Use of the K Matrix in Nuclear Reaction Theories*

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Two recent formulations of nuclear reaction theory have been based on the reactance operator K. Expressions given in these formulations for the reactance operator are identical in form with those that hold for the transition operator T, except that the scattering Green's function is replaced by the standing-wave Green's function. We have used formal scattering theory to find the formal solutions to the Lippmann-Schwinger equations for K and T. We find that the solution for K differs from that used in these two formulations. The discrepancy appears to be significant for one of the formulations but not for the other.

I. INTRODUCTION

RECENTLY, Bloch and Gillet[1] (BG) have proposed a formulation of nuclear reaction theory based on the reactance operator K. An alternative formulation of nuclear reaction theory in terms of K has been suggested by MacDonald and Mekjian[2] (MM). In both of these treatments, the expressions given for K are identical in form with corresponding expressions which hold for the transition operator T when the standing-wave Green's function is replaced by the scattering Green's function. By using formal scattering theory, suitably generalized for nuclear reactions, we will show that the relationship between K and T is somewhat more complicated than this. We apply our results to the simple solvable case of scattering by a separable potential. It appears that the discrepancy we have found is significant for the BG treatment but not for the MM treatment.

In Sec. II, we define the reactance operator K and transition operator T as solutions of their corresponding Lippmann-Schwinger equations. Formal solutions to these equations are derived.

In Sec. III a generalized Heitler equation is derived. This is used to provide a direct relationship between K and T. It is also used to provide formal solutions for K and T in terms of reduced reactance and transition operators $\hat{K}$ and $\hat{T}$, respectively.

Section IV is devoted to a comparison of our formal solutions for K with those used by BG and MM. Our conclusions are illustrated by applying the formalism to the simple solvable case of scattering by a separable potential.

II. FORMAL SOLUTION IN TERMS OF THE INTERACTION

The Hamiltonian H for a given system of nucleons can be decomposed into two parts

$$H=H_\alpha+V_\alpha=H_\beta+V_\beta+\cdots \tag{1}$$

in a different way for each channel available to the system. The channel Hamiltonian H_α is the kinetic energy and the internal interactions for the nuclides that constitute channel α while the channel interaction V_α is the sum of the mutual interactions of these nuclides. Because of the necessity of dealing with distinct channels we are forced to generalize somewhat the standard formulation of formal scattering theory[3] which applies only to elastic and inelastic scattering.

We first define the scattering Green's function

$$G_\alpha(E)=(E-H_\alpha+i\epsilon)^{-1} \tag{2}$$

and the standing-wave Green's function

$$\Gamma_\alpha(E)=(E-H_\alpha)[(E-H_\alpha)^2+\epsilon^2]^{-1}. \tag{3}$$

E is the energy of the system and ϵ is a small positive constant which will be allowed to approach zero in the final expressions. The difference between these two Green's functions is just

$$\Gamma_\alpha(E)-G_\alpha(E)=i\pi\Delta_\alpha(E), \tag{4a}$$

$$\Delta_\alpha(E)=\epsilon\pi^{-1}[(E-H_\alpha)^2+\epsilon^2]^{-1}. \tag{4b}$$

We note that

$$\lim_{\epsilon\to0}\Gamma_\alpha(E)=\text{Princ. part}\frac{1}{E-H_\alpha}, \tag{5a}$$

$$\lim_{\epsilon\to0}\Delta_\alpha(E)=\delta(E-H_\alpha). \tag{5b}$$

The transition operator T and the reactance operator K will now be defined as solutions of the Lippmann-Schwinger (LS) integral equation.[4]

$$T_{\alpha\beta}^{(+)}(E)=V_\alpha+V_\alpha G_\beta(E)T_{\beta\beta}^{(+)}(E), \tag{6a}$$

$$T_{\alpha\beta}^{(-)}(E)=V_\beta+T_{\alpha\alpha}^{(-)}(E)G_\alpha(E)V_\beta, \tag{6b}$$

$$K_{\alpha\beta}^{(+)}(E)=V_\alpha+V_\alpha\Gamma_\beta(E)K_{\beta\beta}^{(+)}(E), \tag{7a}$$

$$K_{\alpha\beta}^{(-)}(E)=V_\beta+K_{\alpha\alpha}^{(-)}\Gamma_\alpha(E)V_\beta. \tag{7b}$$

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[1] C. Bloch, Lectures of the Varenna Summer School, 1965 (unpublished). C. Bloch and V. Gillet, Phys. Letters **16**, 62 (1965).

[2] W. MacDonald and A. Mekjian (to be published).

[3] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964); T. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall Publishing Company, Inc., Englewood Cliffs, New Jersey, 1962); Roger Newton, *Scattering Theory of Particles and Waves* (McGraw-Hill Book Company, Inc., New York, 1966).

[4] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

The transition operator is to be distinguished from the elements of the T-matrix $\mathcal{T}$ which are in fact the transition amplitudes for the various possible reactions. The two are directly related:

$$\mathcal{T}_{ab}^{(+)}(E_a,E_b)=\langle\phi_a(E_a)|T_{\alpha\beta}^{(+)}(E_b)|\phi_b(E_b)\rangle, \quad (8a)$$

$$\mathcal{T}_{ab}^{(-)}(E_a,E_b)=\langle\phi_a(E_a)|T_{\alpha\beta}^{(-)}(E_a)|\phi_b(E_b)\rangle, \quad (8b)$$

where ϕ_a is an eigenstate of H_α with eigenvalue E_a and ϕ_b is an eigenstate of H_β with eigenvalue E_b. The K matrix is related to the reactance operator in a similar fashion. Thus, solution of Eq. (6) provides a determination of the transition amplitudes. Since, as we shall see in Sec. III, the transition operator can be written in terms of the reactance operator, solution of Eq. (7) also determines the transition amplitudes.

To simplify notation we will suppress the argument E in what follows.

The formal solution of Eqs. (6) and (7) can be achieved by formally summing the infinite series that results from iterating these equations.

$$\begin{aligned}T_{\alpha\beta}^{(+)}&=V_\alpha(1-G_\beta V_\beta)^{-1}\\&=V_\alpha+V_\alpha G V_\beta,\end{aligned} \quad (9a)$$

$$\begin{aligned}T_{\alpha\beta}^{(-)}&=(1-V_\alpha G_\alpha)^{-1}V_\beta\\&=V_\beta+V_\alpha G V_\beta,\end{aligned} \quad (9b)$$

$$\begin{aligned}K_{\alpha\beta}^{(+)}&=V_\alpha(1-\Gamma_\beta V_\beta)^{-1}\\&=V_\alpha+V_\alpha\bar{\Gamma}_\beta V_\beta,\end{aligned} \quad (10a)$$

$$\begin{aligned}K_{\alpha\beta}^{(-)}&=(1-V_\alpha\Gamma_\alpha)^{-1}V_\beta\\&=V_\beta+V_\alpha\bar{\Gamma}_\alpha V_\beta.\end{aligned} \quad (10b)$$

The operator G that appears in Eq. (9) has a simple interpretation.

$$\begin{aligned}G&=(1-G_\beta V_\beta)^{-1}G_\beta=G_\beta(1-V_\beta G_\beta)^{-1}\\&=(G_\beta^{-1}-V_\beta)^{-1}\\&=(E-H+i\epsilon)^{-1}.\end{aligned} \quad (11)$$

The operator G is just the scattering Green's function for the system Hamiltonian H.

The operator $\bar{\Gamma}_\beta$ that appears in Eq. (10),

$$\bar{\Gamma}_\beta=(1-\Gamma_\beta V_\beta)^{-1}\Gamma_\beta=\Gamma_\beta(1-V_\beta\Gamma_\beta)^{-1}, \quad (12)$$

cannot be simplified in the same way because the inverse of Γ_β does not exist. However, we can simplify Eq. (12) somewhat by introducing the operator

$$q_\beta=G_\beta^{-1}\Gamma_\beta=(E-H_\beta)[E-H_\beta-i\epsilon]^{-1} \quad (13)$$

in terms of which we find

$$\bar{\Gamma}_\beta=(E-H_\beta-q_\beta V_\beta+i\epsilon)^{-1}q_\beta. \quad (14)$$

The operator q_β is the "off-the-energy-shell" projection operator for H_β. It gives 0 when operating on the eigenstate of H_β having eigenvalue E, and it gives 1 when operating on any other eigenstate of H_β. The interpretation of $\bar{\Gamma}_\beta$ remains somewhat obscure. One thing, however, is clear; $\bar{\Gamma}_\beta$ is not the standing-wave Green's function for the system Hamiltonian H, which is given by

$$\Gamma=(E-H)[(E-H)^2+\epsilon^2]^{-1}. \quad (15)$$

We will now present an alternative procedure for constructing a formal solution for $K_{\alpha\beta}$. This solution will be in terms of V_α, V_β, Γ, and Δ, where

$$\begin{aligned}\Delta&=-i\pi^{-1}(\Gamma-G)\\&=\epsilon\pi^{-1}[(E-H)^2+\epsilon^2]^{-1}.\end{aligned} \quad (16)$$

We start by observing that from Eqs. (2) and (11)

$$G^{-1}=G_\alpha^{-1}-V_\alpha. \quad (17)$$

It follows that

$$G_\alpha=G-GV_\alpha G_\alpha=G-G_\alpha V_\alpha G. \quad (18)$$

Now we use Eq. (4) to replace G_α and Eq. (16) to replace G. Equating separately the Hermitian and anti-Hermitian terms in the resulting equation gives the following two relationships:

$$\begin{aligned}\Gamma_\alpha&=\Gamma-\Gamma V_\alpha\Gamma_\alpha+\pi^2\Delta V_\alpha\Delta_\alpha\\&=\Gamma-\Gamma_\alpha V_\alpha\Gamma+\pi^2\Delta_\alpha V_\alpha\Delta,\end{aligned} \quad (19a)$$

$$\begin{aligned}\Delta_\alpha&=\Delta-\Delta V_\alpha\Gamma_\alpha-\Gamma V_\alpha\Delta_\alpha\\&=\Delta-\Gamma_\alpha V_\alpha\Delta-\Delta_\alpha V_\alpha\Gamma.\end{aligned} \quad (19b)$$

Substituting Eq. (19a) into Eq. (7), the LS equation for K, gives

$$K_{\alpha\beta}^{(+)}=V_\alpha+V_\alpha\Gamma V_\beta+\pi^2 V_\alpha\Delta V_\beta\Delta_\beta K_{\beta\beta}^{(+)}, \quad (20a)$$

$$K_{\alpha\beta}^{(-)}=V_\beta+V_\alpha\Gamma V_\beta+\pi^2 K_{\alpha\alpha}^{(-)}\Delta_\alpha V_\alpha\Delta V_\beta. \quad (20b)$$

By using Eqs. (19b) and (7) we can reduce the last term on the right to the required form. We observe that

$$\begin{aligned}\Delta_\beta K_{\beta\beta}^{(+)}&=\Delta(1-V_\beta\Gamma_\beta)K_{\beta\beta}^{(+)}-\Gamma V_\beta\Delta_\beta K_{\beta\beta}^{(+)}\\&=(1+\Gamma V_\beta)^{-1}\Delta V_\beta,\end{aligned} \quad (21a)$$

$$\begin{aligned}K_{\alpha\alpha}^{(-)}\Delta_\alpha&=K_{\alpha\alpha}^{(-)}(1-\Gamma_\alpha V_\alpha)\Delta-K_{\alpha\alpha}^{(-)}\Delta_\alpha V_\alpha\Gamma\\&=V_\alpha\Delta(1+V_\alpha\Gamma)^{-1},\end{aligned} \quad (21b)$$

which, when substituted into Eq. (20), yields the final result

$$K_{\alpha\beta}^{(+)}=V_\alpha+V_\alpha\Gamma V_\beta+\pi^2 V_\alpha\Delta V_\beta(1+\Gamma V_\beta)^{-1}\Delta V_\beta, \quad (22a)$$

$$K_{\alpha\beta}^{(-)}=V_\beta+V_\alpha\Gamma V_\beta+\pi^2 V_\alpha\Delta(1+V_\alpha\Gamma)^{-1}V_\alpha\Delta V_\beta. \quad (22b)$$

We note that the formal solution for K, Eq. (10) or (22), cannot be gotten from the formal solution for T, Eq. (9), by simply replacing the scattering Green's function G by the standing-wave Green's function Γ.

III. FORMAL SOLUTION IN TERMS OF THE REDUCED OPERATORS

Comparing Eqs. (6) and (7), we see that T and K are solutions of LS equations having the same interactions but different Green's functions. This carries the implication of a set of relationships connecting K and T which we will now derive. Solving Eq. (6) for V gives

$$V_\alpha = T_{\alpha\beta}^{(+)}(1+G_\beta T_{\beta\beta}^{(+)})^{-1}, \tag{23a}$$

$$V_\beta = (1+T_{\alpha\alpha}^{(-)}G_\alpha)^{-1}T_{\alpha\beta}^{(-)}. \tag{23b}$$

This expression for V is now substituted into Eq. (10), the formal solution for K. After a few algebraic manipulations we arrive at the following result:

$$K_{\alpha\beta}^{(+)} = T_{\alpha\beta}^{(+)}[1+(G_\beta-\Gamma_\beta)T_{\beta\beta}^{(+)}]^{-1}, \tag{24a}$$

$$K_{\alpha\beta}^{(-)} = [1+T_{\alpha\alpha}^{(-)}(G_\alpha-\Gamma_\alpha)]^{-1}T_{\alpha\beta}^{(-)}, \tag{24b}$$

Eq. (24) is just the formal solution of the Heitler equation[5]

$$T_{\alpha\beta}^{(+)} = K_{\alpha\beta}^{(+)} - i\pi K_{\alpha\beta}^{(+)}\Delta_\beta T_{\beta\beta}^{(+)}, \tag{25a}$$

$$T_{\alpha\beta}^{(-)} = K_{\alpha\beta}^{(-)} - i\pi T_{\alpha\alpha}^{(-)}\Delta_\alpha K_{\alpha\beta}^{(-)}. \tag{25b}$$

The inverse of this relationship results when we solve Eq. (7) for V,

$$V_\alpha = K_{\alpha\beta}^{(+)}(1+\Gamma_\beta K_{\beta\beta}^{(+)})^{-1}, \tag{26a}$$

$$V_\beta = (1+K_{\alpha\alpha}^{(-)}\Gamma_\alpha)^{-1}K_{\alpha\beta}^{(-)}, \tag{26b}$$

and substitute the result into Eq. (9), the formal solution for T, namely,

$$T_{\alpha\beta}^{(+)} = K_{\alpha\beta}^{(+)}[1+(\Gamma_\beta-G_\beta)K_{\beta\beta}^{(+)}]^{-1}, \tag{27a}$$

$$T_{\alpha\beta}^{(-)} = [1+K_{\alpha\alpha}^{(-)}(\Gamma_\alpha-G_\alpha)]^{-1}K_{\alpha\beta}^{(-)}. \tag{27b}$$

This is just the formal solution to the equation

$$K_{\alpha\beta}^{(+)} = T_{\alpha\beta}^{(+)} + i\pi T_{\alpha\beta}^{(+)}\Delta_\beta K_{\beta\beta}^{(+)}, \tag{28a}$$

$$K_{\alpha\beta}^{(-)} = T_{\alpha\beta}^{(-)} + i\pi K_{\alpha\alpha}^{(-)}\Delta_\alpha T_{\alpha\beta}^{(-)}. \tag{28b}$$

From Eqs. (25) and (28) we see that K and T are related to each other by LS equations having Green's functions equal to $\pm(G_\alpha-\Gamma_\alpha)$. We will call the relationships expressed by Eqs. (24), (25), (27), and (28) the Heitler relation. These relations hold between any two operators that satisfy LS equations with the same interaction but different Green's functions.

Note that our results depend on the existence of the inverses that appear in Eqs. (9), (10), (23), and (26). Thus our results will be valid only for those values of the energy E for which the operators $\Gamma_\alpha V_\alpha$, $(V_\alpha\Gamma_\alpha)^\dagger$, $(-\Gamma_\alpha K_{\alpha\alpha}^{(+)})$, and $(-K_{\alpha\alpha}^{(-)}\Gamma_\alpha)^\dagger$ do not have eigenvalue 1.

We now introduce the reduced transition operator $\hat{T}$ and the reduced reactance operator $\hat{K}$ which are defined by the following LS equations:

$$\hat{T}_{\alpha\beta}^{(+)} = V_\alpha + V_\alpha G_\beta P\hat{T}_{\beta\beta}^{(+)}, \tag{29a}$$

$$\hat{T}_{\alpha\beta}^{(-)} = V_\beta + \hat{T}_{\alpha\alpha}^{(-)}PG_\alpha V_\beta, \tag{29b}$$

$$\hat{K}_{\alpha\beta}^{(+)} = V_\alpha + V_\alpha\Gamma_\beta P\hat{K}_{\beta\beta}^{(+)}, \tag{30a}$$

$$\hat{K}_{\alpha\beta}^{(-)} = V_\beta + \hat{K}_{\alpha\alpha}^{(-)}P\Gamma_\alpha V_\beta. \tag{30b}$$

The reduced operators $\hat{K}$ and $\hat{T}$ satisfy LS equations identical to those for K and T except that the Green's functions have been multiplied by a projection operator $P^2=P$. The Heitler relation can be used to express K and T in terms of $\hat{K}$ and $\hat{T}$, respectively. The result is

$$T_{\alpha\beta}^{(+)} = \hat{T}_{\alpha\beta}^{(+)}(1-G_\beta Q\hat{T}_{\beta\beta}^{(+)})^{-1} = \hat{T}_{\alpha\beta}^{(+)} + \hat{T}_{\alpha\beta}^{(+)}\hat{G}_\beta^{(+)}Q\hat{T}_{\beta\beta}^{(+)}, \tag{31a}$$

$$T_{\alpha\beta}^{(-)} = (1-\hat{T}_{\alpha\alpha}^{(-)}QG_\alpha)^{-1}\hat{T}_{\alpha\beta}^{(-)} = \hat{T}_{\alpha\beta}^{(-)} + \hat{T}_{\alpha\alpha}^{(-)}Q\hat{G}_\alpha^{(-)}\hat{T}_{\alpha\beta}^{(-)}, \tag{31b}$$

$$K_{\alpha\beta}^{(+)} = \hat{K}_{\alpha\beta}^{(+)}(1-\Gamma_\beta Q\hat{K}_{\beta\beta}^{(+)})^{-1} = \hat{K}_{\alpha\beta}^{(+)} + \hat{K}_{\alpha\beta}^{(+)}\tilde{\Gamma}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}, \tag{32a}$$

$$K_{\alpha\beta}^{(-)} = (1-\hat{K}_{\alpha\alpha}^{(-)}Q\Gamma_\alpha)^{-1}\hat{K}_{\alpha\beta}^{(-)} = \hat{K}_{\alpha\beta}^{(-)} + \hat{K}_{\alpha\alpha}^{(-)}Q\tilde{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}, \tag{32b}$$

where Q is the projection operator complementary to P,

$$Q = 1-P = Q^2. \tag{33}$$

The reduced operators provide us with the opportunity of finding a solution $\hat{K}$ or $\hat{T}$ valid in one portion of Hilbert space, and then on the basis of Eq. (31) or (32) treating transitions between the two portions of Hilbert space to low order in perturbation theory.[2,6]

The quantities $\hat{G}_\alpha$ that appear in Eq. (31) are enhanced scattering Green's functions.

$$\hat{G}_\beta^{(+)} = (1-G_\beta Q\hat{T}_{\beta\beta}^{(+)})^{-1}G_\beta = (E-H_\beta-Q\hat{T}_{\beta\beta}^{(+)}+i\epsilon)^{-1}, \tag{34a}$$

$$\hat{G}_\alpha^{(-)} = G_\alpha(1-\hat{T}_{\alpha\alpha}^{(-)}QG_\alpha)^{-1} = (E-H_\alpha-\hat{T}_{\alpha\alpha}^{(-)}Q+i\epsilon)^{-1}. \tag{34b}$$

In applications Q is often chosen so that it commutes with H_β or H_α and $\hat{T}$ is approximated by $V+VGPV$ so that

$$\hat{G}_\beta^{(+)}Q \approx Q(E-H_\beta-QV_\beta Q-QV_\beta G_\beta PV_\beta Q)^{-1}Q \approx Q\hat{G}_\beta^{(-)}. \tag{35}$$

Then the real and imaginary parts of

$$QV_\beta G_\beta PV_\beta Q = QV_\beta\Gamma_\beta PV_\beta Q - i\pi QV_\beta\Delta_\beta PV_\beta Q \tag{36}$$

can be interpreted as the level shift and level width operators, respectively.

The quantities $\tilde{\Gamma}_\alpha$ that appear in Eq. (32) are similar to the $\tilde{\Gamma}_\alpha$ we encountered in the previous section.

$$\tilde{\Gamma}_\beta^{(+)} = (1-\Gamma_\beta Q\hat{K}_{\beta\beta}^{(+)})^{-1}\Gamma_\beta = (E-H_\beta-q_\beta Q\hat{K}_{\beta\beta}^{(+)}+i\epsilon)^{-1}q_\beta, \tag{37a}$$

$$\tilde{\Gamma}_\alpha^{(-)} = \Gamma_\alpha(1-\hat{K}_{\alpha\alpha}^{(-)}Q\Gamma_\alpha)^{-1} = q_\alpha(E-H_\alpha-\hat{K}_{\alpha\alpha}^{(-)}Qq_\alpha+i\epsilon)^{-1}, \tag{37b}$$

where q_α is defined in Eq. (13). Again we note that the Green's function $\tilde{\Gamma}$ is not the standing wave Green's function associated with the scattering Green's function

[5] W. Heitler, Proc. Cambridge Phi.. Soc. 37, 291 (1941).

[6] H. Feshbach, Ann. Phys. (N. Y.) 5, 357 (1958); 19, 287 (1962); L. S. Rodberg, Phys. Rev. 124, 210 (1961); W. M. MacDonald, Nucl. Phys. 54, 393 (1964); W. Tobocman, Phys. Rev. 136, B1825 (1964).

$\hat{G}$. We can express K in terms of $\hat{K}$ and $\hat{\Gamma}$ and $\hat{\Delta}$ where

$$\hat{\Gamma}_\beta^{(+)}=(E-H_\beta-Q\hat{K}_{\beta\beta}^{(+)})\times[(E-H_\beta-Q\hat{K}_{\beta\beta}^{(+)})^2+\epsilon^2]^{-1}, \quad (38a)$$

$$\hat{\Gamma}_\alpha^{(-)}=(E-H_\alpha-\hat{K}_{\alpha\alpha}^{(-)}Q)\times[(E-H_\beta-\hat{K}_{\alpha\alpha}^{(-)}Q)^2+\epsilon^2]^{-1}, \quad (38b)$$

and

$$\hat{\Delta}_\beta^{(+)}=\epsilon\pi^{-1}[(E-H_\beta-Q\hat{K}_{\beta\beta}^{(+)})^2+\epsilon^2]^{-1}, \quad (39a)$$

$$\hat{\Delta}_\alpha^{(-)}=\epsilon\pi^{-1}[(E-H_\alpha-\hat{K}_{\alpha\alpha}^{(-)}Q)^2+\epsilon^2]^{-1}, \quad (39b)$$

by following the procedure we used to derive Eq. (22). This will now be done.

First we rewrite Eq. (32) in the form of an LS equation.

$$K_{\alpha\beta}^{(+)}=\hat{K}_{\alpha\beta}^{(+)}+K_{\alpha\beta}^{(+)}\Gamma_\beta Q\hat{K}_{\beta\beta}^{(+)}, \quad (40a)$$

$$K_{\alpha\beta}^{(-)}=\hat{K}_{\alpha\beta}^{(-)}+\hat{K}_{\alpha\alpha}^{(-)}Q\Gamma_\alpha K_{\alpha\beta}^{(-)}, \quad (40b)$$

From Eqs. (2) and (34) we get

$$Q\hat{K}_{\beta\beta}^{(+)}=G_\beta^{-1}-\hat{G}_\beta^{(+)-1}, \quad (41a)$$

$$\hat{K}_{\alpha\alpha}^{(-)}Q=G_\alpha^{-1}-\hat{G}_\alpha^{(-)-1}. \quad (41b)$$

Operate with G_α and $\hat{G}_\alpha$ on both sides of Eq. (41).

$$G_\beta=\hat{G}_\beta^{(+)}-\hat{G}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}G_\beta=\hat{G}_\beta^{(+)}-G_\beta Q\hat{K}_{\beta\beta}^{(+)}\hat{G}_\beta^{(+)}, \quad (42a)$$

$$G_\alpha=\hat{G}_\alpha^{(-)}-\hat{G}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}QG_\alpha=\hat{G}_\alpha^{(-)}-G_\alpha\hat{K}_{\alpha\alpha}^{(-)}Q\hat{G}_\alpha^{(-)}. \quad (42b)$$

Now we use Eq. (4) to replace G_α and we use

$$\hat{G}_\alpha^{(+)}=\hat{\Gamma}_\alpha^{(+)}-i\pi\hat{\Delta}_\alpha^{(+)}, \quad (43a)$$

$$\hat{G}_\alpha^{(-)}=\hat{\Gamma}_\alpha^{(-)}-i\pi\hat{\Delta}_\alpha^{(-)}, \quad (43b)$$

to replace $\hat{G}_\alpha$. Equating the Hermitian and anti-Hermitian parts separately yields

$$\Gamma_\beta=\hat{\Gamma}_\beta^{(+)}(1-Q\hat{K}_{\beta\beta}^{(+)}\Gamma_\beta)+\pi^2\hat{\Delta}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}\Delta_\beta =(1-\Gamma_\beta Q\hat{K}_{\beta\beta}^{(+)})\hat{\Gamma}_\beta^{(+)}+\pi^2\Delta_\beta Q\hat{K}_{\beta\beta}^{(+)}\hat{\Delta}_\beta^{(+)}, \quad (44a)$$

$$\Gamma_\alpha=\hat{\Gamma}_\alpha^{(-)}(1-\hat{K}_{\alpha\alpha}^{(-)}Q\Gamma_\alpha)+\pi^2\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q\Delta_\alpha =(1-\Gamma_\alpha\hat{K}_{\alpha\alpha}^{(-)}Q)\hat{\Gamma}_\alpha^{(-)}+\pi^2\Delta_\alpha\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Delta}_\alpha^{(-)}, \quad (44b)$$

$$\Delta_\beta=\hat{\Delta}_\beta^{(+)}(1-Q\hat{K}_{\beta\beta}^{(+)}\Gamma_\beta)-\hat{\Gamma}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}\Delta_\beta =(1-\Gamma_\beta Q\hat{K}_{\beta\beta}^{(+)})\hat{\Delta}_\beta^{(+)}-\Delta_\beta Q\hat{K}_{\beta\beta}^{(+)}\hat{\Gamma}_\beta^{(+)}, \quad (45a)$$

$$\Delta_\alpha=\hat{\Delta}_\alpha^{(-)}(1-\hat{K}_{\alpha\alpha}^{(-)}Q\Gamma_\alpha)-\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q\Delta_\alpha =(1-\Gamma_\alpha\hat{K}_{\alpha\alpha}^{(-)}Q)\hat{\Delta}_\alpha^{(-)}-\Delta_\alpha\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Gamma}_\alpha^{(-)}. \quad (45b)$$

Equation (44) is now substituted into Eq. (40).

$$K_{\alpha\beta}^{(+)}=\hat{K}_{\alpha\beta}^{(+)}+\hat{K}_{\alpha\beta}^{(+)}\hat{\Gamma}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}+\pi^2K_{\alpha\beta}^{(+)}\Delta_\beta Q\hat{K}_{\beta\beta}^{(+)}\hat{\Delta}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}, \quad (46a)$$

$$K_{\alpha\beta}^{(-)}=\hat{K}_{\alpha\beta}^{(-)}+\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}+\pi^2\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q\Delta_\alpha K_{\alpha\beta}^{(-)}. \quad (46b)$$

The last term on the right of Eq. (46) is reduced with the help of Eqs. (45) and (40).

$$K_{\alpha\beta}^{(+)}\Delta_\beta=\hat{K}_{\alpha\beta}^{(+)}\hat{\Delta}_\beta^{(+)}-K_{\alpha\beta}^{(+)}\Delta_\beta Q\hat{K}_{\beta\beta}^{(+)}\hat{\Gamma}_\beta^{(+)} =\hat{K}_{\alpha\beta}^{(+)}\hat{\Delta}_\beta^{(+)}(1+Q\hat{K}_{\beta\beta}^{(+)}\hat{\Gamma}_\beta^{(+)})^{-1}, \quad (47a)$$

$$\Delta_\alpha K_{\alpha\beta}^{(-)}=\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}-\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q\Delta_\alpha K_{\alpha\beta}^{(-)} =(1+\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q)^{-1}\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}. \quad (47b)$$

Substituting Eq. (47) into Eq. (46) gives us the final result.

$$K_{\alpha\beta}^{(+)}=\hat{K}_{\alpha\beta}^{(+)}+\hat{K}_{\alpha\beta}^{(+)}\hat{\Gamma}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}+\pi^2\hat{K}_{\alpha\beta}^{(+)}\hat{\Delta}_\beta^{(+)}(1+Q\hat{K}_{\beta\beta}^{(+)}\Gamma_\beta^{(+)})^{-1}\times Q\hat{K}_{\beta\beta}^{(+)}\hat{\Delta}_\beta^{(+)}Q\hat{K}_{\beta\beta}^{(+)}, \quad (48a)$$

$$K_{\alpha\beta}^{(-)}=\hat{K}_{\alpha\beta}^{(-)}+\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}+\pi^2\hat{K}_{\alpha\alpha}^{(-)}Q\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q\times(1+\hat{\Gamma}_\alpha^{(-)}\hat{K}_{\alpha\alpha}^{(-)}Q)^{-1}\hat{\Delta}_\alpha^{(-)}\hat{K}_{\alpha\beta}^{(-)}. \quad (48b)$$

Equation (48) is the formal solution of the reactance-operator LS equation in terms of the reduced reactance operators. Equation (31) is the formal solution of the transition-operator LS equation in terms of the reduced transition operators.

IV. COMPARISON WITH PREVIOUS RESULTS

Both BG and MM use expressions for the reactance operator K which are of the form of Eq. (48) with the last term deleted. Let us restrict ourselves to the case where only elastic and inelastic scattering are possible, as do the above-mentioned authors. Then Eq. (1) will be replaced by

$$H=H_0+V_0 \quad (49)$$

and Eqs. (7) and (48) become

$$K=V_0+V_0\Gamma_0K=K_1+K_2, \quad (50a)$$

$$K_1=\hat{K}+\hat{K}\hat{\Gamma}Q\hat{K}, \quad (50b)$$

$$K_2=\pi^2\hat{K}\hat{\Delta}(1+Q\hat{K}\hat{\Gamma})^{-1}Q\hat{K}\hat{\Delta}Q\hat{K}, \quad (50c)$$

where now

$$\hat{K}=V_0+V_0\Gamma_0P\hat{K}, \quad (51a)$$

$$\Gamma_0=(E-H_0)[(E-H_0)^2+\epsilon^2]^{-1}, \quad (51b)$$

$$\hat{\Gamma}=(E-H_0-Q\hat{K})[(E-H_0-Q\hat{K})^2+\epsilon^2]^{-1}, \quad (51c)$$

$$\hat{\Delta}=\epsilon\pi^{-1}[(E-H_0-Q\hat{K})^2+\epsilon^2]^{-1}. \quad (51d)$$

Both BG and MM retain K_1 and drop K_2. We will refer to K_2 as the discrepancy between our result at the results of these authors.

It would appear that the discrepancy K_2 is not significant for the MM formalism but is non-negligible for the BG formalism. In the MM treatment Q is taken to be the projection operator onto the discrete eigenstates of H_0. Thus the factors $\hat{\Delta}Q$ that appear in K_2 will go to zero in the $\epsilon=0$ limit except at those values of E which coincide with the eigenvalues of one of the discrete eigenstates of H_0 or of $H_0+Q\hat{K}$. A discrepancy which is nonvanishing only at isolated energies in the continuum will not affect a practical calculation.

The BG treatment, on the other hand, takes Q to be a projection operator onto the continuum eigenstates of H_0. For this choice of Q there does not seem to be any reason for discarding K_2.

To illustrate these conclusions let us apply our formalism to a simple model. We will consider a system such that H_0 has a discrete spectrum consisting of a single state

$$(W_0-H_0)|0\rangle=0, \quad (52)$$

and a continuous spectrum of states

$$(W-H_0)|W\rangle=0. \quad (53)$$

The interaction will be taken to be separable.

$$V_0=|\gamma\rangle\langle\gamma|. \quad (54)$$

The reactance operator is then just

$$K=|\gamma\rangle\langle\gamma|(1+\Gamma_0 K) = (1-\langle\gamma|\Gamma_0|\gamma\rangle)^{-1}V_0. \quad (55)$$

Similarly, the reduced reactance operator is

$$\hat{K}=|\gamma\rangle\langle\gamma|(1+\Gamma_0 P\hat{K}) = (1-\langle\gamma|\Gamma_0 P|\gamma\rangle)^{-1}V_0. \quad (56)$$

Having explicit expressions for both K and $\hat{K}$, we can immediately display their relationship.

$$K=\hat{K}k, \quad (57a)$$

$$k=(1-\langle\gamma|\Gamma_0 P|\gamma\rangle)(1-\langle\gamma|\Gamma_0|\gamma\rangle)^{-1}. \quad (57b)$$

Consider next the expression for K_1:

$$K_1=\hat{K}+\hat{K}\hat{\Gamma}Q\hat{K}=\hat{K}k_1, \quad (58a)$$

$$k_1=(1-\langle\gamma|\Gamma_0 P|\gamma\rangle+\langle\gamma|\hat{\Gamma}Q|\gamma\rangle)\times(1-\langle\gamma|\Gamma_0 P|\gamma\rangle)^{-1}, \quad (58b)$$

and the expression for K_2.

$$K_2=\pi^2\hat{K}\hat{\Delta}(1+Q\hat{K}\hat{\Gamma})^{-1}Q\hat{K}\hat{\Delta}Q\hat{K}=\hat{K}k_2, \quad (59a)$$

$$k_2=\pi^2\langle\gamma|\hat{\Delta}Q|\gamma\rangle^2(1-\langle\gamma|\Gamma_0 P|\gamma\rangle+\langle\gamma|\hat{\Gamma}Q|\gamma\rangle)^{-1}\times(1-\langle\gamma|\Gamma_0 P|\gamma\rangle)^{-1}. \quad (59b)$$

Evidently, we expect the following relationship:

$$k=k_1+k_2. \quad (60)$$

For convenience let us introduce the following abbreviations:

$$w(E)=\langle\gamma|\Gamma_0(E)P|\gamma\rangle, \quad (61a)$$

$$x(E)=\langle\gamma|\Gamma_0(E)Q|\gamma\rangle, \quad (61b)$$

$$y(E)=\langle\gamma|\hat{\Gamma}(E)Q|\gamma\rangle, \quad (61c)$$

$$z(E)=\pi\langle\gamma|\hat{\Delta}(E)Q|\gamma\rangle. \quad (61d)$$

Then we can write

$$k=(1-w)(1-w-x)^{-1}, \quad (62a)$$

$$k_1=(1-w+y)(1-w)^{-1}, \quad (62b)$$

$$k_2=z^2(1-w+y)^{-1}(1-w)^{-1}. \quad (62c)$$

In the MM treatment we take

$$Q=|0\rangle\langle 0|, \quad (63a)$$

$$P=\int dW|W\rangle\langle W|. \quad (63b)$$

This leads to

$$w=\int dW\langle W|V_0|W\rangle(E-W)[(E-W)^2+\epsilon^2]^{-1}, \quad (64a)$$

$$x=(1-w)U(E-W_0)[(E-W_0)^2+\epsilon^2]^{-1}, \quad (64b)$$

$$y=(1-w)U(E-W_0-U)[(E-W_0-U)^2+\epsilon^2]^{-1}, \quad (64c)$$

$$z=(1-w)U\epsilon[(E-W_0-U)^2+\epsilon^2]^{-1}, \quad (64d)$$

$$U=\langle 0|V_0|0\rangle(1-w)^{-1}. \quad (64e)$$

Combining Eqs. (62) and (64) gives us

$$k=[(E-W_0)^2+\epsilon^2]\times[(E-W_0-U)(E-W_0)+\epsilon^2]^{-1}, \quad (65a)$$

$$k_1=[(E-W_0)(E-W_0-U)+\epsilon^2]\times[(E-W_0-U)^2+\epsilon^2]^{-1},$$

$$k_2=\epsilon^2U^2[(E-W_0)(E-W_0-U)+\epsilon^2]^{-1}\times[(E-W_0-U)^2+\epsilon^2]^{-1}. \quad (65c)$$

The relationship $k=k_1+k_2$ is readily verified. This corroborates our claim that the expression used by BG and MM is incomplete.

We note that in the limit as ϵ approaches zero, k_2 will vanish for all values of E except two. These are $E=W_0$ for which

$$k(W_0)=1, \quad (66a)$$

$$k_1(W_0)=0, \quad (66b)$$

$$k_2(W_0)=1, \quad (66c)$$

and $E=W_0+U$ for which

$$k(W_0+U)=U^2\epsilon^{-2}=\infty, \quad (67a)$$

$$k_1(W_0+U)=1, \quad (67b)$$

$$k_2(W_0+U)=U^2\epsilon^{-2}=\infty. \quad (67c)$$

Since k_2 is nonvanishing only at isolated energies, it will not have an observable effect.

Now let us turn to the BG treatment in which

$$Q=\int dW|W\rangle\langle W|, \quad (68a)$$

$$P=|0\rangle\langle 0|. \quad (68b)$$

This leads to

$$w=\langle 0|V_0|0\rangle(E-W_0)[(E-W_0)^2+\epsilon^2]^{-1}, \qquad (69a)$$

$$x=\int dW\langle W|V_0|W\rangle(E-W)[(E-W)^2+\epsilon^2]^{-1}, \qquad (69b)$$

$$y=\oint dU\{(U|V_0|U)-(U|0\rangle\langle 0|V_0|U)\}(E-U)\times[(E-U)^2+\epsilon^2]^{-1}, \qquad (69c)$$

$$z=\pi\{(E|V_0|E)-(E|0\rangle\langle 0|V_0|E)\}, \qquad (69d)$$

where we have used the round bras and kets to represent the eigenstates of $H_0+Q\hat{K}\equiv\hat{H}_0$. In Eq. (69c) the symbol $\oint$ denotes that a sum over the discrete eigenstates of $\hat{H}_0$ is to be added to the integral over the continuum.

We see that the expressions for w, x, y, and z become more complicated in the BG treatment of this system than the corresponding expressions for the MM treatment. The relationship $k=k_1+k_2$ is not readily verified. However, it is apparent that $z(1-w+y)^{-1}$ will not in general vanish, so that there is no justification for the neglect of k_2.

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Shell-Model Theory of Nuclear Reactions in Deformed Nuclei*†

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A theoretical study has been made of the intermediate-structure resonances observed in neutron scattering and proton capture by F^{19} in terms of simple excitations of the compound nucleus using Feshbach's formalism for nuclear and photonuclear reactions. The simple excitations were taken to be particle-hole states in the deformed-level scheme, including the excitation of rotational bands. The average resonance widths, and spacing, and average total cross section ($\Delta E=0.5$ MeV) observed for $n+F^{19}$ with neutron energy between 0.5 and 2.5 MeV is reasonably well reproduced by the model. In particular, the agreement with the average total cross section indicates that an optical potential can be derived from the above model. In proton capture by F^{19}, though the calculated widths of the resonances are of the correct order of magnitude, the relative spacing of the resonances and magnitude of the cross sections is not in agreement with experiment.

I. INTRODUCTION

RECENT neutron-scattering experiments[1] have revealed resonances in average cross sections with widths (≈ 200 keV) that are too large to be due to compound-nucleus formation, yet too small to be described by an optical model. It has been suggested[2–4] that such resonances might be due to the excitation of particularly simple states of the compound system. In light nuclei, we can hope to describe these simple excitations in terms of single-particle excitations (e.g., particle-hole states). Such calculations have already been performed for neutron scattering on N^{15} and C^{12},[5,6] using a particle-hole description for the excited states of the compound system. Widths of the order of 100–500 keV are obtained, in good agreement with experiment.

Another possible example of such intermediate structures, as these resonances have now been called, has been observed in neutron scattering on F^{19}.[7] In the same energy region above the ground state of Ne^{20} (16–20 MeV) the cross section for $F^{19}(p,\gamma)Ne^{20}$ is known[8] and manifests resonances of a similar nature to the observed structure in $n+F^{19}$. We take the point of view that the appearance of similar resonance structure

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