interesting questions such as: How do the particles move relative to the core?

The answer, moreover, leads to a *global* description of the nonresonant interaction and provides the foundation for physical deductions. There is nothing false about the local approach, nor is there any inconsistency in describing the motions of n particles with respect to n different coordinate systems—but this approach is hardly conducive to physical comprehension.

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Static Axisymmetric Interior Solution in General Relativity

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A class of static, axisymmetric, interior solutions of Einstein's field equations in general relativity is obtained. The solutions are physically reasonable, and can be interpreted as sources for Weyl and Levi-Civita's general exterior solution that is flat at infinity, by making the metric components and their derivatives continuous at the boundary of the matter. For a particular model, a correlation is exhibited between the structure of the source material and the exterior field, which resembles closely, but not exactly, that of the corresponding Newtonian model.

SINCE Einstein's equations are nonlinear differential equations of the second order, problems arise in the physical interpretation of general relativity which have no counterpart in Newtonian theory. The problem considered here is that of establishing a correlation between the angular dependence of the field quantities at large distances from the source, and the actual distribution of source material.

This is, of course, trivial in Newtonian gravitation because of the linearity; the multipole structure of the linearized gravitational field in general relativity and its relation to a source distribution has also been exhibited.[1] The concept of multipole moment is far from clear in Einstein's theory and although there are various studies of asymptotic multipole moments,[2–4] these are within a framework of a formalism not easily adapted to examination of source distributions.

This paper deals with the static case only. A class of interior solutions is examined and is matched with Weyl and Levi-Civita's general exterior solution which is flat at infinity.

1. THE EXTERIOR SOLUTION

Weyl and Levi-Civita[5] showed that a static axisymmetric vacuum solution of Einstein's gravitational field equations may always be put into the canonical form

$$ds^2=e^{2u}dt^2-e^{2k-2u}(dr^2+dz^2)-r^2e^{-2u}d\varphi^2,$$

where t is the timelike coordinate, r, z, and φ are the spacelike coordinates, u and k are functions of r and z only, and $r=0$ is the axis of symmetry.

The empty space-time field equations are then

$$\frac{1}{r}\frac{\partial k}{\partial z}=2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z},$$

$$\nabla^2 u=0,$$

$$\frac{1}{r}\frac{\partial k}{\partial r}=\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2,$$

where

$$\nabla^2\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}$$

is the flat-space Laplacian operator in cylindrical polar coordinates. Then the solutions for which u and k vanish at infinity may be written in the form

$$u=-\sum_{n=0}^{n=\infty}a_n\rho^{-(n+1)}P_n(\cos\theta),$$

$$k=-\sum_{l,m=0}^{\infty}\sum a_la_m(l+1)(m+1)(l+m+2)^{-1}\times\rho^{-(l+m+2)}[P_lP_m-P_{l+1}P_{m+1}],$$

where $r=\rho\sin\theta$, $z=\rho\cos\theta$, $P_n(\cos\theta)$ are Legendre polynomials and the a_n's are arbitrary constants.

For this form of k, a Minkowskian tangent space will exist at all points on $r=0$ away from the origin; the singularity in the line element at these points is a coordinate one.

[1] R. Sachs and P. Bergman, Phys. Rev. **112**, 674 (1958).
[2] E. Newman and T. Unti, J. Math. Phys. **6**, 1806 (1965).
[3] D. Lamb, J. Math. Phys. **7**, 458 (1966).
[4] A. Janis and E. Newman, J. Math. Phys. **6**, 902 (1965).
[5] P. Bergmann, *An Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1942).

2. THE INTERIOR SOLUTION

It is assumed that the exterior solution represents space-time outside an isolated body whose boundary is $\rho=d$, where d is a constant. Interior solutions can now easily be constructed which may, but need not, be axisymmetric or static. In this paper, however, only axisymmetric and static interior solutions are investigated.

Inside the boundary of the body a possible choice for u and k is

$$u=\sum_{n=0}^{n=\infty}\tfrac{1}{2}\{(2n+1)(\rho/d)^2-(2n+3)\}a_n\rho^n d^{-(2n+1)}P_n,$$

$$k=-\sum_{l,m=0}^{\infty}\sum a_l a_m(l+1)(m+1)(l+m+2)^{-1}$$
$$\times d^{-2(l+m+2)}(p-2)^{-1}[\{2(l+m)+p+2\}\rho^{l+m+2}$$
$$-\{2(l+m)+4\}\rho^{l+m+p}d^{2-p}][P_lP_m-P_{l+1}P_{m+1}],$$

where p is an integer $\geqslant 3$.

For this choice, the components of g_{ab} and their first partial derivatives are continuous across the boundary $\rho=d$. If the appropriate series converge, then u and k will be regular on the boundary of the body and therefore regular everywhere.

The nonzero components of the energy tensor are [with $(t,r,z,\varphi)=(x^0,x^1,x^2,x^3)$ and $c=1$]

$$T_{12}=2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}-\frac{1}{r}\frac{\partial k}{\partial z},$$

$$T_{00}=\kappa^{-1}e^{4u-2k}\left[2\nabla^2u-\frac{\partial^2k}{\partial r^2}-\frac{\partial^2k}{\partial z^2}-\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2\right],$$

$$T_{11}=-T_{22}=-\kappa^{-1}\left[\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2-\frac{1}{r}\frac{\partial k}{\partial r}\right],$$

$$T_{33}=\kappa^{-1}r^2e^{-2k}\left[-\frac{\partial^2k}{\partial r^2}-\frac{\partial^2k}{\partial z^2}-\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2\right].$$

Then, the four eigenvectors λ^a and eigenvalues λ of the energy tensor defined by

$$(T_{ab}-\lambda g_{ab})\lambda^b=0$$

are

$$\kappa^{-1}e^{2u-2k}\left[2\nabla^2u-\frac{\partial^2k}{\partial r^2}-\frac{\partial^2k}{\partial z^2}-\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2\right],\qquad(1)$$

$$\pm\kappa^{-1}e^{2u-2k}\left[\left\{\left(\frac{\partial u}{\partial r}\right)^2-\left(\frac{\partial u}{\partial z}\right)^2-\frac{1}{r}\frac{\partial k}{\partial r}\right\}^2+\left\{2\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}-\frac{1}{r}\frac{\partial k}{\partial z}\right\}^2\right]^{1/2},\qquad(2)$$

$$\kappa^{-1}e^{2u-2k}\left[\frac{\partial^2k}{\partial r^2}+\frac{\partial^2k}{\partial z^2}+\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right].\qquad(3)$$

The eigenvector corresponding to (1) is timelike, and the other three eigenvectors are spacelike. Under the conditions of symmetry assumed here, it would not be reasonable to require the material of the body to constitute a perfect fluid; it will be assumed merely that the material is physically reasonable, so that the three spacelike eigenvalues, corresponding to pressures or stresses, are required to be much smaller than the timelike eigenvalue, corresponding to density. This is achieved if, for every n, $a_nd^{-(1+n)}$ is small.

The condition that the density of matter be positive is that

$$G_{00}<0,$$

or

$$2\nabla^2u>\frac{\partial^2k}{\partial r^2}+\frac{\partial^2k}{\partial z^2}+\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2.$$

This inequality, which must hold throughout the volume in which matter is present, may be satisfied in various ways by the constants a_n, $n\geqslant 0$. One possibility is that $a_n=0$ for $n>0$ and a_0/d is positive and sufficiently small; this is discussed in the next section. If $a_n=0$ for all $n>N$, where N is a positive integer, then the inequality will certainly hold if $a_n\geqslant 0$ for $n\leqslant N$ and it is possible for some a_n's to be negative.

3. A SPECIAL CASE

In this section, the particularly simple case is discussed in which $a_0>0$, $a_n=0$ for all $n>0$, and $p=4$.

The four eigenvalues of T_{ab} may now be written as

$$\lambda_0\kappa=e^{2u-2k}(6a_0/d^3-15a_0^2r^2/d^6-3a_0^2z^2/d^6+3a_0^2/d^4),$$
$$\lambda_1\kappa=-\lambda_2\kappa=e^{2u-2k}(1-\{r^2+z^2\}/d^2)3a_0^2/d^4,$$
$$\lambda_3\kappa=e^{2u-2k}(15a_0^2r^2/d^6+3a_0^2z^2/d^6-3a_0^2/d^4),$$

where the three spacelike eigenvectors, corresponding to λ_1, λ_2, and λ_3, are now in the r, z, and φ directions, respectively. λ_1 is always positive, and represents a tension; similarly, λ_2 is always negative and represents a pressure. The density given by λ_0 can be seen to be greatest on the axis of symmetry, $r=0$. It is then to be expected that pressures exist near the axis $r=0$ in the φ direction; this is confirmed by the expression for λ_3.

There are various methods for studying the shape of the body. A "radar distance" may be calculated by emitting photons from the origin of coordinates and allowing them to travel to the surface of the body and back again.

Thus for a null geodesic $z=0$, $\varphi=$const, whose r, t equation is determined by

$$g_{ab}dx^adx^b=0,$$

the time taken for the round trip is (with $c=1$)

$$T=2\int_0^d e^{k-2u}dr$$
$$=2d\left[1+\frac{8}{3}\frac{a_0}{d}+\frac{33}{10}\left(\frac{a_0}{d}\right)^2+O\left(\frac{a_0}{d}\right)^3\right].$$

Similarly, for $r=0$, $\varphi=\text{const}$,

$$T=2d\left[1+\frac{8}{3}\frac{a_0}{d}+\frac{36}{10}\left(\frac{a_0}{d}\right)^2+O\left(\frac{a_0}{d}\right)^3\right].$$

Therefore the radar radius is greater along the axis of symmetry than in the equatorial plane.

Another possible comparison is of proper distances measured along geodesics in a spacelike surface $t=\text{const}$. Both the axis of symmetry $r=0$, $t=\text{const}$, and any radial line $z=0$, $\varphi=\text{const}$, $t=\text{const}$, are geodesics to which this comparison may be applied.

Along the axis of symmetry,

$$L=\int_0^d \frac{ds}{dz}dz=\int_0^d e^{k-u}dz$$
$$=e^{3a_0/2d}d\left[1-\frac{1}{6}\frac{a_0}{d}+\frac{1}{40}\left(\frac{a_0}{d}\right)^2+O\left(\frac{a_0}{d}\right)^3\right].$$

For any radius in $z=0$

$$L=e^{3a_0/2d}d\left[1-\frac{1}{6}\frac{a_0}{d}-\frac{11}{40}\left(\frac{a_0}{d}\right)^2+O\left(\frac{a_0}{d}\right)^3\right].$$

This comparison agrees with the previous one: L is greater along the axis of symmetry than for any radius in $z=0$.

Schwarzschild's solution can be transformed into these coordinates and compared with this case: The transformation is given by

$$e^{2u}=1-2m/\rho',$$
$$e^{-2k}=1+(m^2\sin^2\theta')/(\rho'^2-2m\rho'),$$
$$r=\sin\theta'\,(\rho'^2-2m\rho')^{1/2},$$
$$z=\cos\theta'\,(\rho'-m),$$
$$\varphi=\varphi',$$
$$t=t',$$

where $(t',\rho',\theta',\varphi')$ are the coordinates for the Schwarzschild line element:

$$ds^2=(1-2m/\rho')dt'^2-(1-2m/\rho')^{-1}d\rho'^2-\rho'^2(d\theta'^2+\sin^2\theta' d\varphi'^2).$$

The equipotentials, which are the 2-spheres $\rho'=\text{const}$, become

$$r^2\rho'^{-1}(\rho'-2m)^{-1}+z^2(\rho'-m)^{-2}=1,$$

which are "ellipsoids" of revolution about the z axis, with semiaxes of coordinate length $(\rho'-m)$ and $\rho'(1-2m/\rho')^{1/2}$. As $\rho'>2m$, the coordinate distance along the axis of symmetry is always greater than the coordinate radius in $z=0$.

Since 2-spheres appear prolate in these coordinates, the boundary of the interior solution under consideration, which is a coordinate 2-sphere, must in fact be oblate.

This result is in agreement with a calculation of the first curvature of a line $\varphi=\text{const}$ on the coordinate 2-sphere. If

$$C^a=\frac{d^2x^a}{ds^2}+\Gamma_{bc}{}^a\frac{dx^b}{ds}\frac{dx^c}{ds},$$

then the first curvature is defined by

$$C=|C^aC_a|^{1/2}$$
$$=e^{u-k}\left|\frac{\partial u}{\partial\rho}-\frac{\partial k}{\partial\rho}-\frac{1}{\rho}\right|$$
$$=d^{-1}[1-2(a_0/d)+\tfrac{3}{2}(a_0/d)^2(1+\sin^2\theta)+O(a_0/d)^3].$$

Again, a calculation by Levy[6] of the timelike geodesics of the exterior metric, and consideration of the perihelion precession, indicates that dynamically the central body must be oblate, with quadrupole moment $+\frac{1}{3}a_0^3$.

The following table summarizes the estimates of the shape of the central body which have been made up to now, together with two additional estimates depending on the distribution of density and of the sum of eigenvalues:

Radar from center:	Prolate;
Spacelike distance:	Prolate;
Distribution of density:	Prolate;
Planetary motion:	Oblate;
Curvature of boundary:	Oblate;
Comparison with Schwarzschild's solution:	Oblate;
Distribution of the sum of eigenvalues:	Oblate.

These effects are all of the second order of smallness in (a_0/d) except the last one, which is of the third order.

Since the distribution of gravitational mass unequivocally determines the planetary perihelion motion, these contrasting results suggest that neither the timelike eigenvalue of T_{ab} nor the sum of the eigenvalues is as reliable an indicator of the *gravitational* mass density as has, up to now, sometimes been thought.

[6] H. Levy, doctoral dissertation, University of London, 1966 (unpublished).

Finally, the mass of the body, defined either by

$$M=\tfrac{1}{2}\kappa\int_0^d \rho^2 T_0{}^0 d\rho \quad \text{or} \quad M=\int_\Omega \lambda_0 d\Omega,$$

where $d\Omega$ is the proper volume element, has the value

$$M=d[(a_0/d)+O(a_0/d)^2].$$

4. GENERALIZATION

If other terms besides a_0 are nonzero, the eigenvectors (2) and (3) will not point in the r and z directions, respectively, but will be slightly displaced.

If $a_1\neq 0$, then its effect will be that of introducing a dipole moment, and the plane $z=0$ will no longer be a plane of symmetry for the body.

A physically interesting situation is that where only a_0 and a_2 are nonzero, and further $a_0{}^2d\sim a_2$. The effect of the a_2 is that of altering the quadrupole moment. From the calculation of planetary perihelion precession[6] it can be seen that, for the central body to be oblate, a_2 must be negative in sign. It is reasonable to expect, therefore, that by making a_2 sufficiently negative, one may change the result of the methods of investigation which indicate that the body is prolate and, at the same time, leave the conclusions of the other methods unchanged.

The radar radius along the axis of symmetry is

$$T=d[1+(8/3)(a_0/d)+(18/5)(a_0/d)^2+\tfrac{4}{3}(a_2/d^3)+O(a_0/d)^3],$$

and in the equatorial plane

$$T=d[1+(8/3)(a_0/d)+(33/10)(a_0/d)^2-\tfrac{2}{3}(a_2/d^3)+O(a_0/d)^3].$$

Thus, if $a_2<-(3/20)a_0{}^2d$, this method indicates that the body is oblate.

A similar conclusion is reached, if $a_2<-(3/10)a_0{}^2d$, when comparing the proper distance along the axis of symmetry with the proper distance along any radial line $z=0$, $\varphi=\text{const}$, $t=\text{const}$.

The comparison with Schwarzschild's solution does not depend on the sign of a_2, and the reasoning concluding that the body is oblate is the same as in Sec. 3. When the first curvature, however, is calculated for a line $\varphi=\text{const}$ on the 2-sphere $t=\text{const}$, $\rho=d$ the opposite conclusion is reached:

$$C=d^{-1}[1-2(a_0/d)+\tfrac{3}{2}(a_0/d)^2(1+\sin^2\theta)-4(a_2/d^3)P_2(\cos\theta)+O(a_0/d)^3].$$

If $a_2<-(1/4)a_0{}^2d$, the body appears prolate.

This is not as surprising a result as it may seem at first. The first curvature is that of the coordinate 2-sphere $t=\text{const}$, $\rho=d$; this 2-sphere is not an equipotential as in the Schwarzschild case, and thus need not represent the gravitational shape of the body.

The eigenvalue λ_0 corresponding to the density (1) is

$$\lambda_0\kappa=6a_0d^{-3}-3a_0{}^2d^{-4}[5-(\rho/d)^2]+70\rho^2d^{-7}a_2P_2(\cos\theta)-12a_0{}^2d^{-6}\rho^2\sin^2\theta+O(a_0/d)^3.$$

Then, if $a_2<-(4/35)a_0{}^2d$, the result of this method will change, and it will indicate that the body is oblate.

The last method considered, that of determining the distribution of the sum of eigenvalues of T_{ab}, becomes a second-order effect if $a_2\neq 0$, and suggests that the body is oblate if a_2 is negative.

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