to an exchanged particle pole, to study the physical origin of the polarization effects that arise, etc.

In addition, some applications arise which are qualitatively different when such a simplified procedure is available. In a sense, one of the most puzzling aspects of the absorptive peripheral model is that it works; it is not well understood. Particularly obscure is the relation to Regge-pole models, where the energy dependence can be correct but the momentum-transfer dependence is essentially arbitrary. With our simpler procedure, it is relatively easy to separate the energy and momentum-transfer dependence, and it may be possible to examine the relations between different approaches to scattering phenomena. In another direction, it is possible to formulate a procedure for performing absorption corrections in a multiparticle final state in terms of the corrections for the various two-body states. Consequently, it should be possible to decide whether multiperipheral processes occur, in the same sense that the absorptive model itself allowed one to conclude with confidence that peripheral collisions often dominated quasi-two-body inelastic processes.

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[1] See, for example, J. D. Jackson, in *Proceedings of the Thirteenth Annual Conference on High-Energy Physics* (University of California Press, Berkeley, California, 1967); Rev. Mod. Phys. 37, 484 (1965); and the following two references.

[2] J. D. Jackson and K. Gottfried, Nuovo Cimento 34, 735 (1964). We will refer to this paper as JG.

[3] L. Durand, III, and Y. T. Chiu, Phys. Rev. 139, B646 (1965). We will refer to this paper as DC.

We obtain our results by approximating the usual absorption correction formulas[2,3] in the impact-parameter representation. This has already been done for heavy-particle exchange by Durand and Chiu[3]; we give their results along with ours for light-particle (pion) exchange. The usual procedure is given in Refs. 2 and 3; we will summarize it here. It is convenient to work with the variables and notation of JG.

We consider the scattering processes $a+b\to c+d$, where the particles have helicities $\lambda_a, \lambda_b, \lambda_c, \lambda_d$. As emphasized in JG, it is essential to treat spins properly. Assuming that the scattering proceeds by the exchange of a single particle e, one proceeds by writing down all the helicity amplitudes[4] for the single-particle-exchange diagram (the Born-approximation helicity amplitudes).

JG define the dimensionless variables ω, ϵ by

$$m_e^2-t=qq'(\omega^2+\epsilon^2),$$
$$\omega=2\sin\tfrac{1}{2}\theta,$$
$$qq'\epsilon^2=m_e^2+(q-q')^2-(m_a^2-m_b^2-m_c^2+m_d^2)^2/4s,$$

where s and t are the usual scalar variables, $s=-(p_a+p_b)^2$, $t=-(p_c-p_a)^2$, and q and q' are the magnitudes of the initial and final three-momenta in the center-of-mass system. They also define the quantum number n, which controls the spin dependence, by $n=|\lambda_c-\lambda_d-\lambda_a+\lambda_b|$. They express the final, absorption-corrected, amplitudes in terms of simple polynomials and the integrals

$$I_n(\omega^2,\epsilon^2)=\epsilon^n\int_0^\infty x\,dx\,J_n(\omega x)K_n(\epsilon x)\exp[i\delta_{\rm in}(x)]\times\exp[i\delta_{\rm fin}(x)],\quad(1)$$

where the factors representing the absorption are

$$\exp[i\delta_{\rm in}(x)]=\left[1-\left(\frac{\sigma_{T,\rm in}}{4\pi A_{\rm in}}\right)\exp\left(\frac{-x^2}{2A_{\rm in}q^2}\right)\right]^{1/2}$$

for the initial state, with a similar factor for the final state. $A_{\rm in}$ and $\sigma_{T,\rm in}$ are the slope of the forward peak ($d\sigma/dt\sim e^{At}$) and the total cross section, respectively.

First we assume that the initial- and final-state absorption is the same. Then we can summarize our procedure as follows. The formulas are quite different for exchange of light (i.e., pion) particles and massive particles (all but pion). Details are given in the Appendix.

The approximation replaces I_n of Eq. (1) by

$$I_n(\omega^2,\epsilon^2)\simeq\frac{\omega^n}{\omega^2+\epsilon^2}A_n(\omega^2,\epsilon^2)=\frac{(2\sin\tfrac{1}{2}\theta)^n qq'}{m_e^2-t}A_n(\omega^2,\epsilon^2),\quad(2)$$

where the "absorption factors" A_n are given below. The coefficient $\omega^n/(\omega^2+\epsilon^2)$ is just the Born approximation in the present notation. The A_n are the product of initial and final absorption corrections, taken equal. The case when they are different is discussed below.

[4] This is the minimum requirement for taking spin into account correctly. A useful treatment is given by S. Gasiorowicz [*Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1966), chap. 27] with the case $\pi N\to\rho N$ worked out in detail.

TABLE I. Numerical values for some quantities that appear in the absorption formulas.

$\omega^2/4\gamma$	$\frac{4\gamma}{\omega^2}\int_0^{\omega^2/4\gamma}\frac{(e^y-1)dy}{y}$	$\exp(-\omega^2/4\gamma)$	$\epsilon^2/4\gamma$	$\int_0^\infty\frac{e^{-v}dy}{y+\epsilon^2/4\gamma}$
0	1.000	1	0.01	4.00
0.05	1.013	0.951	0.02	3.40
0.10	1.026	0.905	0.03	3.04
0.15	1.039	0.861	0.04	2.76
0.20	1.052	0.819	0.07	2.30
0.25	1.066	0.779	0.10	2.01
0.30	1.080	0.741	0.20	1.49
0.35	1.095	0.705	0.30	1.22
0.40	1.110	0.670	0.40	1.04
0.45	1.125	0.638	0.50	0.92
0.50	1.140	0.607	0.60	0.82
0.60	1.173	0.549	0.70	0.74
0.70	1.207	0.497	0.80	0.69
0.80	1.241	0.449	0.90	0.64
0.90	1.279	0.407	1.00	0.60
1.00	1.318	0.368	1.25	0.51
1.25	1.425	0.287	1.50	0.45
1.50	1.546	0.223	2.00	0.36
1.75	1.661	0.174		
2.00	1.842	0.135		
2.50	2.232	0.082		
3.00	2.750	0.050		
4.00	4.198	0.018		

For pion exchange one has ($C=\sigma_T/4\pi A$, $\gamma=1/2Aq^2$)

$n=0$:

$$A_0(\omega^2,\epsilon^2)=1-[C(\omega^2+\epsilon^2)/4\gamma]\exp(-\omega^2/4\gamma)Z(\omega^2,\epsilon^2),$$
$$Z=\int_0^\infty dy\exp(-y)/(y+\epsilon^2/4\gamma)+\int_0^{\omega^2/4\gamma}dy\times[\exp(y)-1]/y.\quad(3)$$

Although $Z(\omega^2,\epsilon^2)$ appears to be complicated, its two terms are both tabulated functions, related to the exponential integral, given in Chap. 5 of Ref. 5. For completeness, we give values for the two integrals in Table I. For small or large arguments they can be approximated by simple expressions [see the Appendix and particularly Eq. (A4)].

The rest of the absorption factors are still simpler.

$n>0$:

$$A_1(\omega^2,\epsilon^2)=1-C(\omega^2+\epsilon^2)[1-\exp(-\omega^2/4\gamma)]/\omega^2,\quad(4)$$
$$A_2(\omega^2,\epsilon^2)=1-C(\omega^2+\epsilon^2)\times\{1-4\gamma[1-\exp(-\omega^2/4\gamma)]/\omega^2\}/\omega^2,\quad(5)$$
$$A_n(\omega^2,\epsilon^2)=1-C[(\omega^2+\epsilon^2)/(2^{n+2}\gamma)]\exp(-\omega^2/8\gamma)\times\int_{-1}^1 dx\,(x+1)^{n-1}\exp(\omega^2x/8\gamma).\quad(6)$$

[5] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications Inc., New York, 1965), Chap. 5.

TABLE II. Values of $I_n(\omega^2,\epsilon^2)$ from Eq. (1) ("Exact") and from our approximations to $I_n(\omega^2,\epsilon^2)$ from Eqs. (2)–(7) ("Approximate").

Pion exchange	Exact	Approximate	Massive exchange	Exact	Approximate
$I_0(0.08,0.024)$	2.71	2.40	$I_0(0.08,0.64)$	0.26	0.36
$I_0(0.24,0.024)$	−1.17	−1.43	$I_0(0.24,0.64)$	0.13	0.14
$I_1(0.08,0.024)$	1.87	1.64	$I_1(0.08,0.64)$	0.16	0.23
$I_1(0.24,0.024)$	0.71	0.53	$I_1(0.24,0.64)$	0.18	0.22
$I_2(0.08,0.024)$	0.61	0.63	$I_2(0.08,0.64)$	0.075	0.10
$I_2(0.24,0.024)$	0.55	0.53	$I_2(0.24,0.64)$	0.16	0.18

The integral in the last line is elementary for all $n\geq 1$ and it is a convenient form for presenting the general pion-exchange absorption factor. The result for $n=1, 2$ are given explicitly.

We note that at the pole ($\omega^2+\epsilon^2=0$) the absorption factors A_n are unity, and in the forward direction ($\omega=0$), where only I_0 is nonvanishing, the approximation of Appendix A becomes exact and the I_0's from Eqs. (1) and (2) are equal. The coefficients of ω^n for $n>0$ are not exact as $\omega\to 0$.

For massive exchange the absorption factors can be found from the work in DC: for all n ($C=\sigma_T/4\pi A$, $\gamma=1/2Aq^2$),

$$A_n(\omega^2,\epsilon^2)=1-C+4\gamma C[n+1-2\omega^2/(\omega^2+\epsilon^2)]/(\omega^2+\epsilon^2). \quad (7)$$

The derivation of this approximation and some discussion of its properties is given in DC.

The approximations for light exchange (pion exchange above) should be valid whenever $\epsilon^2/4\gamma\ll 1$ (see Appendix), while those for massive exchange should be valid[3] for $(\omega^2+\epsilon^2)/4\gamma\gg 1$. For most processes and energies of interest, one has γ within about a factor of 2 of 0.05. For pion exchange $\epsilon^2\lesssim 0.03$, while for vector meson exchange $\epsilon^2\gtrsim 0.60$, for typical energies. For K exchange at 5 GeV/c one has $\epsilon^2\simeq 0.25$, $\gamma\simeq 0.03$, so $(\omega^2+\epsilon^2)/4\gamma\simeq 2+8\omega^2$, and the approximation is not particularly good, though it can still be used. When $(\omega^2+\epsilon^2)/4\gamma$ is smaller than about 3 the massive exchange approximation can become a poor one, so the question of its validity should be examined in some detail.

To give some ideal of the validity of the approximation, we quote some values of $I_n(\omega^2,\epsilon^2)$ in Table II. The values under "Exact" are calculated from Eq. (1), those under "Approximate" from Eq. (2). They are calculated with $C=1$, $A=7.5/(\text{GeV}/c)^2$, at 3 GeV/c ($\gamma=0.055$). Table III gives a comparison of the differential cross section and ρ density matrix elements for $\pi^-p\to\rho^-p$ at 3 GeV/c, using $C=1$, $A=7.5/(\text{GeV}/c)^2$ for our approximate calculation, with the "exact" absorption-model fits to the data.

In general, one will be able to fit data with the approximate factors as well as with the exact ones, but with slightly different values of the parameters σ_T, A, and the coupling constants involved. So long as one does not want to interpret the resulting values of these parameters, the approximate version of the corrections should be adequate. If an interpretation is desirable one should presumably work with the exact corrections, and perhaps even directly with the partial-wave expansions of the helicity amplitudes[6] rather than with their approximate impact-parameter representation. Our approximate factors may be useful in establishing that the data can be understood within the framework of the absorptive peripheral model before one carries out the full absorption calculation.

If the initial and final absorption should be taken to be different, one may proceed as follows: From the discussion in the Appendix one sees that the integral we actually approximate is

$$\int xdx\, J_nK_n \exp(-\gamma x^2),$$

which is what arises from the factor $\exp(i\delta_{\text{in}})\exp(i\delta_{\text{fin}})=1-C\exp(-\gamma x^2)$ when the initial and final corrections are the same. When they are not the same, the factor in the integral is $\alpha=[1-C_{\text{in}}\exp(-\gamma_{\text{in}}x^2)]^{1/2}[1-C_{\text{fin}}\times\exp(-\gamma_{\text{fin}}x^2)]^{1/2}$. For large x we can expand the radicals so that

$$\alpha\simeq 1-\tfrac{1}{2}C_{\text{in}}\exp(-\gamma_{\text{in}}x^2)-\tfrac{1}{2}C_{\text{fin}}\exp(-\gamma_{\text{fin}}x^2).$$

Using this form for α in the integral, we obtain an approximate expression for I_n. We now assume that we can reverse the expansion of the radicals and we finally obtain the expression for I_n involving approximate absorption factors,

$$I_n\simeq[\omega^n/(\omega^2+\epsilon^2)]A_{n,\text{in}}{}^{1/2}A_{n,\text{fin}}{}^{1/2}, \quad (8)$$

where the $A_{n,\text{in}}{}^{1/2}$ or $A_{n,\text{fin}}{}^{1/2}$ are the square roots of the absorption factors given in Eqs. (3)–(7). Thus, for example, $A_{1,\text{in}}{}^{1/2}(\omega^2,\epsilon^2)=\{1-C_{\text{in}}(\omega^2+\epsilon^2)[1-\exp(-\omega^2/4\gamma_{\text{in}})]/\omega^2\}^{1/2}$, for pion exchange [from Eq. (4)], where $C_{\text{in}}=\sigma_{T,\text{in}}/4\pi A_{\text{in}}$ and $\gamma_{\text{in}}=1/2A_{\text{in}}q_{\text{in}}{}^2$, all the quantities involved being appropriate for the initial scattering

TABLE III. Comparison of the approximate differential cross section and density matrix elements for $\pi^\pm p(\pi)\rho^\pm p$ at 2.75 and 3 GeV/c with the absorption model fitted to the experimental data [estimated from V. Hagopian, W. Selove, J. Alitti, J. P. Baton, and M. Neveu-René, Phys. Rev. **145**, 1128 (1966) and Ref. 1]. The approximate results were calculated with $C=1$ and $A=7.5/(\text{GeV}/c)^2$. No effort was made to vary these parameters to improve the fit to the data.

	$\omega^2=0.08$ ($\cos\theta=0.96$) Approximate results	"Exact" fit to data	$\omega^2=0.24$ ($\cos\theta=0.88$) Approximate results	"Exact" fit to data
$m_\pi{}^2d\sigma/dt$ (mb)	0.066	0.072	0.022	0.020
ρ_{00}	0.64	0.70	0.57	0.59
$\text{Re}\rho_{10}$	−0.19	−0.16	−0.11	−0.14
ρ_{1-1}	0.06	0.07	0.16	0.10

[6] J. Donohue, Ph.D. thesis, University of Illinois, 1966 (unpublished); and H. Högaasen, J. Högaasen, R. Keyser, and B. E. Y. Svensson, Nuovo Cimento **42A**, 323 (1966).

state. The result in Eq. (8) reduces to that in (2) when the initial- and final-state absorption become equal. We have emphasized the case when initial and final absorption is taken the same because the result of Eqs. (8) is not something we can derive in general, though it has the expected behavior for weak absorption and for equal initial and final absorption and appears to be a sensible conjecture for the approximation to I_n when initial and final absorption differ.

Let us close by summarizing the procedure for performing approximate absorption corrections simply. First one calculates the helicity amplitudes for the process of interest and follows the procedure of JG or DC to write the modified helicity amplitudes in terms of the factors I_n of Eq. (1). Then one replaces the I_n by the approximate form of Eq. (8), with the absorption factors given by Eqs. (3)–(6) for pion exchange or by Eq. (7) for massive exchange. Now one has the approximate absorption corrected helicity amplitudes, and the cross section and density matrix elements are computed in the conventional manner.[2,3]

I have profited from instructive conversations with Dr. P. K. Williams, Dr. Y. T. Chiu, Professor L. Durand, Professor M. Ross, and Professor J. D. Jackson.

APPENDIX

The derivation of the absorption factors in the text is not particularly instructive; it is a matter of manipulating Bessel function identities.

For the case of equal initial and final absorption we have

$$I_n(\omega^2,\epsilon^2)=\epsilon^n\int_0^\infty x\,dx\,J_n(\omega x)K_n(\epsilon x)[1-C\exp(-\gamma x^2)]$$
$$=\omega^n/(\omega^2+\epsilon^2)-CL_n(\omega^2,\epsilon^2), \tag{A1}$$

where the first term results from an identity given in Ref. 2, and

$$L_n(\omega^2,\epsilon^2)=\epsilon^n\int_0^\infty x\,dx\,J_n(\omega x)K_n(\epsilon x)\exp(-\gamma x^2) \tag{A2}$$

is the integral we want to evaluate. Using the identities[7]

$$\epsilon^nK_n(x)=\int_0^\infty dy\,J_n(xy)y^{n+1}/(y^2+\epsilon^2),$$

and

$$\int_0^\infty x\,dx\,J_n(\omega x)J_n(yx)\exp(-\gamma x^2)$$
$$=\left[\exp\left(-\frac{(\omega^2+y^2)}{4\gamma}\right)\right]\frac{I_n(\omega y/2\gamma)}{2\gamma},$$

we obtain

$$L_n(\omega^2,\epsilon^2)=\left[\exp\left(-\frac{\omega^2}{4\gamma}\right)\right]\left(\frac{1}{2\gamma}\right)\int_0^\infty y^{n+1}dy$$
$$\times\exp\left(-\frac{y^2}{4\gamma}\right)\frac{I_n(\omega y/2\gamma)}{y^2+\epsilon^2}. \tag{A3}$$

Now, for light-particle exchange we have ϵ^2 small. For example, for pion exchange and momenta above about 2 GeV/c, $\epsilon^2\lesssim0.03$. (At 5 GeV/c for pion exchange $\epsilon^2\simeq0.01$.) On the other hand, the integrand peaks at $y^2\lesssim4\gamma\lesssim0.20$. [At 5 GeV/$c$, for $A=7.5/(\text{GeV}/c)^2$, $4\gamma=0.125$.] Thus, for $n\geq1$, where the integrand is small for $y^2\simeq\epsilon^2$, we can simply drop the ϵ^2 in the denominator of Eqs. (A3). The integral can then be evaluated using the identity

$$\int_0^\infty I_\nu(ay)[\exp(-p^2y^2)]y^{\mu-1}dy$$
$$=(a/2p)^\nu(1/2p^\mu)\Gamma((\mu+\nu)/2)[1/\Gamma(\nu+1)]$$
$$\times{}_1F_1((\mu+\nu)/2,\,\nu+1,\,a^2/4p^2).$$

For our case the hypergeometric function is expressible in terms of elementary functions (recall $n\geq1$)

$${}_1F_1(n,n+1;z)=ne^{z/2}\left(\frac{1}{2^n}\right)\int_{-1}^1 e^{zy/2}(y+1)^{n-1}dy,$$

giving the results in the text in Eq. (6).

The condition for this approximation to be valid is then that $y^{n+1}[\exp(-y^2/4\gamma]/(y^2+\epsilon^2)\simeq y^{n-1}\exp(-y^2/4\gamma)$ for values of y^2 large enough to contribute appreciably to the integral and smaller than 4γ because of the exponential. Since $4\gamma\ll1$, the integrand will peak at $y^2\simeq4\gamma$, so we can take $\epsilon^2/4\gamma\ll1$ as our condition for the validity of the approximation. When $n=0$ we cannot drop ϵ^2 because the integrand in (A3) would be singular at $y=0$. We proceed instead by defining Z by

$$L_0(\omega^2,\epsilon^2)=[\exp(-\omega^2/4\gamma)](1/4\gamma)Z(\omega^2,\epsilon^2).$$

Then

$$dZ(\omega^2,\epsilon^2)/d\omega=(1/2\gamma)\int_0^\infty y^2dy\,[\exp(-y^2/4\gamma)]$$
$$\times I_1(\omega y/2\gamma)/(y^2+\epsilon^2)$$

can be evaluated by the procedure above, and we can obtain Z from

$$Z(\omega^2,\epsilon^2)=Z(0,\epsilon^2)+\int_0^\omega dy\,dZ(y^2,\epsilon^2)/dy.$$

[7] All the identities we use are obtainable from W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954).

This gives the result in the text for $A_0^2(\omega^2,\epsilon^2)$, with

$$Z(\omega^2,\epsilon^2)=\int_0^\infty dy\frac{e^{-y}}{y+\epsilon^2/4\gamma}+\int_0^{\omega^2/4\gamma} dy\frac{e^y-1}{y}.$$

These functions are tabulated in Ref. 5; their names are

$$\int_0^{\omega^2/4\gamma} dy\frac{e^y-1}{y}=E_i(\omega^2/4\gamma)-\ln(\omega^2/4\gamma)-0.577216\cdots,$$

$$\int_0^\infty dy\frac{e^{-y}}{y+\epsilon^2/4\gamma}=\exp(\epsilon^2/4\gamma)]E_1(\epsilon^2/4\gamma).$$

To make the paper self-contained for simple applications, we have included a few values for these integrals in Table I.

For large or small arguments they may be approximated as follows:

For $\omega^2/4\gamma$ somewhat less than unity (satisfied, for example, for momenta less than about 4 GeV/c at angles $\cos\theta\gtrsim 0.9$), one can obtain

$$\int_0^x dy\ (e^y-1)/y\simeq x+x^2/2!2+x^3/3!3+\cdots,$$

and the first two terms are a good approximation. For very large $\omega^2/4\gamma$, one can use $E_i(x)\to(e^x/x)(1+\cdots)$.

Similarly, for small x,

$$e^{-x}\int_0^\infty\frac{e^{-y}dy}{y+x}\simeq x-\tfrac{1}{4}x^2+\cdots-\ln x-0.577216\cdots,$$

and for large x

$$\int_0^\infty\frac{e^{-y}dy}{y+x}\simeq 1/x+\cdots.$$

Thus, for the case of most interest with small $\omega^2/4\gamma$ and small $\epsilon^2/4\gamma$, we have

$$A_0(\omega^2,\epsilon^2)\simeq 1-C(\omega^2+\epsilon^2)[\exp(-\omega^2/4\gamma)](1/4\gamma)[(\omega^2/4\gamma)+(\omega^2/4\gamma)^2/4+\epsilon^2/4\gamma-\ln(\epsilon^2/4\gamma)-0.577\cdots]. \quad \text{(A4)}$$

Finally we summarize some of the properties of the absorption factors.

As $\omega^2/4\gamma\to\infty$ (large momentum transfers), all $A_n\to 1-C$.

At $\omega^2/4\gamma=0$ (forward direction) the approximate I_0 is the same as the Born approximation for the pion exchange case, but not for massive exchange. For $n>0$ all I_n vanish in the forward direction.

At the pole ($\omega^2=-\epsilon^2$ or $t=m_{\text{exch}}^2$) all A_n for pion exchange approach unity, so the residues of the amplitude are equal to those for the Born approximation, while the result for massive exchange has a modified singularity structure, with high-order poles appearing (consequently the A_n for massive exchange not only do not approach unity at the pole, but the residue is infinite there).

At high energies ($s\to\infty$, $\omega^2/4\gamma\to -At/2$, $\epsilon^2/4\gamma\to Am_{\text{exch}}^2/2$) all the absorption factors have the interesting property that they become pure functions of t (independent of s to the extent that the absorption parameters A, σ_T are independent of s), so that the amplitude has the energy dependence of the Born approximation.