$$T_\alpha^{(1)}(i) \propto \boldsymbol{\sigma}(i)\cdot\{\nabla\phi_\alpha(\mathbf{r}_i)-(\mu/m)\mathbf{p}_i\phi(\mathbf{r}_i)\}\tau_-(i)\,, \quad \text{(B1)}$$

where $\boldsymbol{\sigma}(i)$, $\tau_-(i)$, $\mathbf{p}_i$ are the spin, isospin, and momentum operators, respectively, of the $i$th nucleon, and $\phi_\alpha(\mathbf{r}_i)$ is the wave function of the pion, in the initial state $\alpha$, at the position of the same nucleon.

To a first approximation, for a pion bound in an atomic state to a nucleus of small change $Z$, the wave function within the nucleus is given by

$$\phi_{nlm}(\mathbf{r}) \propto r^l Y_{lm}(\theta,\phi)\,. \quad \text{(B2)}$$

This is equivalent to neglecting the pion momentum.

Now consider the operator $\tilde{T}_\alpha$ for the pion in an $s$ state ($\alpha=n,0,0$) in the approximation (B2).

$$\tilde{T}_S \propto \boldsymbol{\sigma}(1)\cdot\mathbf{p}_1\tau_-(1)+\boldsymbol{\sigma}(2)\cdot\mathbf{p}_2\tau_-(2)\,,$$

which can also be written

$$\tilde{T}_S \propto \tfrac{1}{4}\{(\boldsymbol{\sigma}(1)+\boldsymbol{\sigma}(2))(\tau_-(1)-\tau_-(2))+(\boldsymbol{\sigma}(1)-\boldsymbol{\sigma}(2))\times(\tau_-(1)+\tau_-(2))\}\cdot(\mathbf{p}_1-\mathbf{p}_2)+\tfrac{1}{4}\{(\boldsymbol{\sigma}(1)+\boldsymbol{\sigma}(2))\times(\tau_-(1)+\tau_-(2))+(\boldsymbol{\sigma}(1)-\boldsymbol{\sigma}(2))\times(\tau_-(1)-\tau_-(2))\}\cdot(\mathbf{p}_1+\mathbf{p}_2)\,. \quad \text{(B3)}$$

It is easily verified that the first term of (B3) gives the selection rules (4.2a), while the second term gives (4.2b). For nucleons moving slowly in the target, we expect the nuclear matrix elements to satisfy

$$\langle f|\mathbf{p}_1-\mathbf{p}_2|i\rangle \gg \langle f|\mathbf{p}_1+\mathbf{p}_2|i\rangle\,, \quad \text{(B4)}$$

which gives the general selection rule (4.1). This is in part due to the final-state kinematics ($K\ll k$) and in part to the role of the two-nucleon correlation, which enhances the transfer of momentum in the relative coordinates, but not, presumably, in the c.m. coordinates.

A similar argument can be constructed for a $p$-state pion, which leads to the rules (4.3).

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# Second-Order Contribution to the Binding Energy of Closed-Shell Nuclei with the Tabakin Potential*

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Hartree-Fock calculations have been performed previously on the binding energy of closed-shell nuclei using Tabakin's separable nonlocal two-nucleon potential. In this paper, we report on an evaluation of the second-order correction to the binding energy of $O^{16}$ and $Ca^{40}$. Including the second-order terms, we obtain binding energies of 6.7 and 10.9 MeV, respectively, for these two nuclei.

## I. INTRODUCTION

TWO distinct kinds of two-nucleon potentials have been used in recent years in nuclear structure studies. The Yale potential of Lassila *et al.*[1] and the Hamada-Johnston potential[2] belong to the first category. These potentials were constructed to fit the properties of the deuteron and two-nucleon scattering data over a large energy range; they both contain a repulsive hard core and hence give rise to two-nucleon wave functions that are very strongly correlated at a small separation distance. In order to use these potentials in nuclear-structure work, one has to go through the painstaking process of evaluating the effective matrix elements (i.e., the $K$-matrix elements of Brueckner's many-body theory[3]). The problems that this kind of calculation encounters for finite nuclei have been overcome in recent years and the effective matrix elements for the harmonic-oscillator shell-model states are now available from the work of Kuo and Brown,[4] Becker and McKellar,[5] and Shakin *et al.*[6] These matrix elements have been used in conventional spectroscopic calculations in Ref. 4 and by Lawson *et al.*[7] and Waghmare and Shakin.[8] Successful results have also been obtained[9,10] in Hartree-Fock (HF) calculations

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[1] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

[2] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).

[3] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958). This work lists all the earlier references by Brueckner and collaborators.

[4] T. T. S. Kuo and G. E. Brown, Nucl. Phys. **85**, 40 (1966).

[5] R. L. Becker and A. D. McKeller, Phys. Letters **21**, 201 (1966).

[6] C. M. Shakin, Y. R. Waghmare, and M. H. Hull, Jr. (to be published).

[7] R. D. Lawson, M. H. MacFarlane, and T. T. S. Kuo, Phys. Letters **22**, 168 (1966).

[8] Y. R. Waghmare, C. M. Shakin, and J. P. Svenne, Bull. Am. Phys. Soc. **11**, 321 (1966).

[9] C. M. Shakin, J. Svenne, and Y. R. Waghmare, Phys. Letters **21**, 209 (1966); also (to be published).

[10] M. K. Pal and A. P. Stamp, Phys. Rev. **158**, 924 (1967).

with the effective matrix elements of the Yale potential; in particular, the binding energy per nucleon is found to be nearly of the right magnitude.

High-energy experiments other than nucleon-nucleon scattering indicate that short-range correlations actually exist between a pair of nucleons.[11] Therefore, Yale or the Hamada-Johnston potentials may be called realistic in the sense that they do give rise to such correlations. It must be mentioned, however, that these correlations do not necessarily imply an *infinite* core. The possibility of replacing the hard core by a softer one, or by a nonlocal repulsion, is of genuine interest, has been investigated,[12] and is under investigation at the present time.

The second kind of two-nucleon potential has been designed from the very beginning to serve this purpose. To this category belong the velocity-dependent potentials of Razavy *et al.* and Green, and the separable nonlocal potential of Tabakin.[13] The last-named author matched the $S$-, $P$-, and $D$-state nucleon-nucleon phase parameters up to 320 MeV with a properly defined set of separable potentials having small off-energy-shell matrix elements. Although the quality of the fit was not as good as that obtained in Refs. 1 and 2, there is scope for further improvement.[14] What is important, however, is the demonstration of the fact that the nucleon-nucleon data can be fairly well reproduced without generating *strong* short-range correlations in the two-nucleon wave function. The main purpose behind the design of such an effective potential is that its separable nature makes an application to nuclear structure calculations a fairly simple task.

HF calculations have been performed[15] for $O^{16}$ and $Ca^{40}$ using the Tabakin potential. The binding energy per nucleon obtained for these two nuclei are low—2.41 and 3.74 MeV, respectively. It would therefore seem that the relative advantage of the Tabakin potential over the $K$-matrix elements of the Yale potential is marred by its inadequacy in producing the right binding energy. This conclusion, however, is not consistent for the following reason: since the $K$-matrix already contains the effect of interaction between a pair of nucleons to all possible orders, one should calculate the higher-order contributions (at least the second-order contribution) of the Tabakin potential before making the comparison.

In the present work, we have used the HF wave functions for the Tabakin potential, calculated in Ref. 15, to evaluate the second-order contribution to the binding energy. The principle we follow is based on Goldstone's linked-cluster perturbation expansion[16] in terms of the two-nucleon potential $v$. In his original work, Tabakin[14] applied the potential to calculate the binding energy of infinite nuclear matter and found that the perturbation expansion in terms of $v$ converges fairly rapidly. Our calculations for finite nuclei also corroborate this conclusion.

It is well known[16] that the HF potential can be defined in an unambiguous manner within the framework of the linked-cluster expansion in terms of $v$, provided the series converges. The use of the HF single-particle energies (in the energy denominators) and wave function in the evaluation of a perturbation theory diagram automatically takes care of all diagrams of the same type having any number of self-energy insertions. This is because a self-energy insertion represents the interaction of a particle with the particles in the Fermi sea (i.e., the HF ground state), and by the definition of the HF potential this contribution is already present in the energy of the single particle.

The principle behind the HF calculation with the Tabakin potential, and its second-order correction described in this paper, therefore, has a firm foundation in the Goldstone theory which seems to converge for the Tabakin potential. The rearrangement of the linked-cluster expansion, leading to the $K$-matrix expansion of Brueckner, gives rise to certain formal difficulties in the definition of the HF potential $V$. The usual procedure, suggested by Bethe,[17] and followed in Refs. 9 and 10, substitutes the $K$-matrix elements instead of the matrix elements of $v$ in defining the quantity $V$. The correct $K$-matrix elements are dependent on the single-particle states themselves and hence the minimization of the ground-state energy does no longer lead to the simple HF equation; there are extra terms in the equation coming from the above-mentioned state dependence. On the other hand, if one tries to bypass this difficulty by using an approximate state-independent Hermitian $K$ (this was done in Refs. 9 and 10) in the theory, then one is left with corrections to the energy that are quite involved and have not been estimated so far.

The formulas used in our second-order calculation are derived in Sec. II. Numerical results are presented in Sec. III. Section IV contains concluding remarks.

## II. THE SECOND-ORDER FORMULAS

Let $m, n$ represent occupied single-particle states in the HF determinant. Then $a, b$ are any two unoccupied single-particle states above the Fermi surface. The HF energies of these states will be denoted by $\epsilon$ with appropriate subscripts. The second-order contribution to the

[11] J. I. Friedman, H. W. Kendal, and P. A. M. Gram, Phys. Rev. **120**, 992 (1960).
[12] C. Bressel, A. K. Kerman, and E. Lomon, Bull. Am. Phys. Soc. **10**, 584 (1965).
[13] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962); A. M. Green, Nucl. Phys. **47**, 671 (1963); F. Tabakin, Ann. Phys. (N.Y.) **30**, 51 (1964).
[14] M. Bolsterli and J. MacKenzie (private communication) have shown that better fits can indeed be obtained.
[15] A. K. Kerman, J. P. Svenne, and F. M. H. Villars, Phys. Rev. **147**, 710 (1966).
[16] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[17] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

ground-state energy is then given by

$$E^{(2)}=-\sum_{m<n}^{\text{occ}}\sum_{a<b}^{\text{unocc}}\frac{|\langle mn|v|ab\rangle_A|^2}{\epsilon_a+\epsilon_b-\epsilon_m-\epsilon_n}. \tag{1}$$

The subscript $A$ denotes that the matrix element is taken with respect to antisymmetric two-body states, i.e., it consists of a direct and an exchange term:

$$\langle mn|v|ab\rangle_A=\langle mn-nm|v|ab\rangle =\sqrt{2}\langle mn|v|ab\rangle. \tag{2}$$

According to our notation, the state with a rounded bracket is antisymmetric, while that with a pointed bra or ket notation is an ordinary product of two-particle wave functions. That is,

$$|mn)=(1/\sqrt{2})\{|m(1)n(2)\rangle-|n(1)m(2)\rangle\}, \tag{3a}$$

and

$$|ab\rangle=|a(1)b(2)\rangle. \tag{3b}$$

With this notation, and the result (2), we obtain from Eq. (1)

$$E^{(2)}=-\sum_{m<n}^{\text{occ}}\sum_{a,b}^{\text{unocc}}\frac{(mn|v|ab\rangle\langle ab|v|mn)}{\epsilon_a+\epsilon_b-\epsilon_m-\epsilon_n}. \tag{4}$$

Our purpose is to evaluate this expression using the HF wave functions and energies calculated in Ref. 15.

Although a straightforward computation of the expression (4) is not difficult, in this work we have followed an approximation for the intermediate states $|ab\rangle$ which makes the computation much simpler. This approximation has been used by other authors[4] and consists of replacing the HF states $|a,b\rangle$ by plane-wave states $|\mathbf{k}_1,\mathbf{k}_2\rangle$ with $k_1, k_2$ larger than the Fermi momentum $k_F$. The amplitude with which $v$ scatters the Fermi-sea particles $m, n$ to unoccupied states is strongly peaked for states $a, b$ which are somewhat above the Fermi surface (see Fig. 1). Therefore, the plane-wave approximation we use for these states is well justified and expected to lead to small errors.

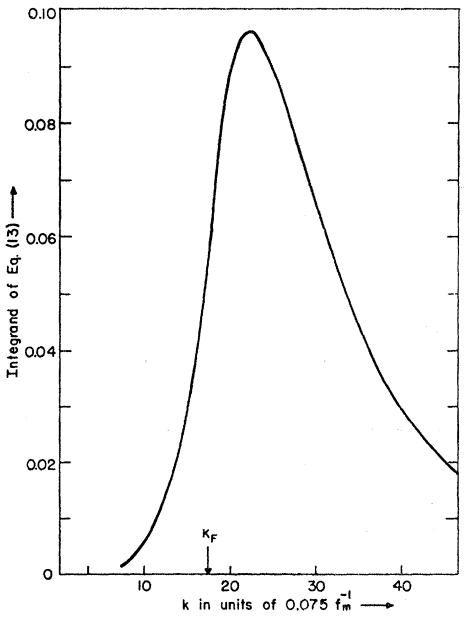


FIG. 1. Plot of the $k$ integrand in Eq. (13) for the attractive part of Tabakin's $^3S_1$ potential ($\lambda=0$, $S=J=T=1$, and $i=j=1$).

Each of the HF states $m, n$ is given in terms of harmonic-oscillator basis states $|\alpha\rangle$ as follows:

$$|m\rangle=\sum_\alpha c_\alpha{}^{(m)}|\alpha\rangle. \tag{5}$$

The coefficients $c_\alpha{}^{(m)}$ are known. For the first-order calculation of the potential energy required in the HF work, one needs the matrix elements $(mn|v|mn)$. These are evaluated by expanding the product of two harmonic-oscillator states of the individual particles, with the help of Moshinsky brackets,[18] in terms of oscillator wave function $|NL\rangle$ and $|nl\rangle$, for the center of mass (c.m.) and relative coordinates, $\mathbf{R}$ and $\mathbf{r}$, respectively. The result is a linear sum of matrix elements of $v$ connecting two states of the type $|NL,(nlSJ):\mathcal{J}\mathfrak{M},TM_T\rangle$. Here $S$ and $T$ are the two-body spin and isospin quantum numbers, $J$ is obtained by coupling $l$ with $S$, while the total two-nucleon angular momentum $\mathcal{J}$ is the result of coupling $L$ with $J$; and $\mathfrak{M}$ and $M_T$ are projection quantum numbers corresponding to $\mathcal{J}$ and $T$, respectively. A central potential gives matrix elements connecting two different values of $n$ with the conservation of all other quantum numbers, while a noncentral potential can change $l$.

The procedure for evaluating the second-order matrix element is exactly similar. Obviously, one obtains the same kind of linear combination of matrix elements connecting the states $|NL,(nlSJ):\mathcal{J}\mathfrak{M},TM_T\rangle$ as in the first-order calculation. The operator, however, is different; instead of the operator $v$ we now have $V^{(2)}$, given by

$$V^{(2)}=\sum_{a,b}^{\text{unocc}}\frac{v|ab\rangle\langle ab|v}{\epsilon_a+\epsilon_b-\epsilon_m-\epsilon_n}, \tag{6}$$

which can, in general, connect states of different $N$, $L$, $n$, and $l$.

According to earlier statements, we shall replace the HF states $|ab\rangle$ by the plane-wave states $|\mathbf{k}_1,\mathbf{k}_2\rangle$, which can be immediately transformed to $|\mathbf{K},\mathbf{k}\rangle$ where $\mathbf{K}$, $\mathbf{k}$ are, respectively, the c.m. and relative momenta. The restriction that $a, b$ be unoccupied states above the Fermi sea requires that $k_1, k_2$ be larger than the Fermi momentum $k_F$. This, in turn, can be taken into account, while working in terms of $\mathbf{K}$ and $\mathbf{k}$, by the Pauli-principle operator $Q(\mathbf{K},\mathbf{k},\mathbf{k}_F)$. An angle-averaged approximation is fairly good for this operator (cf. Kuo

[18] T. A. Brody and M. Moshinsky, *Tables of Transformation Brackets* (Monografias del Instituto de Fisica, Mexico, 1960).

and Brown[4]), and is given by

$$Q(K,k,k_F)=0 \quad \text{if} \quad k^2+\tfrac{1}{4}K^2<k_F^2,$$
$$=1 \quad \text{if} \quad k-\tfrac{1}{2}K>k_F,$$
$$=(k^2+\tfrac{1}{4}K^2-k_F^2)/Kk \text{ otherwise.} \quad (7)$$

We shall further simplify the calculation by using an approximate average value $\Delta$ for the unperturbed pair energy $(\epsilon_m+\epsilon_n)$. Including the normalization constant $(2\pi)^{-3/2}$ of a plane-wave state explicitly, we get

$$V^{(2)}=\frac{1}{(2\pi)^6}\int d^3K \int d^3k\, Q(K,k,k_F) \times \frac{v|\mathbf{K},\mathbf{k}\rangle\langle\mathbf{K},\mathbf{k}|v}{(\hbar^2/2m)(2k^2+\frac{1}{2}K^2)+\Delta}, \quad (8)$$

where $m$ is the nucleon mass. We have to evaluate the matrix element of this operator between the states $|NL,(nlSJ):\mathcal{J}\mathfrak{M},TM_T\rangle$ and $|N'L',(n'l'SJ):\mathcal{J}\mathfrak{M},TM_T\rangle$.

Since the potential $v$ does not depend on the c.m. coordinate, we can easily work out the c.m.-dependent part of the matrix element. We have

$$\langle NLM|K\rangle=\int d^3R\, \mathfrak{R}_{NL}(R)Y_M{}^{L*}(\hat{R})e^{i\mathbf{K}\cdot\mathbf{R}}$$
$$=4\pi i^L Y_M{}^{L*}(\hat{K})\langle NL|j_L\rangle, \quad (9a)$$

where

$$\langle NL|j_L\rangle=\int_0^\infty \mathfrak{R}_{NL}(R)j_L(KR)R^2dR,$$
$$=(-1)^N(\pi/2)^{1/2}\mathfrak{R}_{NL}(K). \quad (9b)$$

In these equations $\mathfrak{R}$ is the harmonic-oscillator radial function, and the symbols $\hat{R},\hat{K}$ stand for the $\theta,\phi$ angles of the corresponding vectors. $\mathfrak{R}_{NL}(K)$ is the oscillator radial function in momentum space with the oscillator constant $(\sqrt{2}/b)$, where $b=(\hbar/m\omega)^{1/2}$ is the oscillator constant of the individual-particle oscillator function.

If we use a result similar to (9a) and (9b) for $\langle\mathbf{K}|N'L'M'\rangle$ then the angle integration of $\mathbf{K}$ can be immediately carried out, yielding

$$\int d\Omega_K\langle NLM|\mathbf{K}\rangle\langle\mathbf{K}|N'L'M'\rangle$$
$$=8\pi^3(-1)^{N+N'}\mathfrak{R}_{NL}(K)\mathfrak{R}_{N'L}(K)\delta_{LL'}\delta_{MM'}. \quad (10)$$

Thus, with the angle-averaged approximation of $Q$, it has been possible to prove that this second-order matrix element conserves the c.m. angular momentum $L$; but unlike the first-order matrix element it is not required to conserve $N$. However, actual numerical evaluation shows that matrix elements nondiagonal in $N$ are very small compared to the diagonal ones. We shall, therefore, assume the conservation requirements of the second-order matrix elements to be the same as those of the first-order ones. Therefore, the linear combinations of matrix elements connecting the states $|NL,(nlSJ):\mathcal{J}\mathfrak{M},TM_T\rangle$, that occur in the two cases, are exactly identical; only the operators are different.

To complete the derivations of the second-order result, we now need the expression for the Tabakin potential $v$, which can be written as follows:

$$v=\sum_{\substack{ll'SJT\\MM_T}}\sum_{i=1}^{2}|g_{lSJT}{}^{(i)}lSJM,TM_T\rangle \times\langle g_{l'SJT}{}^{(i)}l'SJM,TM_T|. \quad (11)$$

For each state specified by $SJM,TM_T$ the potential consists, in general, of two terms $(i=1,2)$. The values of $l$ and $l'$ are consistent with $S$, $J$, and parity of the state; that is, for $S=0$ and $S=1$, parity$=(-1)^J$, we have $l=l'=J$; while for $S=1$ and parity$=-(-1)^J$ each of $l$ and $l'$ can be equal to $J\pm1$. It is clear that in coordinate space the potential $v$ for each state is nonlocal and consists of the factors $g_{lSJT}{}^{(i)}(r)$ and $g_{l'SJT}{}^{(i)}(r')$. Tabakin used a slightly different notation for these factors, but the correspondence between the two notations is obvious.

If we substitute the expression (11) for $v$ in Eq. (8) and evaluate the matrix elements between the two states then obviously one factor of each $v$ will be integrated with the $|nl\rangle$ and $|n'l'\rangle$ parts, respectively, of the two states; while the other factor will be integrated with the relative plane-wave states $|\mathbf{k}\rangle$. The integration over the angles of $\mathbf{k}$ in the latter factors can be carried out easily, yielding

$$\int d\Omega_k\langle g_{l_1SJT}{}^{(i)}l_1SJM,TM_T|\mathbf{k}\rangle\langle\mathbf{k}|g_{l_2SJT}{}^{(j)}l_2SJM,TM_T\rangle$$
$$=\delta_{l_1l_2}16\pi^2\bar{g}_{l_1SJT}{}^{(i)}(k)\bar{g}_{l_2SJT}{}^{(j)}(k), \quad (12)$$

where the momentum transforms of the potential,

$$\bar{g}_{lSJT}{}^{(i)}(k)=\int_0^\infty g_{lSJT}{}^{(i)}(r)j_l(kr)r^2dr,$$

are given in Tabakin's paper.

Collecting all these results we obtain finally

$$\langle NL,(nlSJ):\mathcal{J}\mathfrak{M},TM_T|V^{(2)}|N'L',(n'L'SJ):\mathcal{J}\mathfrak{M},TM_T\rangle$$
$$=\delta_{N,N'}\delta_{L,L'}\binom{2}{\pi}\sum_{i,j}\langle nl|g_{lSJT}{}^{(i)}\rangle\langle g_{l'SJT}{}^{(j)}|n'l'\rangle$$
$$\times\int_0^\infty k^2dk\, f_{NL}(k,k_F,\Delta)\sum_{\lambda}g_{\lambda SJT}{}^{(i)}(k)g_{\lambda SJT}{}^{(j)}(k), \quad (13)$$

where

$$f_{NL}(k,k_F,\Delta)$$
$$=\int_0^{2k_F}K^2dK\,\mathfrak{R}_{NL}{}^2(K)\frac{Q(K,k,k_F)}{(\hbar^2/2m)(\frac{1}{2}K^2+2k^2)+\Delta} \quad (14)$$

TABLE I. Second-order matrix elements (MeV) of the Tabakin potential between relative two-body states $\langle nlSJ|$ and $|n'l'SJ\rangle$. The isospin $T$ is given by $l+S+T=$odd integer. The $(n,n')$ values are specified in the first column. $(l,S,J)$ values are specified in the other columns according to the spectroscopic notation $2S+1l_j$. The sign following each such notation is the over-all sign of the numbers in that column. The entries on the first and second line against each $n$-$n'$ value correspond to $b=(2.6)^{1/2}$ and $(3.75)^{1/2}$ F, respectively. The maximum value of $(n,n')$ used in the two cases are (i) 3 ($S$ and $P$ states), 2 ($D$ states), and (ii) 4 ($S$ states), 3 ($P$ and $D$ states). Average occupied pair energy $\Delta=20.0$ MeV. The matrix elements missing from this table are given in Table II.

| $n$-$n'$ | $^3S_1$, − | $^3D_1$, − | $^3S_1$-$^3D_1$, + | $^1S_0$, − | $^3P_0$, − | $^3P_1$, − | $^3P_2$, − | $^1P_1$, − | $^3D_2$, − | $^3D_3$, − | $^1D_2$, − |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0-0 | 1.417 | 0.762 | 0.423 | 0.397 | 0.032 | 0.501 | 0.096 | 0.911 | 0.324 | 0.040 | 0.045 |
| 0-0 | 1.115 | 0.456 | 0.317 | 0.348 | 0.043 | 0.258 | 0.048 | 0.385 | 0.167 | 0.021 | 0.020 |
| 0-1 | 1.214 | 0.698 | 0.291 | 0.221 | −0.014 | 0.688 | 0.140 | 1.542 | 0.370 | 0.047 | 0.060 |
| | 1.053 | 0.485 | 0.276 | 0.258 | 0.001 | 0.384 | 0.073 | 0.714 | 0.213 | 0.027 | 0.030 |
| 0-2 | 1.014 | 0.666 | 0.255 | 0.073 | −0.038 | 0.731 | 0.162 | 1.779 | 0.382 | 0.049 | 0.067 |
| | 0.942 | 0.483 | 0.245 | 0.159 | −0.031 | 0.441 | 0.089 | 0.903 | 0.235 | 0.029 | 0.035 |
| 0-3 | 0.847 | | | 0.034 | −0.053 | 0.736 | 0.172 | 1.878 | | | |
| | 0.832 | 0.468 | 0.219 | 0.072 | −0.055 | 0.467 | 0.099 | 1.012 | 0.244 | 0.031 | 0.039 |
| 1-1 | 1.042 | 0.648 | 0.221 | 0.130 | 0.089 | 0.946 | 0.204 | 2.612 | 0.424 | 0.054 | 0.079 |
| | 0.994 | 0.519 | 0.247 | 0.194 | 0.018 | 0.573 | 0.112 | 1.324 | 0.273 | 0.035 | 0.043 |
| 1-2 | 0.872 | 0.620 | 0.101 | 0.052 | 0.139 | 1.005 | 0.236 | 3.014 | 0.438 | 0.056 | 0.089 |
| | 0.089 | 0.519 | 0.218 | 0.122 | 0.030 | 0.658 | 0.137 | 1.676 | 0.301 | 0.038 | 0.051 |
| 1-3 | 0.730 | | | −0.003 | 0.169 | 1.012 | 0.252 | 3.181 | | | |
| | 0.786 | 0.504 | 0.193 | 0.059 | 0.038 | 0.697 | 0.153 | 1.878 | 0.312 | 0.040 | 0.057 |
| 2-2 | 0.733 | 0.594 | 0.123 | 0.036 | 0.229 | 1.068 | 0.273 | 3.479 | 0.452 | 0.058 | 0.100 |
| | 0.798 | 0.521 | 0.176 | 0.081 | 0.075 | 0.756 | 0.168 | 2.120 | 0.331 | 0.042 | 0.061 |
| 2-3 | 0.616 | | | 0.023 | 0.280 | 1.075 | 0.291 | 3.671 | | | |
| | 0.706 | 0.507 | 0.155 | 0.045 | 0.107 | 0.800 | 0.187 | 2.377 | 0.344 | 0.044 | 0.068 |
| 3-3 | 0.520 | | | 0.042 | 0.345 | 1.082 | 0.310 | 3.874 | | | |
| | 0.625 | 0.493 | 0.115 | 0.032 | 0.155 | 0.847 | 0.209 | 2.665 | 0.357 | 0.046 | 0.075 |

and

$$\langle nl|g_{lSJT^{(i)}}\rangle=\int_0^\infty r^2dr\ \mathfrak{R}_{nl}(r)g_{lSJT^{(i)}}(r)$$
$$=(-1)^n\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty k^2dk\,\mathfrak{R}_{ln}(k)\bar{g}_{lSJT^{(i)}}(k). \tag{15}$$

The function $\mathfrak{R}_{nl}(k)$ is the oscillator radial function in momentum space for the relative momentum $k$, and uses an oscillator well parameter $(\sqrt{2}b)^{-1}$.

In the actual calculation we made a further approximation. It was found from the preliminary results that the dependence of $f(k,k_F,\Delta)$ on the quantum numbers $(NL)$ is very weak. We, therefore, carried out an average of this quantity over the various $(NL)$ quantum numbers that enter into our calculations. The averaged value was used in Eq. (13), and the resultant matrix element was interpreted as that connecting the relative state $|nlSJM,TM_T\rangle$ with $|n'l'SJM,TM_T\rangle$. The same averaging with respect to the $(NL)$ quantum numbers was done in Ref. 15 while calculating the potential energy in the first order.

## III. NUMERICAL RESULTS FOR $O^{16}$ AND $Ca^{40}$

We used the values of the oscillator parameter $b$ equal to $(2.6)^{1/2}$ F and $(3.75)^{1/2}$ F for $O^{16}$ and $Ca^{40}$, respectively (1 F$=10^{-13}$ cm). The averaged occupied pair energy $\Delta$ was taken to be 20.0 MeV. The value of the Fermi momentum $k_F$ was assumed to be 1.3 F$^{-1}$.

The integrand of Eq. (13) is plotted as a function of $k$ in Fig. 1. It is noticed that the peak of the integrand is at a value of $k$ which is larger than $k_F$, and the area under the curve below $k_F$ is a very small part of the total area. This justifies our plane-wave approximation for the intermediate states.

The second-order matrix elements between various relative two-nucleon states, calculated with the help of Eq. (13), using an average value of $f_{NL}(k,k_F,\Delta)$, are tabulated in Tables I and II for the two values of $b$, mentioned above.

The calculation of the two-nucleon matrix elements from these relative matrix elements was done by using a code written by J. P. Svenne for the work of Ref. 15. The second-order contribution to the binding energy was obtained from these two-nucleon matrix elements and the expansion coefficients $c_\alpha^{(m)}$ of the HF states. The results for $O^{16}$ and $Ca^{40}$ obtained in this way are, respectively, 4.3 and 7.2 MeV per nucleon. Adding these to the first-order results[15] for these two nuclei (2.4 and 3.7 MeV, respectively) we obtain the very satisfactory values of 6.7 and 10.9 MeV per nucleon, respectively.

## IV. DISCUSSIONS AND CONCLUSIONS

The values of $b$, used in this work, were found to be the most appropriate for $O^{16}$ and $Ca^{40}$ in the HF calculations of Ref. 15. Results of the type compiled in Tables I and II for various values of $b$ and all possible shell-model relative states $(nl)$ and $(n'l')$ occurring in structural calculations are available but will not be presented here. The averaging of the quantity $f_{NL}(k,k_F,\Delta)$, over the various values of $NL$, becomes a worse and worse approximation as the number of nodes $N$ is increased gradually. In general, this quantity decreases as $N$ is increased. Hence, in order that the tables of matrix elements be useful throughout the periodic table, the matrix elements between various relative states should be computed for different values of $NL$ individually. Table III gives some idea of the dependence of the matrix elements on the value of $b$ and the number of nodes $N$ considered in the averaging of $f(k,k_F,\Delta)$.

The dependence of the matrix elements on the parameter $\Delta$ was not investigated in the present work in great detail. As we go through the periodic table, the average occupied pair energy can change quite significantly, and the effect of this change on the matrix elements should, therefore, be studied. A rough guide to the value of $\Delta$ is provided by the HF energies themselves. A discussion of this point is given in Ref. 19.

Finally, we make a few comments about the binding energies for $O^{16}$ and $Ca^{40}$ that we have found here. The first-order potential energies (per nucleon) in these two nuclei are, respectively, 21.8 and 25.6 MeV. Comparing these values with the second-order results, mentioned in Sec. III, we conclude that the convergence of the perturbation series is fairly rapid; this is in agreement with what was found by Tabakin for infinite nuclear matter. The third- and higher-order terms may be estimated, at this rate, to contribute about 1 MeV or less in $O^{16}$. If we make a similar estimate of the higher-order contributions and add this to the value we have already found for $Ca^{40}$, the resultant number gives an overbinding if we assume that the contribution is attractive. However, we probably have overestimated our second-order term for $Ca^{40}$ by using the same value of $\Delta$ as for $O^{16}$. The value of $\Delta$ should be somewhat larger and this would produce less binding.

The conclusion we have reached from the results contained in this paper is that the Tabakin potential can produce very satisfactory values of the binding energy of finite nuclei, provided one takes into account at least the second-order contribution. The type of detailed numerical agreement one obtains in this way is comparable to what has been obtained with the effective matrix elements of the Yale potential.

TABLE II. See caption of Table I for explanations.

| $n$-$n'$ | 0-4 | 1-4 | 2-4 | 3-4 | 4-4 | $b$ (F) |
|---|---|---|---|---|---|---|
| $^3S_1$, − | 0.733 | 0.694 | 0.623 | 0.553 | 0.491 | $(3.75)^{1/2}$ |
| $^1S_0$, − | 0.002 | 0.009 | 0.015 | 0.021 | 0.026 | $(3.75)^{1/2}$ |
| $n$-$n'$ | 1-0 | 2-0 | 2-1 | 3-0 | 3-1 | 3-2 | $b$ (F) |
| $^3S_1$-$^3D_1$, + | 0.332 | 0.239 | 0.147 | 0.159 | 0.084 | 0.006 | $(2.6)^{1/2}$ |
| | 0.287 | 0.240 | 0.202 | 0.193 | 0.157 | 0.133 | $(3.75)^{1/2}$ |

TABLE III. Matrix elements for the $^3S_1$ state with $n=n'=0$ showing the dependence on the oscillator parameter $b$, and the number of nodes $N$ taken for averaging $f(k,k_F,\Delta)$. The matrix elements are in MeV, and have a minus sign (attractive, $\Delta=20.0$ MeV).

| $b^2$ (F$^2$) | 4.37 | 3.75 | 3.0 | 2.6 | Number of nodes |
|---|---|---|---|---|---|
| Matrix element | 0.690 | 0.759 | 0.856 | 0.942 | $N=8$ |

| $b^2$ (F$^2$) | Number of nodes ($N$) | Value of the matrix element |
|---|---|---|
| 4.37 | 8 | 0.690 |
| | 4 | 1.006 |
| 3.75 | 8 | 0.759 |
| | 4 | 1.115 |
| 3.0 | 8 | 0.856 |
| | 3 | 1.340 |
| 2.6 | 8 | 0.942 |
| | 3 | 1.417 |

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[19] W. Bassichis, A. K. Kerman, and J. P. Svenne (to be published).

[20] M. K. Pal, J. P. Svenne, and A. K. Kerman, in Proceedings of the International Conference on Nuclear Physics, Oak Ridge, Tennessee, 1966 (unpublished).