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Method for Finding the General-Relativistic Effective Potential*

JONG K. JAEN
Physics Department, University of Florida, Gainesville, Florida
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A method of determining exactly the general-relativistic effective potential for a given nongravitational central potential is presented by a formalism based on the extended Hamilton–Jacobi theory of point-particle mechanics to the general-relativistic gravidynamics. The general result is applied to the Newton-gravitational, Coulomb-electrostatic, and Yukawa-nuclear potentials to give further insight. A numerical estimate is made for the gravitational effect superimposed on the original, nongravitational, central potential.

I. INTRODUCTION

In order to develop a simpler understanding of the general-relativistic behavior of a point particle in a nongravitational central potential field superposed on a spherically symmetric, static gravitational field which vanishes at infinity, a theory is developed for finding a corresponding general-relativistic effective potential. By "the general-relativistic effective potential" we mean the Newtonian-mechanical potential which gives exactly the same orbit as that found by the general theory of relativity. In other words, our main problem is to find the central potential $\phi_e(r)$ such that the spatial orbital equation $\mathbf{r}=\mathbf{r}(t)$ obtained by solving the general-relativistic equation of motion

$$m_0c^2\left(\frac{d^2x^\lambda}{ds^2}+\begin{Bmatrix}\lambda\\ \mu\quad\nu\end{Bmatrix}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right)=f^\lambda,\quad(\lambda=1,2,3,4)\quad(1)$$

for a given nongravitational 4-force f^λ with a central character,[1] is equivalent to that found by solving the corresponding Newtonian equation of motion given by

$$m_0\, d^2\mathbf{r}/dt^2=-\boldsymbol{\partial}\phi_e.\quad(2)$$

Here m_0 is the proper mass of the moving point-particle, c is the special-relativistic photon speed in vacuum, $\{\lambda,\mu\nu\}$ is the Christoffel three-index symbol in the coordinate system x^λ $(\lambda=1,2,3,4)$ defined by the invariant line element ds of the form

$$ds^2=g_{\lambda\mu}dx^\lambda dx^\mu\quad(3)$$

in the Riemannian space-time, and the summation convention is adopted here and henceforth. The first three coordinates (x^1,x^2,x^3) represent the spatial coordinates, and x^4 represents the pure-imaginary time-coordinate defined by $x^4=ict$. The coordinate time t is regarded as the Newtonian time in this theory. Thus, Eq. (3) reduces to the corresponding Lorentz form, $ds^2=dl^2-c^2dt^2$, with its spatial part dl^2 if the gravitational source vanishes. We call the space-time defined by this Lorentzian line element the coordinate space-time corresponding to Eq. (3). Then, $\mathbf{r}$ and $\boldsymbol{\partial}$ in Eq. (2) represent the spatial position vector of the moving point-particle and the spatial gradient operator in this coordinate space-time, respectively.

Our method is based on the Hamilton-Jacobi theory, extended to the general-relativistic gravidynamics by using the length s of the worldline as an extra parameter in addition to the four coordinates x^λ $(\lambda=1,2,3,4)$. The parameter s plays a similar role to that of the time parameter in the usual Hamilton-Jacobi theory. We can construct the gravidynamical Hamilton-Jacobi theory of a particle moving in a given nongravitational potential field ϕ in analogy to that of a charged point-particle moving in an electrostatic potential field superimposed on a gravitational field. The Hamiltonian H of the particle is found in the following form:

$$H\equiv p_\lambda\dot{x}^\lambda-L=g^{\lambda\mu}(p_\lambda-g_\lambda{}^4\phi)(p_\mu-g_\mu{}^4\phi)/2m_0c^2.\quad(4)$$

This is obtained by applying the Legendre transformation given by

$$p_\lambda\equiv\partial L/\partial\dot{x}^\lambda=m_0c^2g_{\lambda\mu}\dot{x}^\mu+g_\lambda{}^4\phi,\quad\text{or}\quad \dot{x}^\lambda=g^{\lambda\mu}(p_\mu-g_\mu{}^4\phi)/m_0c^2,\quad(5)$$

by using the Lagrangian L defined by

$$L\equiv m_0c^2g_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu/2+g_\lambda{}^4\dot{x}^\lambda\phi.\quad(6)$$

Here $\dot{x}^\lambda\equiv dx^\lambda/ds$, and we assume that the nongravitational central potential ϕ is given by the temporal component of a four-vector potential field $(0,0,0,i\phi)$ in the particular coordinate system under consideration. The Hamilton-Jacobi equation for the generation function W is

$$2m_0c^2\partial_0W+g^{\lambda\mu}(\partial_\lambda W-g_\lambda{}^4\phi)(\partial_\mu W-g_\mu{}^4\phi)=0,\quad(7)$$

where $\partial_0\equiv\partial/\partial s$ and $\partial_\lambda\equiv\partial/\partial x^\lambda$. As is well known, the gravidynamics of the moving particle is given by[2,3]

$$\begin{gathered}\partial W/\partial\alpha_\lambda=\beta^\lambda,\quad p_\lambda=\partial_\lambda W,\\ (\beta^\lambda=\text{const}),\quad(\lambda=1,2,3,4),\end{gathered}\quad(8)$$

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[1] For example, in an electromagnetic field, $f^\lambda=-ie(dx^\mu/ds)F_\mu{}^\lambda$, where $F_\mu{}^\lambda$ is the mixed component of the antisymmetric intensity-tensor of the electromagnetic field, and e is the charge of the moving particle.

[2] Y. S. Hagihara, *Foundations of Celestial Mechanics* (Kawaide Shobou, Tokyo, Japan, 1950), Vol. 1, p. 148.

[3] A somewhat similar formalism to ours is seen in L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 312.

constructed from the complete solution to Eq. (7) which is of the form

$$W \equiv W(s; x^1, x^2, x^3, x^4; \alpha_1, \alpha_2, \alpha_3, \alpha_4), \tag{9}$$

where $\det(\partial(\partial_\lambda W)/\partial\alpha_\mu) \neq 0$. It should be pointed out that the function W in Eq. (9) has the dimension of (energy$\times$length) and is pure-imaginary, since $ds^2 < 0$ along the worldline of the particle under consideration.

II. GENERAL-RELATIVISTIC EFFECTIVE POTENTIAL

Let us suppose that the given nongravitational potential field ϕ is static and spherically symmetric about the origin of a static and spherically symmetric coordinate system set up in a gravitational field, and that a test particle is moving under the influence of this superimposed field. Since the orbital plane of this test particle passes through the origin (the symmetric center), it is convenient to choose this orbital plane as the spatial coordinate plane on which we set up a two-dimensional polar-coordinate system (r,θ). Then, the gravidynamics of the test particle can be described by a three-dimensional line element of the standard form[4] given by

$$ds^2 = dr^2/g(r) + r^2 d\theta^2 - f(r)c^2 dt^2, \tag{10}$$

where $g(r) > 0$ and $f(r) > 0$ for all values of r. Thus, Eq. (7) takes the simpler form

$$\frac{\partial W}{\partial s} + \frac{1}{2m_0c^2}\left[g(r)\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{f(r)}\left(\frac{\partial W}{\partial x^4} - \phi(r)\right)^2\right] = 0. \tag{11}$$

In Eq. (11), the coefficients $g(r)$ and $f(r)^{-1}$ produce the general-relativistic effect for the motion of the test particle under consideration. By the general procedure for solving the Hamilton-Jacobi equation, we look for the complete solution W in the separated form

$$W = Ex^4 + \alpha s + i\alpha'\theta + iR(r). \tag{12}$$

The three constants E, α, and α' corresponding to α_λ in Eq. (9), and the one variable function $R(r)$ must be real in accordance with the pure-imaginary character of the generation function W.

The substitution of Eq. (12) into Eq. (11) gives

$$(dR/dr)^2 = 2m_0c^2\alpha/g(r) + \{E - \phi(r)\}^2/f(r)g(r) - \alpha'^2/r^2 g(r), \tag{13}$$

and therefore Eq. (12) is found in the form

$$W = Ex^4 + \alpha s + i\alpha'\theta \pm i\int dr\left[\frac{2m_0c^2\alpha}{g(r)} + \frac{\{E-\phi(r)\}^2}{f(r)g(r)} - \frac{\alpha'^2}{r^2 g(r)}\right]^{1/2}. \tag{14}$$

Equation (14) is the complete solution of Eq. (11) corresponding to Eq. (9). The orbital equation of the test particle is found from one of the first set of equations in Eq. (8). It is as follows:

$$\beta = \partial W/\partial\alpha' = i\theta \pm i\alpha' \int dr\{r^2 g(r)\}^{-1} \times [2m_0c^2\alpha/g(r) + \{E-\phi(r)\}^2/f(r)g(r) - \alpha'^2/r^2 g(r)]^{-1/2},$$

or, by putting $u = 1/r$ and choosing the coordinate axes so that $\beta = 0$,

$$\theta = \pm\alpha' \int \frac{du}{g(u^{-1})}\left[\frac{2m_0c^2\alpha}{g(u^{-1})} + \frac{\{E-\phi(u^{-1})\}^2}{g(u^{-1})f(u^{-1})} - \frac{\alpha'^2 u^2}{g(u^{-1})}\right]^{-1/2}. \tag{15}$$

The dimension of each of the three dynamical constants α, α', and E is easily obtained by comparing the dimensions on both sides of Eq. (15), or Eq. (12) and the dimension of the function W. For example, the constant E has the dimension of energy. For our later use of the constant E, we shall find its physical meaning by the aid of the second set of Eqs. (8) and (5). We substitute Eq. (12) into the second set of Eq. (8) with $\lambda = 4$, and refer to the first of Eq. (5). We then have

$$E = \partial_4 W = p_4 = m_0c^2 f(r)\dot{x}^4 + \phi(r),$$

or, by eliminating $\dot{x}^4$ $(\equiv dx^4/ds)$ with Eq. (10) and defining $\dot{r} \equiv dr/dt$ and $\dot{\theta} \equiv d\theta/dt$,

$$E = m_0c^2 f(r)(f(r) - \{\dot{r}^2/g(r) + r^2\dot{\theta}^2\}/c^2)^{-1/2} + \phi(r). \tag{16}$$

Since the right-hand side of Eq. (16) expresses the special-relativistic total mechanical energy plus the proper mass energy of the particle if the gravitation vanishes, i.e., $f(r) = g(r) = 1$, it can be interpreted as the general-relativistic total mechanical energy plus the proper mass energy of the particle. This total energy is conserved on the orbit, since E in the left-hand side of Eq. (16) was introduced originally as a constant. By this constant of the motion, we may define a conserved quantity m characteristic to the orbit by

$$E = mc^2. \tag{17}$$

Hereafter, we call this newly defined quantity m the general-relativistic orbital mass of the particle under consideration. In analogy to the velocity-dependent mass of a particle in the special theory of relativity, we may define another mass m' by

$$m' \equiv m_0 f(r)[f(r) - \{\dot{r}^2/g(r) + r^2\dot{\theta}^2\}/c^2]^{-1/2} \tag{18}$$

[4] C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1955), p. 323, Eq. (63).

by using the first term of Eq. (16). This mass m' varies with both the velocity and the position of the particle in the gravitational field. In particular, the rest mass m_0' in the gravitational field is given by

$$m_0' = m_0[f(r)]^{1/2}, \tag{19}$$

and it depends upon the position of the particle. Thus, the gravitational potential energy $\phi'(r)$ of a particle at rest is

$$\phi'(r) = m_0c^2\{[f(r)]^{1/2} - 1\}. \tag{20}$$

The combination of Eqs. (16), (17), and (18) yields a relationship between two masses m and m', given by

$$m = m' + \phi(r)/c^2. \tag{21}$$

All the above results are consistent with those derivable from another approach.[5] Classically, the total energy E is a continuous function of the geometrical structure of the orbit. However, according to the postulate of the old quantum theory, the only orbits which can actually be realized in nature are those satisfying the Bohr-Sommerfeld quantum condition.[6] We will use this concept of the quantized orbit in the subsequent discussion.

We are now in a position to find the general-relativistic effective potential $\phi_e(r)$ of Eq. (2) by using the general-relativistic orbital equation given by Eq. (15), which is the orbital solution of Eq. (1) obtained indirectly through the extended Hamilton-Jacobi theory. As is well known in Newtonian mechanics, the relationship between the central potential $\phi_e(r)$ and the orbital equation is given by[7]

$$\frac{d\phi_e}{du} = -\frac{l(l+1)\hbar^2}{m}\left[u - \frac{d^2\theta}{du^2}\bigg/\left(\frac{d\theta}{du}\right)^3\right], \tag{22}$$

where $l(l+1)\hbar^2$ is the square of the magnitude of the angular momentum on the quantized orbit. Furthermore, in Eq. (22) the orbital mass m, invariant on the orbit, is taken instead of the proper mass m_0, since it is more reasonable to take the orbital mass m in place of the proper mass m_0 on the orbit in Eq. (2). Note also that the introduction of the quantization rule is not in contradiction with the fundamental postulate of Newtonian classical mechanics at all.[6]

The substitution of Eq. (15) into Eq. (22) gives, after a simple integration,

$$\phi_e(r) = -\frac{l(l+1)\hbar^2}{2m\alpha'^2}\frac{g(r)}{f(r)}\{E - \phi(r)\}^2 - \frac{m_0}{m}\frac{l(l+1)\hbar^2c^2\alpha}{\alpha'^2}g(r) - \frac{l(l+1)\hbar^2}{2m}\frac{1-g(r)}{r^2}. \tag{23}$$

[5] C. Møller, *Selected Problems in General Relativity Brandeis Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1960), p. 59.

[6] M. Born, *The Mechanics of the Atom* (Frederick Ungar Publishing Company, New York, 1960), p. 99.

[7] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), p. 72, Eq. (3-34).

In order to determine the two constants α and α' in terms of the remaining constants, we now impose two boundary conditions upon Eq. (23): (a) both $g(r)$ and $f(r)$ approach unity as $r \to \infty$, and (b) $\phi_e(r)$ approaches the special-relativistic effective potential[8] $\phi(r) - \phi(r)^2/2mc^2$ as $r \to \infty$. By noting that the last term of Eq. (23) goes to zero and the first term produces the special-relativistic potential under the use of conditions (a) and (b), we obtain the final form of the exact general-relativistic effective potential given by

$$\phi_e(r) = \frac{g(r)}{f(r)}\left[\phi(r) - \frac{\phi(r)^2}{2mc^2}\right] + \tfrac{1}{2}mc^2\left[g(r) - \frac{g(r)}{f(r)}\right] + \frac{l(l+1)\hbar^2}{2m}\frac{g(r)-1}{r^2}. \tag{24}$$

Note that the original potential $\phi(r)$ must be represented by the functional form peculiar to the coordinate system given by the line element of Eq. (10). The first term is the special-relativistic effective potential coupled with the Riemannian space-time structure of the field. The second term, dependent upon the orbital mass, and the third term, dependent upon both the orbital mass and angular momentum, represent the pure gravitational effect on the test point-particle under consideration.

There is another kind of the effective potential, i.e., the special-relativistic mechanical potential $V(r)$ which gives exactly the same orbit as the general-relativistic mechanical orbit of the test particle. It is found by equating the special-relativistic effective potential $V(r) - V(r)^2/2mc^2$ to the general-relativistic effective potential $\phi_e(r)$. That is,

$$V - V^2/2mc^2 = \phi_e.$$

This is an algebraic, quadratic equation with V unknown. Thus, V is given by the root

$$V(r) = mc^2[1 - (1 - 2\phi_e/mc^2)^{1/2}], \tag{25}$$

where $\phi_e(r)$ is replaced by Eq. (24). The remaining root is discarded because it does not give the result $V = \phi$ when $g = f = 1$ in Eq. (24). We should note here that $V(r)$ exists only in the region where the general-relativistic effective potential ϕ_e satisfies the condition

$$1 - 2\phi_e/mc^2 \geqslant 0.$$

That is, $\phi_e(r)$ is no larger than one half the total energy, including the proper mass energy, of the moving particle. The general-relativistic mechanics reduces to the special-relativistic mechanics by using the potential $V(r)$ of Eq. (25) in the central-potential problem.

[8] Jong K. Jaen, Nuovo Cimento, Suppl. (to be published).

III. APPLICATION OF THE GENERAL RESULT

We apply the general result of Eq. (24) to the three familiar cases of the pure-gravitational, Coulomb-electrostatic, and Yukawa-nuclear potential fields.

A. Pure-Gravitational Field

In this case, Eq. (24) reduces to the simpler form

$$\phi_e(r)=\tfrac{1}{2}mc^2\left[g(r)-\frac{g(r)}{f(r)}\right]+\frac{l(l+1)\hbar^2}{2m}\frac{g(r)-1}{r^2}. \quad (26)$$

We use the Schwarzschild gravitational field (exterior) given by[9]

$$g(r)=1-2R_0/r+\tfrac{1}{3}\lambda r^2=f(r), \quad (27)$$

where $2R_0$ is the gravitational radius of the spherical mass fixed at the symmetric center (the origin of the coordinate system), and λ is the cosmological constant. Substitution of Eq. (27) into Eq. (26) gives

$$\phi_e(r)=-\frac{mc^2R_0}{r}-\frac{l(l+1)\hbar^2R_0}{m}\frac{1}{r^3}+\frac{\lambda mc^2}{6}r^2 \quad (28)$$

by dropping off the additional constant produced by the cosmological term.[10] The first term represents the Newtonian gravitation, and the second term represents the general-relativistic correction. The last term is the cosmological correction which is negligible for terrestrial phenomena. Unfortunately, this effective potential of Eq. (28) does not contain the special-relativistic correction term $-\frac{1}{2}mc^2(R_0/r)^2$ of the Newtonian gravitational potential.

Now let us consider the Yilmaz gravitational field given by[11]

$$g(r)=\exp(-2R_0/r)=f(r). \quad (29)$$

This gives the following gravitational law:

$$\begin{aligned}\phi_e(r)&=-\tfrac{1}{2}mc^2(1-e^{-2R_0/r})\left[1+\frac{l(l+1)\hbar^2}{m^2c^2}\frac{1}{r^2}\right]\\&=-\frac{mc^2R_0}{r}+\frac{mc^2R_0{}^2}{r^2}\\&\qquad-\left\{\frac{l(l+1)\hbar^2R_0}{m}+\tfrac{2}{3}mc^2R_0{}^3\right\}\frac{1}{r^3}+\cdots. \quad (30)\end{aligned}$$

The first two terms are in agreement with the special-relativistic effective potential of the Newtonian gravitational potential, apart from the lack of the numerical coefficient $\frac{1}{2}$ and the opposite sign in the second term. Using the line element given by Schild,[12] we can obtain a gravitational effective potential which also contains the special-relativistic correction terms stated above.

B. Coulomb-Electrostatic Potential Field

Next, we consider the effective potential between two charged particles of charges (q,e) and mass (M,m), respectively, and assume that the particle of charge q and mass M is at the origin of the coordinate system given by the line element of Eq. (10). According to Weyl[13] and Reissner,[14] in this coordinate system the electrostatic potential $\phi(r)$ takes exactly the Coulomb form given by

$$\phi(r)=qe/r, \quad (31)$$

and the fundamental metric tensor is given by

$$g(r)=1-2R_0/r+kq^2/r^2=f(r), \quad (32)$$

where $2R_0$ is the gravitational radius of the particle M, and $k\equiv 8\pi G/c^4\approx 2.073\times 10^{-48}$ cm^{-1}g^{-1} sec^2 with the Newtonian gravitational constant G. The electrostatic energy term kq^2/r^2 is negligible at large relative distances, but is quite dominant inside of the spherical region of radius $4\pi q^2/Mc^2$, compared to the mass energy term $2R_0/r$.

The substitution of Eqs. (32) and (31) into Eq. (24) gives exactly the effective potential

$$\phi_e(r)=\frac{\sigma_1}{r}-\frac{\sigma_2}{r^2}-\frac{\sigma_3}{r^3}+\frac{\sigma_4}{r^4}, \quad (33)$$

where the four constants σ_λ ($\lambda=1, 2, 3, 4$) may be expressed in terms of the classical electrostatic radius $r_0\equiv e^2/mc^2$ and the first Bohr radius $a\equiv\hbar^2/me^2$ of the moving test particle. They are as follows:

$$\begin{aligned}&\sigma_1\equiv qe(1-eR_0/qr_0), \quad \sigma_2\equiv\tfrac{1}{2}q^2r_0(1-ke^2/r_0{}^2),\\&\sigma_3\equiv l(l+1)e^2aR_0, \qquad \sigma_4\equiv\tfrac{1}{2}kl(l+1)e^2q^2a.\end{aligned} \quad (34)$$

In Eq. (33), the σ_1 term is the original Coulomb potential with a small parameter correction due to the general-relativistic effect, and the σ_2 term agrees with the special-relativistic modification, apart from the small parametric correction. This term is always attractive since $\sigma_2>0$. The σ_3 and σ_4 terms are entirely a result of the general-relativistic influence on the orbital angular momentum of the moving particle. These two terms vanish simultaneously for the ground state of the two-particle system under consideration. If the orbital angular momentum exists, the σ_3 term is always attractive, and the σ_4 term is always repulsive. Thus, we see that the relativistic effect is Newtonian-classically equivalent to adding two attractive and one repulsive potentials to the original potential with a small parametric correction in this central-potential

[9] K. Schwarzschild, Sitzber. Berlin Akad. Wiss. 424 (1916).

[10] This result agrees with that obtained by another method. See A. Trautman, *Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965), p. 155, Eq. (6-96).

[11] H. Yilmaz, Phys. Rev. **111**, 1417 (1958).

[12] A. Schild, Am. J. Phys. **28**, 778 (1960).

[13] H. Weyl, Ann. Physik **54**, 117 (1917).

[14] H. Reissner, Ann. Physik **50**, 106 (1916).

problem. In particular, the presence of the repulsive term σ_4/r^4 proves that no two charged particles with a relative angular momentum can adhere.

In the two-electron system, the potential of Eq. (33) has a potential hill of height $\phi_e(r_0)\approx\frac{1}{2}mc^2(1-2R_0/r_0+ke^2/r_0^2)\approx 0.256$ MeV at a distance approximately equal to the classical electron radius $r_0\equiv e^2/mc^2\approx 2.82\times 10^{-5}$ Å. There is a tremendously deep potential well of depth $\approx -5.6\times 10^{35}/l(l+1)$ MeV in the vicinity of the distance $\approx 1.34[l(l+1)]^{1/2}\times 10^{-23}$ Å if $l\neq 0$. It is instructive to note that the radius of the potential well is much larger than the electron gravitational radius, 1.2×10^{-47} Å, which is the general-relativistic limit of an electron size. This unusual potential well is produced by the general-relativistic gravitational effect due to the electrostatic energy distribution of the electron at the origin. As is shown in Eq. (33), the general-relativistic gravitational effect in the two-electron system is almost negligible in the region at or beyond the distance of the classical electron radius r_0 and is very dominant within the spherical region of the radius r_0 if there exists a relative angular momentum between the two electrons. This effect shows again that the tunneling effect allows the possibility of a bound state between two electrons even when they are initially in a repulsive state, just as in the case of the special-relativistic effective potential. Would we not explain even more the present belief in the stable electron crystal[15] by a nonrelativistic quantum-mechanical treatment[16] with the use of the effective potential given by Eq. (33)?

C. Yukawa-Nuclear Potential Field

The symmetric stress-energy density $T^{\lambda\mu}$ of the Yukawa-nuclear potential field $\phi(r)$ [$\equiv\xi\Psi(r)$, where ξ is a coupling parameter] which satisfies the law of conservation

$$T^{\lambda\mu}{}_{;\mu}=0 \tag{35}$$

is given by

$$T^{\lambda\mu}\equiv g^{\lambda\sigma}g^{\mu\omega}\Psi_{;\sigma}\Psi_{;\omega}-\tfrac{1}{2}g^{\lambda\mu}(g^{\sigma\omega}\Psi_{;\sigma}\Psi_{;\omega}+a^2\Psi^2)\,, \tag{36}$$

where $a\equiv m_0c/h$, m_0 is the rest mass of π^0 meson, and the semicolon represents absolute differentiation.

Following the discussion of Yilmaz[11] and DeWitt,[17] we may substitute Eq. (36) into Einstein's field equation [e.g., Eq. (16.60) in Ref. 16]. By taking a nucleon at the origin, we may set up a spherical coordinate system with the spherically symmetric, isotropic line element

$$ds^2=-e^{\alpha(r)}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+e^{\beta(r)}c^2dt^2. \tag{37}$$

In this coordinate system, Einstein's field equation yields the following three simultaneous differential equations:

$$\tfrac{1}{2}\beta''+\tfrac{1}{4}\beta'(\beta'-\alpha')-\alpha'/r=-k\Psi'^2-\tfrac{1}{2}ka^2\Psi^2e^{\alpha}, \tag{38}$$

$$\tfrac{1}{2}\beta''+\tfrac{1}{4}\beta'(\beta'-\alpha')+\beta'/r=-\tfrac{1}{2}ka^2\Psi^2e^{\alpha}, \tag{39}$$

$$\frac{1}{r^2}+\frac{\beta'-\alpha'}{r}-\frac{1}{r^2}e^{\alpha}=-\tfrac{1}{2}ka^2\Psi^2e^{\alpha}, \tag{40}$$

where a prime denotes d/dr and a double prime denotes d^2/dr^2. Any one of these three equations can be derived from the remaining two equations.[18] The conservation law given by Eq. (35) yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)+\frac{\beta'-\alpha'}{2}\frac{d\Psi}{dr}=a^2\Psi e^{\alpha}. \tag{41}$$

The three unknown functions $\alpha(r)$, $\beta(r)$, and $\Psi(r)$ are determined by the three independent equations, i.e., Eq. (41) and any two equations among Eqs. (38), (39), and (40). Equation (41) differs from the Klein-Gordon equation with the original Yukawa potential in the two terms $\frac{1}{2}(\beta'-\alpha')\Psi'$ and $a^2\Psi[\exp(\alpha)-1]$. Thus, the unknown functions $\alpha(r)$ and $\beta(r)$ are found to be

$$e^{-\alpha}=r^{-1}\exp\left(-k\int dr\, r\Psi'^2/2\right)\times\left[\int dr(1-ka^2r^2\Psi^2/2)\times\exp\left(k\int dr\, r\Psi'^2/2\right)-2R_0\right], \tag{42}$$

$$e^{-\beta}=r^{-1}\exp\left(k\int dr\, r\Psi'^2/2\right)\times\left[\int dr(1-ka^2r^2\Psi^2/2)\times\exp\left(k\int dr\, r\Psi'^2/2\right)-2R_0\right]. \tag{43}$$

Here, we have taken the gravitational radius $2R_0$ of the nucleon as the constant of integration, in order that these solutions reduce to Schwarzschild's exterior solution when Ψ goes to zero. Using Eq. (43), Eq. (41) takes an alternative form given by

$$\left(\frac{e^{-\alpha}}{r^2}\right)\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)-\left(\frac{e^{-\alpha}}{r}-\frac{1}{r}+\tfrac{1}{2}ka^2r\Psi^2\right)\frac{d\Psi}{dr}=a^2\Psi\,, \tag{44}$$

where $\exp(-\alpha)$ is replaced by Eq. (42). Equation (44) is an integrodifferential equation from which the modified Yukawa-nuclear potential may be determined. The concrete function forms of the two components of the fundamental metric tensor given by Eqs. (42) and

[15] W. J. Carr, Jr., Phys. Rev. **122**, 1437 (1961).

[16] The relativistic quantum-mechanical treatment would be possible by using the effective potential given by Eq. (25).

[17] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach Science Publishers, Inc., New York, 1965), p. 137, Eq. (16.62).

[18] See, e.g., Ref. 4, p. 324.

(43) are found by substituting the solution of this integrodifferential equation into them.

Now we are in a position to determine the general-relativistic effective potential $\phi_e(r)$ of the modified Yukawa-nuclear potential $\phi(r)$ by using Eq. (24). Although the line element of Eq. (37) is defined with the sign opposite to the line element fo Eq. (10), the potential $\phi_e(r)$ is still found by taking $g(r)=\exp(-\alpha)$ and $f(r)=\exp(\beta)$ into Eq. (24). It is as follows:

$$\phi_e(r)=[\phi(r)-\phi(r)^2/2mc^2]\exp\left(-k\int dr\, r\Psi'^2\right)$$
$$+\tfrac{1}{2}mc^2\left[\exp(-\alpha)-\exp\left(-k\int dr\, r\Psi'^2\right)\right]$$
$$+\{l(l+1)\hbar^2/2m\}[\exp(-\alpha)-1]/r^2, \quad (45)$$

where $\phi(r)\equiv\xi\Psi(r)$, and $\exp(-\alpha)$ is replaced by Eq. (42). The last two terms represent the gravitational potential caused by the mass energy and nuclear potential source energy distribution of the nucleon. The first term is the nuclear potential coupled with the metric structure of the Riemannian space-time around the nucleon.

Next we are interested in finding the first-order approximate solution (with respect to the small physical constant k) to the integrodifferential equation given by Eq. (44). We expand the exact solution $\Psi(k,r)$ and $\alpha(k,r)$ in the following power series:

$$\Psi=\sum_{j=0}^{\infty}k^j\Psi_j(r), \quad \alpha=\sum_{j=1}^{\infty}k^j\alpha_j(r), \quad (46)$$

where $\Psi_j\equiv(\partial^j\Psi/\partial k^j)_{k=0}/(j!)$, $\alpha_j\equiv(\partial^j\alpha/\partial k^j)_{k=0}/(j!)$, and we have taken $\alpha_0=0$ in order to make the line element of Eq. (37) reducible to that of pseudo-Euclidean space-time if $k=0$. We substitute both expressions of Eq. (46) into Eq. (44) and compare the coefficients of powers in k. This gives as the zero- and first-order approximations,

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\Psi_0}{dx}\right)=\Psi_0, \quad (47)$$

and

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\Psi_1}{dx}\right)-\Psi_1=\alpha_1\Psi_0-\left(\frac{\alpha_1}{x}-\tfrac{1}{2}x\Psi_0^2\right)\frac{d\Psi_0}{dx}, \quad (48)$$

where $x\equiv ar$. The physically meaningful solution to Eq. (47) is given by the original Yukawa-nuclear potential

$$\Psi_0=-\eta e^{-x}/x, \quad (49)$$

where η is the coupling constant for which the numerical value is given later. In order to find the function α_1, we follow a procedure similar to the above by substituting both expressions of Eq. (46) into Eq. (42) and calculate the concrete form of α_1 by using Eq. (49). We then obtain[19]

$$\alpha_1=(2aR_0/kx)-\tfrac{1}{2}\eta(x^{-1}+x^{-2})\exp(-2x). \quad (50)$$

Finally, the concrete form of the function Ψ_1 is found simply from the linear differential equation

$$\frac{d^2}{dx^2}(x\Psi_1)-x\Psi_1=-2\eta aR_0k^{-1}(x^{-1}+x^{-2}+x^{-3})\exp(-x)$$
$$+\tfrac{1}{2}\eta^3(2x^{-1}+3x^{-2}+2x^{-3}+x^{-4})\exp(-3x), \quad (51)$$

obtained by substituting Eqs. (49) and (50) into Eq. (48).

In this way, we find to a first-order approximation the following modified Yukawa-nuclear potential in the coordinate system given by the line element of Eq. (37):

$$\begin{aligned}\phi&\equiv\xi\Psi\approx\xi(\Psi_0+k\Psi_1)\\&=-\epsilon x^{-1}e^{-x}[1-aR_0\{\ln x+e^{2x}\,\mathrm{Ei}(2x)-x^{-1}\}\\&\quad-\tfrac{1}{12}k\eta^2\{x^{-2}-4e^{2x}\,\mathrm{Ei}(2x)+14e^{4x}\,\mathrm{Ei}(4x)\}e^{-2x}], \quad (52)\end{aligned}$$

where $\epsilon\equiv\xi\eta$, and $\mathrm{Ei}(z)$ is the exponential-integral function defined by

$$\mathrm{Ei}(z)\equiv\textstyle\int_z^\infty dy\, y^{-1}\exp(-y).$$

The constant η^2 is given by[20] $\eta^2\equiv\epsilon a/4\pi\approx7.25\times10^7$ g cm sec^{-2} for a nucleon. The two scalar constants aR_0 and $k\eta^2$ are dimensionless.

Let us study the asymptotic behavior of the potential in Eq. (52) at large distances $x\sim\infty$ and in the vicinity of the origin $x\sim0$. The asymptotic expression at very large distances is easily found to be

$$\phi\approx-\epsilon x^{-1}e^{-x}(1-aR_0\ln x). \quad (53)$$

This is obtained by referring to the asymptotic series[21] of $\mathrm{Ei}(z)$ and dropping all terms much smaller than $x^{-1}\exp(-x)$. Since numerically $aR_0\approx0.93\times10^{-39}$, for a nucleon, the second term in Eq. (53) can certainly be neglected in terrestrial phenomena. This substantiates the validity of Yukawa-nuclear potential at large distances of terrestrial size. This approximate potential becomes repulsive beyond the boundary radius of attraction given by $x\approx\exp(a^{-1}R_0^{-1})$, or $r\approx10^w$ cm for a nucleon where $w=4.65\times10^{38}$. This numerical value of the boundary radius is almost infinite in comparison with the radius of the universe arrived at in modern cosmology.

The asymptotic behavior of the potential very near the origin can be investigated by the approximate expression[22] $\mathrm{Ei}(z)\approx-\gamma-\ln z$ for a small value of z,

[19] The constant k is not actually contained in α_1, since k is also contained in R_0 to make a cancelation with the k in the denominator.

[20] G. Wentzel, *Quantum Theory of Fields*, translated by J. M. Jauch (Interscience Publishers, Inc., New York, 1949), p. 37.

[21] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, Inc., New York, 1965), p. 231, Eq. (5.1.56).

[22] Reference 21, p. 229, Eq. (5.1.11).

where $\gamma=0.5772\cdots$ is Euler's constant. Since numerically $k\eta^2\approx 1.503\times 10^{-40}$ for a nucleon, in the neighborhood of $x\approx 10^{-20}$ we have, to a very good approximation.

$$\phi\approx -\epsilon x^{-1}e^{-x}(1+aR_0x^{-1}-\tfrac{1}{12}k\eta^2x^{-2}e^{-2x}). \quad (54)$$

In this approximate potential, the second term represents the correction due to the mass energy, and the third term is the modification due to the nuclear potential energy source of the nucleon. The ratio $12aR_0x^{-1}/k\eta^2x^{-2}e^{-2x}$ of the two terms is of the order 10^{-19} in this region. Thus, the correction term due to the mass energy is certainly negligible in this region, so that we have the simpler form

$$\phi\approx -\epsilon x^{-1}e^{-x}(1-\tfrac{1}{12}k\eta^2x^{-2}e^{-2x}). \quad (55)$$

The potential of Eq. (55) has a tremendous depth[23] $\phi_{\min}\approx -3.14\times 10^{21}$ MeV at $x_{\min}\approx 1.22\times 10^{-20}$, or $r_{\min}\approx 1.44\times 10^{-33}$ cm. It becomes $\phi=0$ at $r\approx 8.31\times 10^{-34}$ cm,[24] and $\phi=\infty$ at $r=0$ (the pole of third order). Beyond the boundary radius $r_{\min}$ of the repulsive force, the potential of Eq. (52) rapidly approaches the Yukawa-nuclear potential $-\epsilon x^{-1}e^{-x}$, since the correction terms are short-range potentials. Therefore, the general-relativistic influence on the Yukawa-nuclear potential is effective only within the short range with a radius of the order 10^{-33} cm.

Finally, we may find the general-relativistic effective nuclear potential to first order by substituting the first-order approximate solution of Eq. (52) into Eqs. (42) and (45).[25] The form of this approximate effective potential is too complicated to be handled analytically. However, a good approximate form in the region $(k\eta^2)^{1/2}(\approx 10^{-20})\ll x\ll 1$ is easily found by using the approximate expression of $\mathrm{Ei}(z)$ for a small value of z. The result is

$$\phi_e\approx\phi_0\left[1-\frac{\phi_0}{2mc^2}\left(1+\frac{k\eta^2m^2c^4}{2\epsilon^2}\right)-\frac{k^2\eta^4l(l+1)h^2}{16m\epsilon^4}\phi_0^3\right]\times\exp(k\eta^2\phi_0^2/2\epsilon^2), \quad (56)$$

where $\phi_0\equiv -\epsilon x^{-1}e^{-x}$. Among the four terms of Eq. (56), the first two terms are the special-relativistic effective potential of the Yukawa-nuclear potential coupled with the Riemannian space-time structure around the nucleon at $x=0$, and the third and fourth terms are the general-relativistic gravitational potentials due to the nuclear-potential source. It is interesting to note that the gravitational action due to the mass is much smaller than that due to the nuclear-potential source of the nucleon in this region. Therefore, the mass-gravitational term has disappeared from Eq. (56). Since $k\eta^2m^2c^4/2\epsilon^2\approx 10^{-38}$ and the last gravitational term is approximately given by $10^{-107}l(l+1)x^{-3}\exp(-3x)$, the two gravitational terms are certainly negligible. Thus, Eq. (56) can be written

$$\phi_e\approx(\phi_0-\phi_0^2/2mc^2)\exp(k\eta^2\phi_0^2/2\epsilon^2). \quad (57)$$

In this potential the gravitational effect is exhibited in the exponential function. Since the exponent of the exponential function has a short-range character and is approximately $0.751\times 10^{-40}x^{-2}\exp(-2x)$, it is important in the region near $x\approx 10^{-20}$ but approaches zero rapidly beyond $x\approx 10^{-20}$. For example, for $x\approx 10^{-15}$ its value is less than 10^{-10}. Therefore, in the region much beyond $x\approx x^{-20}$, Eq. (57) reduces to the special-relativistic effective potential given by

$$\phi_e\approx -\epsilon x^{-1}e^{-x}[1+(\epsilon/2mc^2)x^{-1}e^{-x}]. \quad (58)$$

Since $\epsilon/2mc^2\approx 3.6\times 10^{-2}$, the second correction term in Eq. (58) is significant in the nuclear-potential problem. For example, the ratio of the suggested potential ϕ_e to the original Yukawa-nuclear potential, i.e., $\phi_e/[-\epsilon x^{-1}\exp(-x)]$ at $x=0.1$ (the neighborhood of the effective distance of the nuclear force) is about 1.326, which represents an increase of 32.6%.

As we have seen, the general-relativistic gravitational effect is quite negligible compared with the nongravitational effect of the Coulomb-electrostatic and Yukawa-nuclear potentials at intermediate distances, but quite dominant in the vicinity very near the potential source. It is possible that we have overlooked this important general-relativistic effect in describing the structure of matter in the microscopic world. Perhaps new insight into the microscopic structure of matter could be gained if the general-relativistic effective potential were used in the quantum-mechanical theory of matter.[26]

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[23] This depth is $\frac{2}{3}$ the numerical value of the original Yukawa-nuclear potential at the same distance.

[24] This value is much larger than the gravitational radius $2R_0\approx 2.2\times 10^{-52}$ cm and much smaller that the Compton wavelength 1.32×10^{-13} cm of a nucleon.

[25] The higher (than first) order approximate solution can be found by an iteration method based on the same idea applied to the first-order approximation.

[26] There is a viewpoint from which one explains the nuclear force as a gravitational force. See J. M. Barnothy and M. F. Barnothy, Bull. Am. Phys. Soc. **12**, 420 (1967).