Multiplying this out and using the recurrence relations for modified Bessel functions, we find

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}$$
$$\times(1-\alpha_1)^2N^{-4}\left\{t^3K_2(t)\int_t^\infty d\xi\,\xi^{-1}K_1(\xi)-t^2K_1^2(t)+N^{-1}t^4\left[\tfrac{1}{2}(3K_2(t)-tK_1(t))\int_t^\infty d\xi\,\xi^{-1}K_1(\xi)\right.\right.$$
$$\left.\left.-\tfrac{1}{2}K_1(t)(4t^{-1}K_1(t)-K_0(t))\right]+O(N^{-2})\right\}. \quad \text{(D2)}$$

When $T>T_c$, an analogous calculation gives the more accurate version of (8.85) of

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+(2\pi)^{-1}z^2(1-z_1)^2(\mathfrak{r}-1)^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-z_2^2)(z_2^2-z^2)^{-2}(1-\alpha_1)z_1^{-2}N^{-4}$$
$$\times\{t'^3K_2(t')-N^{-1}t'^4[K_2(t')+\tfrac{1}{2}t'K_1(t')]+O(N^{-2})\}$$
$$+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-\alpha_1)^2N^{-4}$$
$$\times\left\{t'^3K_2(t')\int_{t'}^\infty d\xi\,\xi^{-1}K_1(\xi)-t'^2K_1^2(t')+N^{-1}t'^4\right.$$
$$\left.\times\left[-\tfrac{1}{2}(t'K_1(t')+K_2(t'))\int_{t'}^\infty d\xi\,\xi^{-1}K_1(\xi)+\tfrac{1}{2}K_1(t')K_0(t')\right]+O(N^{-2})\right\}. \quad \text{(D3)}$$

By inspecting these equations, we see that as $t\to 0$ the terms of order N^{-4} remain finite while the terms of order N^{-5} vanish. This vanishing of the next leading order term at $T=T_c$ has already been seen in the bulk problem as presented in I where, while the leading term in S_N is proportional to $N^{-1/4}$, there is no $N^{-5/4}$ term and the next nonvanishing term is of order $N^{-9/4}$.

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Critical Temperatures of Anisotropic Ising Lattices.* I. Lower Bounds

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The critical or Curie temperature of the anisotropic rectangular Ising ferromagnet is known from Onsager's exact solution to vanish asymptotically as

$$kT_c/2J_x\sim[\ln(1/\eta)-\ln\ln(1/\eta)]^{-1}+\cdots,$$

when $\eta=J_y/J_x$, the ratio of exchange energies for bonds parallel to the y and x axes, approaches zero. An extension of the Peierls argument yields a simple interpretation of this slow decrease and provides, from first principles, a rigorous lower bound of precisely the same asymptotic form. For the anisotropic simple cubic lattice, a lower bound, also of this asymptotic form, is established in terms of $\eta=(J_y+J_z)/J_x$.

I. INTRODUCTION

THE problem of the Ising ferromagnet of spin $\frac{1}{2}$ with nearest-neighbor interaction has been studied extensively. It is well known that the one-dimensional model in the presence of an external magnetic field and various two-dimensional models in zero field are exactly soluble.[1] In particular, the spontaneous magnetization below the critical point has been calculated for both square and "rectangular" lattices.[2,3]

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† Alfred P. Sloan Research Fellow.

[1] L. Onsager, Phys. Rev. **65**, 117 (1944). A convenient review article has been written by G. F. Newell and E. W. Montroll, Rev. Mod. Phys. **25**, 353 (1953).
[2] C. N. Yang, Phys. Rev. **85**, 808 (1952).
[3] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).

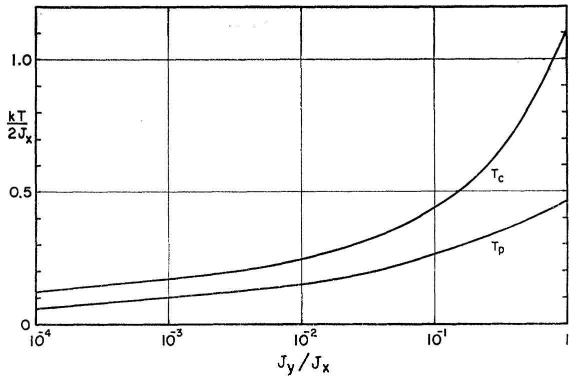

FIG. 1. Curie temperature of an anisotropic Ising ferromagnet on a square lattice as a function of J_y/J_x. The upper curve is the exact result using Eq. (2) and the lower curve is the modified Peierls estimate from the bound (11).

Many years ago Peierls[4] gave a simple argument for the existence of spontaneous magnetization in the two-dimensional square lattice. The argument was modified by Griffiths[5] and independently by Dobrushin[6] to provide a rigorous proof, which could be generalized immediately to a simple cubic lattice. Recently the argument has been extended to more general cases by Ginibre, Grossman, and Ruelle,[7] by Dobrushin,[8] and by Griffiths.[9]

In this paper we are concerned with the "rectangular" Ising lattice in which the spins ($\sigma_i = \pm 1$) lie on a simple square lattice but the interaction Hamiltonian is

$$\mathcal{H} = -J_x \sum_{ij}{}^{x} \sigma_i \sigma_j - J_y \sum_{kl}{}^{y} \sigma_k \sigma_l, \qquad (1)$$

where the first sum is over nearest-neighbor pairs joined by a bond parallel to the x axis, and the second over such pairs joined by a bond parallel to the y axis. The exact transition or Curie temperature T_c obtained by Onsager for this case, and given by the equation

$$(\sinh 2J_x/kT_c)(\sinh 2J_y/kT_c) = 1, \qquad (2)$$

has the interesting feature that it decreases only very slowly if J_x remains constant and J_y decreases to zero. The situation is shown by the upper curve in Fig. 1. For $J_y = 10^{-3} J_x$, T_c has decreased to only about $\frac{1}{7}$ its value at $J_y = J_x$. As $\eta = J_y/J_x$ approaches zero, one finds from (2):

$$\frac{kT_c}{2J_x} = (1/\ln\eta^{-1}) \left\{ 1 + \frac{\ln \ln\eta^{-1}}{\ln\eta^{-1}} + O(1/\ln\eta^{-1}) \right\}. \qquad (3)$$

Since the Peierls argument yields a lower bound on the spontaneous magnetization, it also yields a lower bound T_p on the Curie temperature, the temperature above which no spontaneous magnetization exists. We shall show firstly that for the "rectangular" lattice a simple modification of the Peierls argument yields a lower bound (lower curve in Fig. 1) which has the form (3) as $\eta \to 0$, apart from a factor of $\frac{1}{2}$ multiplying the right-hand side. The argument, discussed in Sec. II, also provides physical insight into the reason for the slow decrease of T_c.

A refinement of the combinatorial estimate entering the Peierls argument is discussed in Sec. III. The revised lower bound to the critical temperature, obtained with the aid of the generating function for random walks with no immediate reversals, has the precise form (3) as $\eta \to 0$ (with no extra factor of $\frac{1}{2}$).

The behavior of the critical temperature of an Ising ferromagnet on a simple cubic lattice as both J_y and J_z go to zero, J_x remaining constant, is discussed in Sec. IV. It is shown that a lower bound of the form (3) is valid with η set equal to $(J_x + J_y)/J_z$ and it is surmised that this form is in fact asymptotically exact. This conclusion is confirmed rigorously in a following paper[10] where general upper bounds for T_c are established.

II. MODIFIED PEIERLS ARGUMENT

Consider a lattice measuring $\sqrt{N} \times \sqrt{N}$ unit cells containing N spins. In any of the 2^N possible configurations the $+$ and $-$ spins may be separated into connected sets by means of noninteracting borders drawn on the dual lattice which is also a square lattice (Fig. 2). [In Ref. 5 the borders were assigned on orientation, but this is clearly unnecessary—see Refs. 6, 7.] We shall, for convenience, consider only configurations for which the $4(\sqrt{N}) - 2$ spins on the boundary of the square are constrained to be $+$, in which case the borders separating $+$ and $-$ spins form closed polygons on the dual lattice. If N_- denotes the number of $-$ spins in any configuration, the Peierls argument consists essentially in showing that at low enough temperatures,

$$N^{-1}\langle N_- \rangle < \tfrac{1}{2} - \epsilon, \qquad (4)$$

where ϵ is a positive constant independent of N, and the angular brackets denote a thermal average:

$$\langle \theta \rangle = \mathrm{Tr}\{\theta e^{-\mathcal{H}/kT}\} / \mathrm{Tr}\{e^{-\mathcal{H}/kT}\}. \qquad (5)$$

In fact (4) is equivalent to saying that the ratio of $+$ to $-$ spins exceeds $\frac{1}{2}$ in the limit of an infinite system; that is, there is a net spontaneous magnetiza-

[4] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 477 (1936).
[5] R. B. Griffiths, Phys. Rev. **136**, A437 (1964). We are indebted to D. G. Kelly for pointing out a minor error in this paper. The summations in Eqs. (13) and (14) should be over the values $b = 2, 3, 4, \cdots$ instead of $b = 4, 6, 8, \cdots$ since there is no requirement that the borders be closed. The final estimate on the right side of (14) is correspondingly altered.
[6] R. L. Dobrushin, Teoriya Veroyatnostei i ee Primeneniya **10**, 209 (1965) [English transl.: Theory Probability Appl. (USSR) **10**, 193 (1965)].
[7] J. Ginibre, A. Grossman, and D. Ruelle, Commun. Math. Phys. **3**, 187 (1966).
[8] R. L. Dobrushin (private communication).
[9] R. B. Griffiths, J. Math. Phys. **8**, 478, 484 (1967).
[10] M. E. Fisher, following paper, Phys. Rev. **160**, 480 (1967).

tion per spin. It has been shown elsewhere[11] that the result (4) indeed indicates a spontaneous magnetization in the thermodynamic sense of a discontinuity in the curve of magnetization versus field. The somewhat unusual boundary conditions are a convenience and not a necessity in carrying out the argument, as was shown in Ref. 5. (See also Ref. 7.) It should be noted that for a "lattice gas" these boundary conditions correspond to the natural requirement that the system be bounded by "hard walls."

Consider a particular border i constructed with $n_x(i)$ "vertical" lines perpendicular to the x axis (and hence intersecting J_x bonds) and $n_y(i)$ "horizontal" lines perpendicular to the y axis. The operator X_i is 1 if this border occurs in a particular configuration and zero otherwise. Since the border encloses at most $\frac{1}{4}n_x(i)n_y(i)$ spins, and since each minus spin is found within at least one border, N_- in a particular configuration has a bound

$$N_-\leq\tfrac{1}{4}\sum_i n_x(i)n_y(i)X_i. \tag{6}$$

An argument identical to that in Ref. 5 yields the bound

$$\langle X_i\rangle\leq\exp\{-2[J_xn_x(i)+J_yn_y(i)]/kT\} \tag{7}$$

for the thermal average of X_i.

The borders, which are closed polygons, may be divided into types j having identical shape and orientation. If there are $p(n,m)$ types with $n_x(j)=n$ and $n_y(j)=m$, then there are at most $Np(n,m)$ borders of the corresponding class. The thermal expectation of (6), together with the bound (7), yields

$$N^{-1}\langle N_-\rangle\leq\tfrac{1}{4}\sum_n\sum_m nmp(n,m)x^ny^m, \tag{8}$$

where

$$x=\exp(-2J_x/kT),\qquad y=\exp(-2J_y/kT). \tag{9}$$

We now seek a suitable simple bound for $p(n,m)$. A border or polygon with a particular shape and orientation may be laid out segment by segment, each segment connected to its predecessor, for example in a sequence $(x\,x\,y\,x\,y\,y\,y\,x\,x\cdots)$, where x and y denote segments perpendicular to their respective axes joining nearest-neighbor points on the dual lattice. Given n x segments and m y segments, there are $(n+m)!/n!m!$ possible sequences. More than one border may be associated with any sequence. In particular, if a y segment follows an x segment there are two ways to place the former: toward positive or negative values of x. The same freedom arises when an x segment follows a y segment. But when an x segment follows an x there is but one possible place for the second segment. If α is the number of times the sequence changes from x to y or vice versa, then at most 2^α borders may be associated with a particular sequence. It is evident that α cannot exceed $2n$. Finally, we should divide by $n+m$ since a particular type of polygon may be drawn by starting at any one of the $n+m$ points separating consecutive segments. The resulting bound is

$$p(n,m)\leq\frac{2^{2n}(n+m)!}{(n+m)n!m!}. \tag{10}$$

If this is inserted in (8) and $nm/(n+m)$ is replaced by the larger quantity n the bound will only be weakened. Similarly the summation may be extended to run over all integers from 0 to ∞ with the result

$$N^{-1}\langle N_-\rangle\leq x(1-y-4x)^{-2}. \tag{11}$$

If the right-hand side of this inequality is equated to $\frac{1}{2}$ and solved for the temperature T, we obtain the bound T_p for the exact critical temperature which is plotted in Fig. 1. For small $\eta=J_y/J_x$ this bound has the form (3) except that the right-hand side of the equation is multiplied by the factor $\frac{1}{2}$. Apart from this factor of $\frac{1}{2}$ our simple argument has provided a lower bound to T_c when η approaches zero quite similar to the exact result. Inspection of the bound (10) for the number of polygons yields some physical insight into the slow logarithmic decrease of the critical temperature. In particular, the number of types of possible polygons or borders for a particular value of $n+m$ becomes very small as n decreases. Putting it another way, and somewhat imprecisely, the entropy associated with borders for a fixed $n+m$ becomes very small as n decreases. This reflects the fact that in constructing a section of border containing only y segments there is no choice as to how to place succeeding segments once the first has been laid down. The remainder then simply follow along a straight line parallel to the x axis. In order to introduce additional freedom, or "entropy," in laying a border, it is necessary to introduce x segments, a procedure relatively costly in energy for $J_x\gg J_y$. Thus the free energy (energy minus T times the entropy) required to lay out a border (again we are speaking imprecisely) decreases rather slowly as J_y is "turned off," since the gain in lowering the

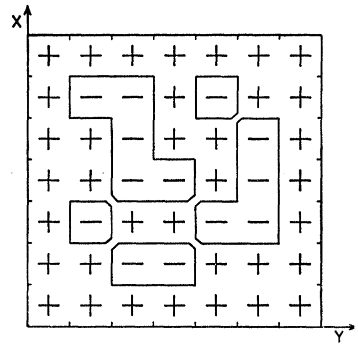

FIG. 2. A configuration for a square containing $N=49$ spins in which the 24 boundary spins are all +.

[11] R. B. Griffiths, Phys. Rev. 152, 240 (1966).

energy by using lots of y segments is offset by the corresponding decrease in entropy. One expects that the greater the free energy required to insert regions of reverse magnetization into the spontaneously magnetized state, the higher will be the Curie temperature at which spontaneous magnetization disappears.

III. IMPROVED LOWER BOUND FOR T_c

One may ask why the ratio T_p/T_c approaches $\frac{1}{2}$ rather than 1 as $\eta\rightarrow 0$ and whether the estimate derived above might not be improved. We shall show that the bound (10) for the number of polygons involves a serious overestimate since in deriving it no use was made of the fact that the borders must form *closed* figures. For example, a closed border certainly involves only an even number of both x and y segments. If one still uses (10) but extends the summation in (8) only over positive even integers m and n then one obtains a new bound T_p' satisfying $T_p'/T_c\rightarrow\frac{2}{3}$ as $\eta\rightarrow 0$. However, much better results are possible as we shall now show.

Consider the case $n=2$, $m\geq 2$, for which, clearly, $p(2, m)=1$ (a rectangular polygon of width 1 and height $\frac{1}{2}m$). Inserting this in (8) yields

$$N^{-1}\langle N_-\rangle \leq \tfrac{1}{2}\xi^2\frac{(1-\delta)^2}{(2-\delta)^2}+R_{4,2}(x, 1-\delta), \qquad (12)$$

where

$$\delta=1-y,$$
$$\xi=x/\delta, \qquad (13)$$

and $R_{4,2}(x, y)$ denotes the remaining sum for $n\geq 4$, $m\geq 2$. For small $\eta=J_y/J_x$, $\delta\approx 2J_y/kT$ approaches zero for T on the order of T_c, and the first term on the right side of (12) becomes a function of ξ alone. We shall show that the same holds for the remainder. Consequently, for small δ and η the bound becomes

$$N^{-1}\langle N^-\rangle\leq F(\xi). \qquad (14)$$

Let ξ_0 be the smallest solution of $F(\xi_0)=\frac{1}{2}$. Then a lower bound T_Q to T_c is obtained from (14):

$$\exp(-2J_x/kT_Q)\simeq\xi_0(2J_y/kT_Q), \qquad (15)$$

and it is easily shown that T_Q has the same asymptotic form (3) as T_c, up to terms of order $[-\ln\eta]^{-1}$ inside the braces.

To establish (14) generally, we argue as follows. The number of different types of polygons $p(n, m)$ is $(n+m)^{-1}$ times $u(n, m)$, the number of closed, self-avoiding random walks of n vertical and m horizontal steps commencing (and terminating) at the origin of the dual lattice. Clearly $u(n, m)$ is less than $q(n, m)$, the number of closed random walks in which immediate reversals are forbidden, but in which all other self-intersections are allowed. If one inserts

$$p(n, m)\leq(n+m)^{-1}q(n, m) \qquad (16)$$

in (8) and replaces $nm/(n+m)$ by n, as before, the result is

$$N^{-1}\langle N_-\rangle\leq\tfrac{1}{4}x\partial Q(x, y)/\partial x, \qquad (17)$$

where

$$Q(x, y)=\sum_n\sum_m q(n, m)x^ny^m \qquad (18)$$

is the generating function for the class of walks in question. This may be calculated as follows.

Let $q(n, m; k, l)$ be the number of random walks with n vertical and m horizontal steps without immediate reversals commencing at the origin and terminating at the point (k, l) on a square lattice. (Consistent with our previous terminology, we suppose the x axis vertical and the y axis horizontal.) The double generating function

$$\hat{Q}(x, y; \theta, \phi)=\sum_n\sum_m\sum_k\sum_l q(n, m; k, l)x^ny^me^{i\theta k}e^{i\phi l} \qquad (19)$$

will be constructed by decomposing the walks into classes according to the number and character of successive sequences of similar steps. Thus the generating function X for walks consisting only of vertical or x steps is evidently

$$X(x, \theta)=xe^{i\theta}+x^2e^{2i\theta}+\cdots+xe^{-i\theta}+x^2e^{-2i\theta}+\cdots \qquad (20)$$

or

$$X(x, \theta)=2x(\cos\theta-x)[1-2x\cos\theta+x^2]^{-1}, \qquad (21)$$

and for horizontal or y steps alone,

$$Y(y, \phi)=2y(\cos\phi-y)[1-2y\cos\phi+y^2]^{-1}. \qquad (22)$$

The total generating function may now be written down by considering all possible sequences of x and y steps so that

$$\hat{Q}(x, y; \theta, \phi)=1+X+Y+XY+YX+XYX+YXY$$
$$+XYXY+YXYX+XYXYX+YXYXY+\cdots. \qquad (23)$$

Provided y is less than 1 and x is sufficiently small, the series (19) and (23) are absolutely convergent and the latter may be summed in the form

$$\hat{Q}=\frac{(1+X)(1+Y)}{1-XY}=\frac{(1-x^2)(1-y^2)}{U-V\cos\theta-W\cos\phi}, \qquad (24)$$

with

$$U=1+x^2+y^2-3x^2y^2,$$
$$V=2x(1-y^2), \qquad W=2y(1-x^2). \qquad (25)$$

For the symmetric case $x=y=z$ this reduces to a formula derived previously by quite different arguments.[12]

The generating function $Q(x, y; k, l)$ for all walks

[12] C. Domb and M. E. Fisher, Proc. Cambridge Phil. Soc. **54**, 48 (1958).

to the point k, l may be found by Fourier inversion [see (19) and (18)]. In particular for the closed walks with $k=l=0$, we have

$$Q(x,y)=(2\pi)^{-2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\,\hat{Q}(x,y;\theta,\phi)$$

$$=\frac{2}{\pi}\frac{(1-x^2)(1-y^2)}{U^2-(V-W)^2}\mathbf{K}\left\{\left[\frac{4VW}{U^2-(V-W)^2}\right]^{1/2}\right\}, \quad (26)$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind. [The integral (26) may be evaluated using the result

$$\int_0^{2\pi}(d\phi/2\pi)(B-C\cos\phi)^{-1}=(B^2-C^2)^{-1/2}, \quad (27)$$

valid provided $|B|>|C|$, a condition which is fulfilled for any $y<1$ by choosing x less than some positive function of y. The substitution $z=-\cos\theta$ brings the remaining integral into a standard elliptic form.[13]]

In the limit $\delta=1-y\ll 1$ and $x<\delta/4$, (26) becomes

$$Q(x,y)\simeq\frac{2}{\pi[1+4x/\delta]^{1/2}}\mathbf{K}\left\{\left[\frac{8x/\delta}{1+4x/\delta}\right]^{1/2}\right\}, \quad (27')$$

a function of $\xi=x/\delta$ alone. Since $\mathbf{K}(w)$ is continuously differentiable and increases monotonically to $+\infty$ as w increases from 0 to 1, it is clear that

$$F(\xi)=\tfrac{1}{4}x[\partial Q(x,y)/\partial x]=\tfrac{1}{4}\xi(\partial Q/\partial\xi) \quad (28)$$

will attain all values between 0 and $+\infty$ for ξ between 0 and $\frac{1}{4}$. Thus there is a ξ_0 less than $\frac{1}{4}$ for which $F(\xi_0)=\frac{1}{2}$, and the bound T_Q is determined by (15) as $\eta\to 0$. One finds, in fact, $\xi_0\simeq 0.227$.

IV. THREE-DIMENSIONAL FERROMAGNET

Consider the general anisotropic Ising ferromagnet on a simple-cubic lattice with nearest-neighbor interactions J_x, J_y, and J_z corresponding to bonds parallel to the x, y, and z axes, respectively. It is of interest to ask how the critical temperature depends on J_y and J_z when these are both much smaller than J_x.[14] The usual Peierls argument[5–7] yields a lower bound for T_c which is of no help in estimating this asymptotic behavior. In a direct attack, following our discussion of the rectangular lattice, one would need a bound analogous to (10) or (16) for the number $p(l,m,n)$ of different types of polyhedra with l, m, and n faces perpendicular to the x, y, and z axes, respectively. This combinatorial problem is more difficult than for planar polygons and we have not obtained any useful results.

We may, however, sidestep such a procedure by using our bound for the two-dimensional lattice and appealing to the theorem that *the critical temperature for finite J_y and J_z is not less than the critical temperature obtained when either of these parameters is equated to zero* (the other remaining fixed).[9] But if, for example, J_z is equated to zero the simple-cubic lattice decomposes into a set of identical square lattices and the critical temperature is given by Onsager's result (2). Evidently, therefore, there is a lower bound with the asymptotic form (3) but with η replaced by

$$\eta'=\max\{J_y,J_z\}/J_x. \quad (29)$$

Now clearly we have

$$\eta'\geq\tfrac{1}{2}(J_y+J_z)/J_x=\tfrac{1}{2}\eta'', \quad (30)$$

so that (3) remains a valid bound if η is replaced by $\frac{1}{2}\eta''$. This in turn may be replaced asymptotically by η'' since the resulting correction to T_c is only of order $(1/\ln\eta^{-1})$ within the braces in (3). Finally therefore we have established a lower bound for the simple-cubic lattice with the asymptotic form (3) but with

$$\eta=(J_y+J_z)/J_x. \quad (31)$$

This is just what one would conclude following our method for the rectangular lattice if the analogous arguments could be pushed through in three dimensions.

It is natural to conjecture that, as in two dimensions, this lower bound is of the same asymptotic form as the *exact* critical temperature (which is not, of course, known explicitly). In support of this conjecture we may cite the results of the special molecular-field approximation of Stout and Chisholm[15] (appropriately modified to the ferromagnetic case) in which the linear chains parallel to the x axis are treated exactly while the weaker interchain y and z interactions are approximated by an "effective" magnetic field. For the rectangular lattice ($J_z=0$) this procedure yields an approximate critical temperature T_M which has the asymptotic form (3) but is always *larger* than the exact result. For the simple-cubic lattice the method yields an approximate critical temperature which is again expected to lie above the true value; it has the asymptotic form (3) but with η given by (31), as conjectured.

Of course, this is no proof since the molecular-field result is not known to be an upper bound, although this has always been observed "empirically." In the following paper,[10] however, rigorous general upper bounds for the critical temperature will be obtained. These prove, in particular, that mean field theory provides an upper bound and they confirm the present conjecture on the asymptotic behavior of T_c for the simple-cubic lattice when $(J_y+J_z)/J_x\to 0$.

[13] W. Gröbner and N. Hofreiter, *Integraltafel: Bestimmte Integrale* (Springer-Verlag, Vienna, 1961), 3rd ed., p. 47; A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, pp. 307–309.

[14] E. I. Nesis {Fiz. Tverd. Tela **7**, 665 (1965) [(English transl.: Soviet Phys.—Solid State **7**, 534 (1965)]} has discussed the case where $J_z\ll J_x=J_y$.

[15] J. W. Stout and R. C. Chisholm, J. Chem. Phys. **36**, 979 (1962).