Number-Conserving Treatment of Pairing Correlations in Even-Even Nuclei*

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A simple method is proposed for treating pairing correlations in even-even nuclei without violating the particle-number conservation. A dispersion formula is derived which makes it possible to calculate the eigenenergies for a system with A particles, once the occupation numbers for the $(A-2)$-particle system are known. This formula is used to calculate the ground-state energies for the even Ni isotopes; the results turn out to be much better than those obtained by the standard BCS method. The ground-state wave function is discussed and compared with the A-particle component of the BCS wave function. The possibility of treating neutron-proton correlations by the same method is also considered, and the general formulation is given for the case of a charge-independent pairing interaction between particles in nondegenerate orbitals. The degenerate model is studied in detail, and the approximate ground-state energy is compared with the exact one. The results obtained suggest that the present method may be usefully applied in the general case of nondegenerate levels.

I. INTRODUCTION

THE problem of treating pairing correlations in nuclei without violating the particle-number conservation has received much attention in the past few years.[1] Recently, new methods closely related to each other have been proposed by several authors.[2–6] These methods, formally very similar to the usual BCS procedure,[7] seem to be very promising as compared with the latter. This is not only because of the conservation of the number of particles, and the improvement of the numerical results,[8] but also because of the better understanding of the approximations involved. Moreover, there is hope that this kind of approach may be useful in dealing with more realistic interactions. In the case of the even-odd nuclei, for instance, it has been possible to treat a pairing-plus-quadrupole force.[9]

In Ref. 2 the low-lying states of a spherical even-odd A nucleus were described as a linear combination of states obtained by adding a particle and a hole to the ground state of the neighboring $(A-1)$ and $(A+1)$ even-even nucleus, respectively. In this paper we propose an analogous method for the study of even-even spherical nuclei, confining ourselves to the simplest stage of approximation. More precisely, the ground state of a system of an even number (A) of identical particles is described as a linear combination of states obtained by adding two particles to the ground state of the $(A-2)$ system.[10] Using this approximation and making use of the equation of motion for a pair of operators $a_\alpha^\dagger a_{\bar{\alpha}}^\dagger$ ($\bar{\alpha}$ is the state which is the time reversal of α), we derive the equations which determine the approximate ground state and its energy for the A-particle system. As a consequence of neglecting the excited states of the $(A-2)$-particle system only pairing-type matrix elements appear in the final equations. For the usual case of a constant pairing force the eigenenergies can be obtained as solutions of a dispersion formula. The accuracy of the approximation, as tested by calculating the ground-state energies for the model of the Ni isotopes,[11] turns out to be very good.

The present method can be straightforwardly applied to the more complicated case of systems of unlike nucleons. In Sec. III we give the general formulation for an even-even system of protons and neutrons moving in a set of single-particle orbits and interacting through a charge-independent pairing force. In this case our treatment would not *a priori* seem to be adequate because of the complete lack of neutron-proton correlation in the ground-state wave function. From a detailed discussion of the degenerate case, however, we shall conclude that such a wave function may in fact

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[1] See Ref. 4 for a full list of references.
[2] E. Salusti, Nuovo Cimento **37**, 199 (1965).
[3] M. Jean, Nuovo Cimento **40**, 1224 (1965).
[4] Giu Do Dang and A. Klein, Phys. Rev. **143**, 735 (1966).
[5] Giu Do Dang and A. Klein, Phys. Rev. **147**, 689 (1966).
[6] M. Jean, X. Campi, and H. Vucetich, in Proceedings of the International Course on Nuclear Physics, Trieste, Italy, 1967 (to be published).
[7] See, e.g., M. Baranger, in *Cargèse Summer School Lectures, 1962*, edited by M. Lévy (W. A. Benjamin, Inc., New York, 1963).
[8] The ground-state energies calculated by Do Dang and Klein (Refs. 4 and 5), and by Jean *et al.* (Ref. 6), are not far from those obtained by projecting out of the BCS wave function the part with the correct number of particles.
[9] D. Prosperi and E. Salusti, Nuovo Cimento **44B**, 233 (1966).
[10] A. Covello and E. Salusti, Bull. Am. Phys. Soc. **12**, 497 (1967). A similar wave function has also been studied by R. W. Richardson (private communication).
[11] A. K. Kerman, R. D. Lawson, and M. H. MacFarlane, Phys. Rev. **124**, 162 (1961).

be useful for the treatment of neutron-proton pairing correlations in even-even nuclei.

II. SYSTEM OF IDENTICAL PARTICLES

A. Formulation of the Method

We consider a system of A identical particles (A even) moving in a set of single-particle orbits and interacting through a two-body residual interaction. The Hamiltonian is then written

$$H=\sum_{\alpha}\epsilon_{\alpha}a_{\alpha}^{\dagger}a_{\alpha}+\tfrac{1}{4}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a_{\alpha}^{\dagger}a_{\beta}^{\dagger}a_{\delta}a_{\gamma}, \quad (1)$$

where α, β, γ, and δ label the single-particle states, and V has the following symmetry properties:

$$V_{\alpha\beta\gamma\delta}=-V_{\beta\alpha\gamma\delta}=-V_{\alpha\beta\delta\gamma}=V_{\gamma\delta\alpha\beta}{}^{*}. \quad (2)$$

Let us denote by $|A-2, I\rangle$ a complete set of eigenstates of H for the system with $(A-2)$ particles, where I stands for all the quantum numbers which specify the state. The ground state $|A,0\rangle$ of the A-particle system can clearly be expanded in terms of states built by applying two creation operators to the eigenvectors $|A-2, I\rangle$. We now make the approximation of neglecting all the excited states ($I\neq 0$) in such an expansion, namely, we put

$$|A,0\rangle=\sum_{\alpha}c_{\alpha}(A)a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}|A-2, 0\rangle, \quad (3)$$

where $a_{\alpha}^{\dagger}$ and $a_{\bar{\alpha}}^{\dagger}$ create time-reversed states, and the coefficients c_{α} depend on A. In other words, thinking of the even-A system as an $(A-2)$ core plus two particles, we assume that its ground-state wave function does not contain any component corresponding to core excitations. Clearly, such an approximation depends on the lowest excitation energy of the core [energy gap between the ground state and the first excited state of the $(A-2)$-particle system], namely, the larger this energy the better the approximation.

We define now the following quantities:

$$X_{\alpha}(A)=\langle A,0|a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}|A-2, 0\rangle, \quad (4)$$

$$e(A)=E_0(A)-E_0(A-2), \quad (5)$$

where $E_0(A)$ and $E_0(A-2)$ are the energies corresponding to the states $|A,0\rangle$ and $|A-2, 0\rangle$, respectively. The equation for the amplitude X_{α} is obtained from the equation of motion for the pair of operators $a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}$. It is

$$eX_{\alpha}=\langle A,0|[H,a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}]|A-2,0\rangle. \quad (6)$$

Once the commutator $[H,a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}]$ is calculated, Eq. (6) can be linearized by introducing the unit operator

$$\sum_{I}|A-2, I\rangle\langle A-2, I|=1 \quad (7)$$

on the right-hand side, and neglecting the excited states[12] $I\neq 0$, consistently with (3). In this way we obtain the following set of linear equations for the X's:

$$eX_{\alpha}=2\epsilon_{\alpha}X_{\alpha}+\tfrac{1}{2}\sum_{\beta}V_{\alpha\bar{\alpha}\beta\bar{\beta}}[1-2\rho_{\alpha}(A-2)]X_{\beta}, \quad (8)$$

where

$$\rho_{\alpha}(A-2)=\langle A-2, 0|a_{\alpha}^{\dagger}a_{\alpha}|A-2, 0\rangle. \quad (9)$$

The amplitudes X_{α} are related to the coefficients c_{α} of (3) through

$$X_{\alpha}(A)=\sum_{\beta}d_{\alpha\beta}(A-2)c_{\beta}{}^{*}(A), \quad (10)$$

where

$$d_{\alpha\beta}(A-2)=\langle A-2, 0|a_{\bar{\beta}}a_{\beta}a_{\alpha}^{\dagger}a_{\bar{\alpha}}^{\dagger}|A-2, 0\rangle. \quad (11)$$

Note that in (1) the residual interaction V was left completely arbitrary, whereas in (8) only pairing-type matrix elements $V_{\alpha\bar{\alpha}\beta\bar{\beta}}$ appear. We may say that as a result of our approximation only pairing correlations are extracted from the interaction.

Equation (8) is equivalent to an eigenvalue problem for the nonsymmetric matrix $\mathbf{T}$ whose elements are

$$T_{\alpha\beta}=2\epsilon_{\alpha}\delta_{\alpha\beta}+\tfrac{1}{2}V_{\alpha\bar{\alpha}\beta\bar{\beta}}[1-2\rho_{\alpha}(A-2)]. \quad (12)$$

The calculation of the ground-state energy and wave function for the A system therefore requires the knowledge of the quantities $\rho_{\alpha}(A-2)$ and $d_{\alpha\beta}(A-2)$. To this end a step-by-step procedure can be used. Since we have

$$\rho_{\alpha}(0)=0, \quad d_{\alpha\beta}(0)=\delta_{\alpha\beta}, \quad (13)$$

it follows from (10) and (11) that

$$X_{\alpha}(2)=c_{\alpha}(2). \quad (14)$$

Also, it is easily seen that

$$\rho_{\alpha}(2)=X_{\alpha}{}^{2}(2)/\sum_{\beta>0}X_{\beta}{}^{2}(2). \quad (15)$$

Hence one can solve Eq. (8) for $A=2$, thus obtaining $e_0(2)$ and $|2,0\rangle$, then calculate the $\rho(2)$'s and the $d(2)$'s and proceed by successive steps up to A. Actually, for $A>2$ each step involves the diagonalization of a non-Hermitian matrix, and therefore the eigenvalues may turn out to be complex. However, the situation becomes much simpler if we consider the case of the usual constant pairing force, namely, if we assume that all the matrix elements $V_{\alpha\bar{\alpha}\beta\bar{\beta}}$ are equal. In this case, as we shall show in the next section, a dispersion formula for the eigenenergies can be derived and the nature of the solutions can be studied graphically.

It is to be noted that for $A=2$ our method is exact, since the expansion (3) and the linearization of Eq. (6) can be made without any approximation. In this case the matrix (12) is obviously symmetric.

[12] We assume that principal-quantum-number excitation is not allowed.

By virtue of the step-by-step method described above, the ground state $|A,0\rangle$ can be straightforwardly related to the vacuum $|0\rangle$. We shall come back to this point in Sec. II C, where the structure of such a wave function will be discussed in detail.

B. Pairing Case

For the usual constant pairing force $V_{\alpha\bar{\alpha}\beta\bar{\beta}}=-G$ $(G>0)$, and the solution of the eigenvalue problem is greatly simplified. In fact, Eq. (8) becomes

$$X_\alpha=-\frac{1}{2}\frac{G[1-2\rho_\alpha(A-2)]}{2\epsilon_\alpha-e}\sum_\beta X_\beta, \tag{16}$$

from which it follows that[13]

$$\frac{1}{G}=\sum_{\alpha>0}\frac{1-2\rho_\alpha(A-2)}{2\epsilon_\alpha-e}. \tag{17}$$

As we have already pointed out, in the simple case of two particles our procedure is exact and Eq. (17) reduces to the well-known[14] dispersion formula

$$\frac{1}{G}=\sum_{\alpha>0}\frac{1}{2\epsilon_\alpha-e}. \tag{18}$$

In this case one has also $d_{\alpha\beta}=\delta_{\alpha\beta}$, and the X's coincide with the c's. For $A>2$ the eigenvalue equation (17) can be solved graphically to give the approximate ground-state energy [relative to the ground state of the $(A-2)$ system] of the system of A particles interacting through a pairing force, once the occupation numbers $\rho_\alpha(A-2)$ for the $(A-2)$-particle system are known.

Actually, Eq. (17) has several solutions, but the eigenvalue $e_0=E_0(A)-E_0(A-2)$ in which we are interested can be easily identified as the one which tends toward the appropriate value of $2\epsilon_\alpha$ when the residual interaction is turned off. The eigenvalues below e_0 correspond to eigenvectors which vanish when $G\to 0$ because of the Pauli principle, and therefore must be considered unphysical. The states corresponding to energies larger than e_0 represent the excited states of seniority zero of the A-particle system, within the approximation of neglecting all the excited states of seniority zero of the $(A-2)$ system. However, these states are not of interest here since the above approximation is not expected to be valid for them.

The Degenerate Model

Let us now consider the degenerate model, in which all the single-particle energies are assumed equal. In this case there are no excited states of seniority zero in the $(A-2)$ system, and therefore one obtains the exact solution for the pairing force. We have

$$\rho(A-2)=(A-2)/2\Omega, \tag{19}$$

where Ω is the number of pair states (number of $\alpha>0$). Then Eq. (17) becomes

$$e=2\epsilon+G(A-2)-G\Omega, \tag{20}$$

and the ground-state energy $E_0(A)$ of the A-particle system can be easily derived. One obtains

$$E_0(A)=A\epsilon-\tfrac{1}{2}GA(1-\tfrac{1}{2}A+\Omega), \tag{21}$$

namely, the exact result.

Numerical Results

We have calculated the ground-state energies for the model of the even Ni isotopes, which is a standard test case since there is the exact solution[11] with which to compare the results. The single-particle energies are (in MeV): $\epsilon_{p_{3/2}}=0.00$, $\epsilon_{f_{5/2}}=0.78$, $\epsilon_{p_{1/2}}=1.56$, $\epsilon_{g_{9/2}}=4.52$, and the pairing constant $G=0.331$ MeV.[15] The number of particles ranges from 2 (Ni^{58}) to 10 (Ni^{66}). Here the occupation numbers $\rho_\alpha(A)$ have been obtained by means of the very simple formula

$$\rho_\alpha(A)=\rho_\alpha(A-2)+X_\alpha^2(A)/\sum_{\beta>0}X_\beta^2(A), \tag{22}$$

which can be easily derived by making use of the same approximation involved in Eq. (8) and by assuming $c_\alpha(A)\approx X_\alpha(A)$.

In Table I we compare our energies to the exact ones obtained by Lawson, Kerman, and MacFarlane[11] and to those obtained by Kisslinger and Sorensen[16] using the BCS method. It appears that the present method gives much better ground-state energies than the standard quasiparticle approximation. Our results are practically as good as those achieved by other particle-number conserving approximations.[4,6,17] In Table II we give the values of the occupation numbers for the various Ni isotopes obtained by using Eq. (22).

TABLE I. Values of the ground-state energy (in MeV) for the model of the Ni isotopes.

$E_0(A)$	$Ni^{58}(2)$	$Ni^{60}(4)$	$Ni^{62}(6)$	$Ni^{64}(8)$	$Ni^{66}(10)$
Exact[a]	−1.49	−2.11	−1.75	−0.51	1.70
BCS[b]	−1.13	−1.51	−1.09	−0.22	2.48
Present work	−1.49	−2.07	−1.59	−0.36	1.75

[a] Reference 11.
[b] Reference 16.

[13] This equation has also been derived by R. R. Chasman, Phys. Rev. **156**, 1197 (1967).
[14] See, e.g., A. M. Lane, *Nuclear Theory* (W. A. Benjamin, Inc., New York, 1964), Chap. 1.
[15] Actually, for $A=6(Ni^{62})$ we used a slightly smaller $G=0.306$ MeV in order to obtain a real solution for Eq. (17).
[16] L. S. Kisslinger and R. A. Sorensen, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **32**, No. 9 (1960).
[17] Y. Nogami and I. J. Zucker, Nucl. Phys. **60**, 203 (1964).

TABLE II. Values of the occupation numbers $\rho_j(A)$ for the Ni isotopes.

j \ A	2	4	6	8	10
$p_{3/2}$	0.342	0.596	0.716	0.831	0.927
$f_{5/2}$	0.082	0.213	0.425	0.619	0.800
$p_{1/2}$	0.036	0.090	0.163	0.282	0.463
$g_{9/2}$	0.007	0.016	0.026	0.040	0.057

C. The Ground-State Wave Function

The ground-state vectors $|A,0\rangle$ and $|A-2,0\rangle$ are related through Eq. (3). In the same way, namely, by neglecting all the excited states of the $(A-4)$-particle system, the ground-state vector $|A-2,0\rangle$ can be related to $|A-4,0\rangle$. Iterating this procedure one finally obtains

$$|A,0\rangle=\prod_{p=1}^{A/2}[\sum_{\alpha} c_\alpha(2p)a_\alpha^\dagger a_{\bar\alpha}^\dagger]|0\rangle, \tag{23}$$

where $|0\rangle$ is the vacuum.

Since the coefficients c_α depend on p, we introduce their average value

$$\bar c_\alpha=\frac{2}{A}\sum_{p=1}^{A/2} c_\alpha(2p) \tag{24}$$

and the relative deviation from the average value

$$\Delta_\alpha(2p)=(1/\bar c_\alpha)[c_\alpha(2p)-\bar c_\alpha]. \tag{25}$$

Equation (23) can therefore be written

$$|A,0\rangle=\prod_p\{\sum_\alpha \bar c_\alpha[1+\Delta_\alpha(2p)]a_\alpha^\dagger a_{\bar\alpha}^\dagger\}|0\rangle=(\sum_\alpha \bar c_\alpha a_\alpha^\dagger a_{\bar\alpha}^\dagger)^{A/2}|0\rangle+(\sum_\alpha \bar c_\alpha a_\alpha^\dagger a_{\bar\alpha}^\dagger)^{\frac{1}{2}(A-2)}\sum_\alpha\sum_p \bar c_\alpha\Delta_\alpha(2p)a_\alpha^\dagger a_{\bar\alpha}^\dagger|0\rangle$$
$$+\tfrac{1}{2}(\sum_\alpha \bar c_\alpha a_\alpha^\dagger a_{\bar\alpha}^\dagger)^{\frac{1}{2}(A-4)}\sum_{p\neq p'}\sum_{\alpha\neq\beta}\bar c_\alpha\bar c_\beta\Delta_\alpha(2p)\Delta_\beta(2p')a_\alpha^\dagger a_{\bar\alpha}^\dagger a_\beta^\dagger a_{\bar\beta}^\dagger|0\rangle+\cdots. \tag{26}$$

The first term in (26),

$$(\sum_\alpha \bar c_\alpha a_\alpha^\dagger a_{\bar\alpha}^\dagger)^{A/2}|0\rangle, \tag{27}$$

is of the well-known form of the A-particle component of the BCS wave function. As has been shown by Bayman,[18] it is possible to derive the BCS theory from a trial function of the form (27) without introducing fluctuations in the number of particles. The second term in (26) vanishes because

$$\sum_p \Delta_\alpha(2p)=0. \tag{28}$$

For the same reason the only nonvanishing contribution to the third term is

$$-\tfrac{1}{2}(\sum_\alpha \bar c_\alpha a_\alpha^\dagger a_{\bar\alpha}^\dagger)^{\frac{1}{2}(A-4)}\times\sum_p\sum_{\alpha\neq\beta}\bar c_\alpha\bar c_\beta\Delta_\alpha(2p)\Delta_\beta(2p)a_\alpha^\dagger a_{\bar\alpha}^\dagger a_\beta^\dagger a_{\bar\beta}^\dagger|0\rangle. \tag{29}$$

We can thus conclude that if the values of the $c_\alpha(2p)$ are close to each other, i.e., if the $\Delta_\alpha(2p)$ are $\ll 1$, the contribution (29) and the higher-order contributions to $|A,0\rangle$ are negligible, and we can approximately identify $|A,0\rangle$ with (27).

Note that the wave function (23) is exact in both limits when G is either very large or very small compared to the single-particle splittings.

III. SYSTEM OF PROTONS AND NEUTRONS

A. Pairing Interaction between Particles in Nondegenerate Orbitals

We shall now study the problem of neutron-proton pairing correlations, making use of the same method applied to the case of identical particles. Here we consider an even-even system of A nucleons (Z protons and N neutrons) and given $T_z=\frac{1}{2}(N-Z)$ interacting through a charge-independent pairing force, which is effective only for $J=0$, $T=1$ pairs. The Hamiltonian is written

$$H=\sum_{\alpha t}\epsilon_\alpha a_{\alpha t}^\dagger a_{\alpha t}-\tfrac{1}{4}G\sum_{\alpha\beta tt'}a_{\alpha t}^\dagger a_{\bar\alpha t'}^\dagger a_{\bar\beta t'}a_{\beta t}, \tag{30}$$

where t is the isobaric-spin index. The two charge states will be explicitly labelled by π and ν.

According to (3), the ground-state vector $|(A,T_z)0\rangle$ can be related either to the ground state of the system with $(A-2)$ particles and T_z-1, or to the ground state of the system with $(A-2)$ particles and T_z+1. We have

$$|(A,T_z)0\rangle=\sum_\alpha c_{\alpha\nu}a_{\alpha\nu}^\dagger a_{\bar\alpha\nu}^\dagger|(A-2,T_z-1)0\rangle \tag{31}$$

and

$$|(A,T_z)0\rangle=\sum_\alpha c_{\alpha\pi}a_{\alpha\pi}^\dagger a_{\bar\alpha\pi}^\dagger|(A-2,T_z+1)0\rangle. \tag{32}$$

Clearly, neither of the wave functions (31) and (32) has the isospin T as a good quantum number.

We now define the following quantities:

$$X_{\alpha\pi}=\langle(A,T_z)0|a_{\alpha\pi}^\dagger a_{\bar\alpha\pi}^\dagger|(A-2,T_z+1)0\rangle, \tag{33}$$

$$e_Z=E_0(A,T_z)-E_0(A-2,T_z+1), \tag{34}$$

where $E_0(A,T_z)$ and $E_0(A-2,T_z+1)$ are the energies corresponding to the states $|(A,T_z)0\rangle$ and

$$|(A-2,T_z+1)0\rangle,$$

respectively. The equation for the amplitude $X_{\alpha\pi}$ is written

$$e_Z X_{\alpha\pi}=\langle(A,T_z)0|[H,a_{\alpha\pi}^\dagger a_{\bar\alpha\pi}^\dagger]|(A-2,T_z+1)0\rangle. \tag{35}$$

[18] B. F. Bayman, Nucl. Phys. 15, 33 (1960).

Proceeding now in the same manner as described in Sec. II A, one obtains the analog of Eq. (8);

$$e_Z X_{\alpha\pi}=2\epsilon_\alpha X_{\alpha\pi}-G\rho_{\alpha\nu}(A-2,\,T_z+1)X_{\alpha\pi} -\tfrac{1}{2}G[1-2\rho_{\alpha\pi}(A-2,\,T_z+1)]\sum_\beta X_{\beta\pi}, \quad (36)$$

where

$$\rho_{\alpha\nu}(A-2,\,T_z+1) =\langle(A-2,\,T_z+1)0|\,a_{\alpha\nu}{}^\dagger a_{\alpha\nu}\,|(A-2,\,T_z+1)0\rangle \quad (37)$$

and

$$\rho_{\alpha\pi}(A-2,\,T_z+1) =\langle(A-2,\,T_z+1)0|\,a_{\alpha\pi}{}^\dagger a_{\alpha\pi}\,|(A-2,\,T_z+1)0\rangle. \quad (38)$$

The second term on the right-hand side of Eq. (36) represents the contribution arising from the neutron-proton interaction.

From Eq. (36), one derives the following dispersion formula for the eigenenergies, which is formally identical to (17):

$$\frac{1}{G}=\sum_{\alpha>0}\frac{1-2\rho_{\alpha\pi}(A-2,\,T_z+1)}{2\tilde{\epsilon}_{\alpha\nu}-e_Z}, \quad (39)$$

with

$$\tilde{\epsilon}_{\alpha\nu}=\epsilon_\alpha-\tfrac{1}{2}G\rho_{\alpha\nu}(A-2,\,T_z+1). \quad (40)$$

Thus the effect of the neutron-proton interaction amounts to a renormalization of the single-particle energies ϵ_α in the eigenvalue equation (36). This equation can be solved graphically, once the two occupation numbers $\rho_{\alpha\pi}(A-2,\,T_z+1)$ and $\rho_{\alpha\nu}(A-2,\,T_z+1)$ are known.

The equation for the energy difference $e_N=E_0(A,T_z)-E_0(A-2,\,T_z-1)$ between the ground states of two isotopes (Z,N) and $(Z,\,N-2)$ can be obtained by exchanging the role of protons and neutrons.

It reads

$$\frac{1}{G}=\sum_{\alpha>0}\frac{1-2\rho_{\alpha\nu}(A-2,\,T_z-1)}{2\tilde{\epsilon}_{\alpha\pi}-e_N}, \quad (41)$$

with

$$\tilde{\epsilon}_{\alpha\pi}=\epsilon_\alpha-\tfrac{1}{2}G\rho_{\alpha\pi}(A-2,\,T_z-1). \quad (42)$$

B. The Ground-State Wave Function

Starting from a system with given A and T_z and stepping down by successive approximations, one obtains the following wave function:

$$|(A,T_z)0\rangle=\prod_{p=1}^{Z/2}\Big[\sum_\alpha c_{\alpha\pi}(2p)a_{\alpha\pi}{}^\dagger a_{\bar{\alpha}\pi}{}^\dagger\Big] \times\prod_{p'=1}^{N/2}\Big[\sum_\beta c_{\beta\nu}(2p')a_{\beta\nu}{}^\dagger a_{\bar{\beta}\nu}{}^\dagger\Big]|0\rangle. \quad (43)$$

We can proceed now as in the case of identical particles and introduce the two average values

$$\bar{c}_{\alpha\pi}=\frac{2}{A-T_z}\sum_{p=1}^{Z/2}c_{\alpha\pi}(2p), \quad (44)$$

$$\bar{c}_{\alpha\nu}=\frac{2}{A+T_z}\sum_{p'=1}^{N/2}c_{\alpha\nu}(2p'), \quad (45)$$

and the relative deviations

$$\Delta_{\alpha\pi}(2p)=(1/\bar{c}_{\alpha\pi})[c_{\alpha\pi}(2p)-\bar{c}_{\alpha\pi}], \quad (46)$$

$$\Delta_{\alpha\nu}(2p')=(1/\bar{c}_{\alpha\nu})[c_{\alpha\nu}(2p')-\bar{c}_{\alpha\nu}]. \quad (47)$$

The wave function (43) can then be written as a sum of terms of increasing order in the Δ's:

$$|(A,T_z)0\rangle=(\sum_\alpha \bar{c}_{\alpha\pi}a_{\alpha\pi}{}^\dagger a_{\bar{\alpha}\pi}{}^\dagger)^{Z/2}(\sum_\beta \bar{c}_{\beta\nu}a_{\beta\nu}{}^\dagger a_{\bar{\beta}\nu}{}^\dagger)^{N/2}|0\rangle +0(\Delta^2)+\cdots. \quad (48)$$

The first term in (48) is of the form of the component with definite A and T_z of a BCS ground state, which is the product of a proton wave function times a neutron wave function. Such a product wave function $|\mathrm{BCS}\rangle_{\pi\nu}=|\mathrm{BCS}\rangle_\pi\times|\mathrm{BCS}\rangle_\nu$, which is an eigenstate of the pairing Hamiltonian $H_{pp}+H_{nn}$ within the usual quasiparticle approximation, has been the starting point of Elliott and Lea[19] for their treatment of pairing correlations between neutrons and protons.

As in the case of identical particles, we now note that if the $\Delta_{\alpha\pi}(2p)$ and $\Delta_{\alpha\nu}(2p)$ are $\ll1$ the second- and higher-order contributions on the right-hand side of (48) are small, and we can approximately write

$$|(A,T_z)0\rangle\approx(\sum_\alpha \bar{c}_{\alpha\pi}a_{\alpha\pi}{}^\dagger a_{\bar{\alpha}\pi}{}^\dagger)^{Z/2} \times(\sum_\beta \bar{c}_{\beta\nu}a_{\beta\nu}{}^\dagger a_{\bar{\beta}\nu}{}^\dagger)^{N/2}|0\rangle. \quad (49)$$

Clearly, both the wave functions (43) and (49), though conserving the number of protons and neutrons, i.e., A and T_z, do not have a definite isobaric spin T. We shall come back to this point in the discussion of the degenerate model.

C. The Degenerate Model

For the degenerate model, we have

$$\rho_\pi(A,T_z)=(A-2T_z)/4\Omega,\ \rho_\nu(A,T_z)=(A+2T_z)/4\Omega. \quad (50)$$

Hence e_Z and e_N are given by

$$e_Z=2\epsilon-(G/2\Omega)(\tfrac{1}{2}A+T_z)+G(\tfrac{1}{2}A-T_z-2)-G\Omega, \quad (51)$$

$$e_N=2\epsilon-(G/2\Omega)(\tfrac{1}{2}A-T_z)+G(\tfrac{1}{2}A+T_z-2)-G\Omega. \quad (52)$$

The pairing contribution $W_0(A,T_z)$ to the ground-state energy of the system with Z protons and N neutrons can

[19] J. P. Elliott and D. A. Lea, Phys. Letters 19, 291 (1965).

TABLE III. Values of σ [see Eq. (55)] in percent for the case $\Omega=10$.

A \ T_z	0	1	2	3	4	5	6	7	8	9	10
2		0									
4	4.29		0								
6		2.76		0							
8	4.21		2.00		0						
10		3.18		1.54		0					
12	4.12		2.50		1.22		0				
14		3.27		2.00		0.98		0			
16	4.00		2.63		1.60		0.77		0		
18		3.23		2.11		1.25		0.57		0	
20	3.85		2.58		1.64		0.91		0.34		0

then be written

$$W_0(A,T_z)=-(G/4\Omega)(\tfrac{1}{4}A^2-T_z^2)-\tfrac{1}{2}GA+\tfrac{1}{2}G(\tfrac{1}{4}A^2+T_z^2)-\tfrac{1}{2}GA\Omega. \quad (53)$$

Here the energy is no longer exact as in the case of identical particles. Actually the ground state (43), which does not contain any neutron-proton correlation, reduces in the degenerate case to an exact eigenstate of the charge-dependent pairing Hamiltonian $H_{pp}+H_{nn}$.

It is now interesting to compare the approximate energy (53) to the exact one, which can be written

$$W_0^{\text{exact}}(A,|T_z|)=-\tfrac{1}{2}G(\tfrac{3}{2}A-|T_z|)+\tfrac{1}{2}G(\tfrac{1}{4}A^2+T_z^2)-\tfrac{1}{2}GA\Omega. \quad (54)$$

Comparing the two expressions (53) and (54) (from now on we assume $T_z\geqslant 0$ for the sake of simplicity) we see that the relative error σ involved in (53) is

$$\sigma=\frac{W_0^{\text{exact}}-W_0}{W_0^{\text{exact}}}=\frac{1}{2\Omega}\frac{A(4\Omega-A)-4T_z(2\Omega-T_z)}{A(4\Omega-A+6)-4T_z(1+T_z)}. \quad (55)$$

It is easily seen that

$$0\leqslant\sigma<1/2\Omega, \quad (56)$$

the equality sign on the left-hand side occurring when $T_z=\frac{1}{2}A$, namely, when one has only one kind of particle.

In Table III we give the relative error σ, in percent, for all the possible values of A and T_z corresponding to $\Omega=10$. It appears that the energy (53) is a very good approximation to the exact ground-state energy (54). This is easily understood on the basis of the results obtained in Ref. 20. There the overlap of the eigenfunctions of $H=H_{pp}+H_{nn}+H_{np}$ (charge-independent pairing Hamiltonian) and $H'=H_{pp}+H_{nn}$ belonging to the eigenvalues (54) and (53), respectively, has been plotted as a function of T for different values of A. For $T>0$ the overlap is remarkably large (close to 100%), showing the lack of neutron-proton correlations in the ground state of even-even systems when the residual interaction is assumed to be a pairing force.

[20] J. Flores and P. A. Mello, Nucl. Phys. **88**, 609 (1966).

Several attempts[21–23] have been made thus far to take into account neutron-proton pairing correlations by means of a generalized Bogoliubov transformation. In this context it has been shown[23] for the degenerate case that the state of lowest energy is obtained when the generalized transformation reduces to a product of two ordinary Bogoliubov transformations on neutrons and proton separately. In other words, the quasiparticle ground state turns out to be the product of a $|\text{BCS}\rangle_\pi$ wave function for protons times a $|\text{BCS}\rangle_\nu$ wave function for neutrons.

The energy corresponding to the $|\text{BCS}\rangle_{\pi\nu}=|\text{BCS}\rangle_\pi\times|\text{BCS}\rangle_\nu$ ground state is written[23] for the degenerate model

$$W_0^{\text{BCS}}(A,T_z)=-(G/4\Omega)(\tfrac{3}{4}A^2+T_z^2)+\tfrac{1}{2}G(\tfrac{1}{4}A^2+T_z^2)-\tfrac{1}{2}GA\Omega. \quad (57)$$

Comparing this expression to the exact one (54), we obtain the relative error in (57), which we call σ':

$$\sigma'=\frac{W_0^{\text{exact}}-W_0^{\text{BCS}}}{W_0^{\text{exact}}}=\frac{1}{2\Omega}\times\frac{3A(4\Omega-A)-4T_z(2\Omega+T_z)}{A(4\Omega-A+6)-4T_z(1+T_z)}. \quad (58)$$

This error is bounded by the following limits:

$$1/2\Omega\leqslant\sigma'<3/2\Omega. \quad (59)$$

Clearly, for a given A the smallest value of σ' is obtained when $T_z=\frac{1}{2}A$, namely, in the case of identical particles. In particular the minimum value $1/2\Omega$ is reached only when $A=2\Omega-2$. The two errors σ and σ' can now be related to each other. One obtains

$$\sigma'=3\sigma+\frac{1}{2\Omega}\frac{16T_z(\Omega-T_z)}{A(4\Omega-A+6)-4T_z(1+T_z)}\geqslant 3\sigma, \quad (60)$$

the equality sign holding only when $T_z=0$ (aside from the trivial case $T_z=\Omega$, where $\sigma=\sigma'=0$).

[21] B. H. Flowers and M. Vujičić, Nucl. Phys. **49**, 586 (1963).
[22] A. Goswami, Nucl. Phys. **60**, 228 (1964).
[23] P. Camiz, A. Covello, and M. Jean, Nuovo Cimento **36**, 663 (1965); **42**, 199 (1966).

TABLE IV. Values of σ' [see Eq. (58)] in percent for the case $\Omega=10$.

A \ T_z	0	1	2	3	4	5	6	7	8	9	10
2		9.00									
4	12.86		8.89								
6		11.38		8.75							
8	12.63		10.57		8.75						
10		11.59		10.00		8.33					
12	12.35		10.83		9.51		8.00				
14		11.45		10.20		9.02		7.50			
16	12.00		10.70		9.60		8.46		6.67		
18		11.13		10.00		8.96		7.71		5.00	
20	11.54		10.32		9.27		8.18		6.55		0

In Table IV we give σ', in percent, for all the values of A and T_z for the case $\Omega=10$. We see then, as has already been pointed out in Ref. 23, that as far as the ground-state energy is concerned, the BCS approximation for a system of protons and neutrons is only slightly worse than the usual BCS approximation for identical particles. It is to be stressed, however, that the overlap of the $|\mathrm{BCS}\rangle_{\pi\nu}$ vacuum with the exact ground state for a system with a given number of particles A, isospin T, and its third component T_z, is fairly poor. In the case $\Omega=4$, for instance, the weight of the component with definite A, T_z and $T=T_z$ in the $|\mathrm{BCS}\rangle_{\pi\nu}$ wave function (with fixed average number of particles $\langle A\rangle=A$ and isospin projection $\langle T_z\rangle=T_z$) is always smaller than $\approx 18\%$,[20] which is much worse than is usually encountered for the case of identical particles.

On the other hand, as we discussed before, the ground-state wave function (43), which does conserve A and T_z has a very large overlap with the exact wave function for $T_z>0$. These considerations, which are valid in the degenerate case, suggest that the wave function (43) may also be useful for the treatment of even-even nuclei with $T_z\neq 0$ in the general case of nondenegerate single-particle levels.

IV. CONCLUSIONS

We have derived an approximate method to treat pairing correlations in even-even nuclei conserving the number of particles. One of the most interesting features of this method is that the eigenvalue problem for any number of particles is reduced to the solution of a dispersion relation. From the numerical results obtained for the model of the Ni isotopes, it appears that our approximation yields ground-state energies which are considerably better than the BCS method, and comparable to those obtained by means of other number-conserving approaches. This is well understood within the framework of the present approach when noting that the ground-state wave function may approximately reduce to the form of the A-particle component of the quasiparticle vacuum.

Concerning the possibility of treating neutron-proton pairing correlations by means of the same method, the study of the degenerate model encourages further work in this direction.

Finally, we wish to conclude by stating that the present treatment of pairing correlations should be considered as a first step towards the treatment of more realistic interactions. It remains to be seen, however, if suitable approximations can be devised to cope with the greater complexity of the problem.

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