Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. IV

BARRY M. McCOY*
Lyman Laboratory, Harvard University, Cambridge, Massachusetts
AND
TAI TSUN WU†
AND
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts‡ and Rockefeller University, New York, New York
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We consider the rectangular Ising model on a half-plane of infinite extent and study some of the consequences connected with the presence of the boundary. Only the spins on the boundary row are allowed to interact with a magnetic field $\mathfrak{H}$. The method of Pfaffians is employed to obtain exact expressions for the partition function. It is found that the free energy is the sum of two terms, one of which is independent of $\mathfrak{H}$ and proportional to the total number of lattice sites, while the other depends on $\mathfrak{H}$ and is proportional to the number of lattice sites on the boundary. This separation makes it possible to define various thermodynamic quantities associated with the boundary. In particular, the boundary magnetization is shown to be discontinuous, in the ferromagnetic case, at zero magnetic field for temperatures below the bulk critical temperature T_c. This discontinuity, which is the spontaneous boundary magnetization, goes to zero as $(1-T/T_c)^{1/2}$ as $T\to T_c-$. For $T=T_c$, the discontinuity is of course absent, and the boundary magnetization behaves as $-\mathfrak{H}\ln\mathfrak{H}$ for small $\mathfrak{H}$. The boundary susceptibility at zero magnetic field in the ferromagnetic case exhibits a logarithmic singularity at $T=T_c$, both above and below transition. An interesting feature is that the ferromagnetic boundary magnetization, although discontinuous for $T<T_c$, may be analytically continued beyond the point $\mathfrak{H}=0$. We interpret this as a hysteresis phenomenon which we study in detail by computing the probability distribution function for the average boundary spin. The correlation function for two spins, both on the boundary row, is also obtained exactly and its asymptotic behavior is given. Finally, we derive an expression for the magnetization in any row and explicitly evaluate it for the second row, i.e., the row next to the boundary.

1. INTRODUCTION

ALTHOUGH a great deal of effort has been spent on the two-dimensional Ising model, the amount of exact results is remarkably limited. For the case of the rectangular lattice without magnetic field, Onsager and Kaufman[1] have given the free energy per lattice site and also the correlation functions for spins at finite distances. In particular, it is readily observed that the expression for the two-spin correlation function becomes rapidly more and more complicated as the separation between the two spins increases. It is for this reason that it is quite difficult to obtain, as first accomplished by Yang,[2] the spontaneous magnetization, which is closely related to the limiting value at infinite separations of the two-spin correlation function. On the contrary, the exact expression for the four-spin correlation function

$$\langle\sigma_{0,0}\sigma_{0,1}\sigma_{M,N}\sigma_{M,N+1}\rangle \tag{1.1}$$

at zero magnetic field, for example, does not become more complex as M and N increases. Indeed, as is well known, the amount of work in writing down the correlation of an even number of spins depends mainly on the minimum distance D on the lattice which is required to join the spins pairwise. This distance is $|M|+|N|$ for $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ and is 2 for (1.1). Roughly speaking, the expression for the correlation is simple when the spins are grouped into nearby pairs.

It is the purpose of this paper to study an aspect of the two-dimensional Ising model where this pairing plays no role. The specific case to be considered is the Ising model on a half-plane of infinite extent. We are primarily interested in the correlation of spins near the boundary. In the simplest case, when two spins are both located on the boundary row, their correlation can be expressed in terms of a single integral, no matter what the distance is between these two spins. Accordingly, it is completely straightforward to calculate both the spontaneous magnetization and indeed the entire asymptotic series. Thus, this calculation follows a rather different route from that of the corresponding quantities for the usual two-dimensional Ising model.[2,3] In the present case, D is reinterpreted to be the minimum distance on the lattice which is required to join the spins either to each other or to any point outside the semi-infinite lattice. Thus, $D=2$ for any two widely separated spins on the boundary row.

* National Science Foundation Predoctoral Fellow. This work forms part of a thesis submitted to the Department of Physics, Harvard University, in partial fulfillment of the requirements for the Ph.D. degree. Present address: Institute for Theoretical Physics, State University of New York, Stony Brook, New York.
† National Science Foundation Senior Postdoctoral Fellow.
‡ Permanent address.
[1] L. Onsager, Phys. Rev. **65**, 117 (1944); B. Kaufman, *ibid.* **76**, 1232 (1949); B. Kaufman and L. Onsager, *ibid.* **76**, 1244 (1949).
[2] C. N. Yang, Phys. Rev. **85**, 808 (1952).
[3] T. T. Wu, Phys. Rev. **149**, 380 (1966). This paper is hereafter referred to as I.

It is interesting and also convenient to let the spins on the boundary row interact with a magnetic field. In Sec. 2, we formulate the problem with a finite number of rows and columns with the periodic boundary conditions applied in the horizontal direction only. By the method of using Pfaffians,[4,5] the partition function can be expressed in terms of a determinant. Since this determinant is nearly cyclic in the variable referring to the columns of the lattice, it can be written as a product of determinants, each of which turns out to have nonvanishing elements only if the two indices differ by 1 or zero. This evaluation of the partition function, and hence the free energy, is given in Sec. 3. When the number of rows and columns is large, the free energy is approximately, with exponentially small errors, the sum of a term proportional to the total number of lattice sites and a term proportional to the number of lattice sites on the boundary. From this we may identify the free energy of the bulk as given by Onsager,[1] the free energy of the boundaries at zero magnetic field, and the additional free energy of the boundary due to the presence of the magnetic field [see Eq. (3.27)]. The boundary entropy and specific heat are obtained in Sec. 4 by differentiating this free energy with respect to temperature. It is found that, contrary to the corresponding bulk quantities, the boundary entropy per boundary site has a logarithmic singularity and the specific heat contains a pole term when the temperature is equal to the bulk critical temperature.

We can also differentiate this free energy with respect to the magnetic field $\mathfrak{H}$ to obtain the magnetization $\mathfrak{M}_1$ of the boundary row. This is carried out in Sec. 5. If the interaction between nearest neighbors on the same row is antiferromagnetic in the sense that an antiparallel neighboring pair of spins has lower energy than a parallel pair, this boundary magnetization so obtained is an analytic function of T and $\mathfrak{H}$ except when $T=T_c$. Here, T_c means the bulk critical temperature of the infinite Ising model without magnetic field. In particular, T_c is independent of $\mathfrak{H}$. In the ferromagnetic case where a parallel neighboring pair on the same row has lower energy, then the boundary magnetization is analytic in T and $\mathfrak{H}$ except

$$T=T_c$$

and

$$T<T_c \quad \text{and} \quad \mathfrak{H}=0. \tag{1.2}$$

We discuss this ferromagnetic case in more detail. Across the line defined by (1.2), $\mathfrak{M}_1$ has a discontinuity associated with spontaneous magnetization. Near $T=T_c$, this spontaneous magnetization is proportional to $(1-T/T_c)^{1/2}$, which is to be compared with the eighth root behavior found by Yang[2] in the bulk case. At $T=T_c$ and if $\mathfrak{H}$ is small, $\mathfrak{M}_1$ is proportional to $-\mathfrak{H}\ln\mathfrak{H}$. The boundary susceptibility at zero magnetic field shows a logarithmic singularity as $T\to T_c$ both from above and below; this is to be contrasted with the usual power behavior in the bulk case, obtained numerically by Baker.[6] In Sec. 5, we also give the behavior of $\mathfrak{M}_1$ when T is near T_c and $\mathfrak{H}$ is small. Even though $\mathfrak{M}_1$ is discontinuous across the line (1.2), for fixed $T<T_c$, $\mathfrak{M}_1$ has the important property that it can be analytically continued in $\mathfrak{H}$ beyond this line. This analytically continued function becomes equal to $\mathfrak{M}_1$ itself for some finite value of $\mathfrak{H}$, which of course depends on T. At least part of the analytically continued curve may be identified with a hysteresis loop.

In order to better understand this interesting phenomenon of the hysteresis loop, we compute in Sec. 6 the probability distribution function for the average boundary spin. This probability distribution function has the properties that (i) its dependence on $\mathfrak{H}$ is simple [see Eq. (6.13)], and (ii) it can be expressed simply in terms of the partition function for complex values of $\mathfrak{H}$ [see Eq. (6.1)]. By the latter property, it can be evaluated by the method of steepest descent. It is verified that at $\mathfrak{H}=0$ it has two maxima below the critical temperature, as expected from the existence of spontaneous magnetization. When $\mathfrak{H}$ is positive and small, both maxima are present but the distribution function is exponentially larger at one of the maxima, say the right one, than at the other, say the left one. When $\mathfrak{H}$ is decreased to negative values with $|\mathfrak{H}|$ still sufficiently small, the distribution is much larger at the left maximum given by $\mathfrak{M}_1$; the analytic continuation of $\mathfrak{M}_1$ gives the position of the lesser maximum at the right. As $\mathfrak{H}$ is further decreased, the position of this lesser maximum moves to smaller average values of the boundary spins; it reaches zero at some negative value of $\mathfrak{H}$, say $-\mathfrak{H}_c$. For $\mathfrak{H}<-\mathfrak{H}_c$, even though further analytic continuation of $\mathfrak{M}_1$ is possible, the distribution function shows only one maximum, with the previous lesser maximum appearing only as a shoulder. We conclude that the portion of the analytic continuation of $\mathfrak{M}_1$ with $|\mathfrak{H}|<\mathfrak{H}_c$ can be identified with the hysteresis loop.

A possible physical interpretation of this mathematical result is as follows. For a system in thermodynamic equilibrium with $T<T_c$ and $\mathfrak{H}\neq 0$, the average values of the boundary spins is almost certainly close to $\mathfrak{M}_1$; these are the stable states. As $\mathfrak{H}$ is reduced from a small positive value to a small negative value, this average value changes sign. Since it is difficult to make transition between states of these opposite values of average boundary spin, it takes a very long time to reach thermodynamic equilibrium even after $\mathfrak{H}$ is made negative. For a time (short compared with the time needed to approach this equilibrium), the average value

[4] P. W. Kasteleyn, Physica **27**, 1209 (1961); H. N. V. Temperley and M. E. Fisher, Phil. Mag. **6**, 1061 (1961).

[5] E. W. Montroll, R. B. Potts, and J. C. Ward, J. Math. Phys. **4**, 308 (1963).

[6] G. A. Baker, Phys. Rev. **124**, 768 (1961); C. Domb and M. F. Sykes, J. Math. Phys. **2**, 63 (1961); J. W. Essam and M. E. Fisher, J. Chem. Phys. **38**, 802 (1963).

of the boundary spins for the system refuses to change sign; instead, it follows the position of the lesser maximum. These are the metastable states. The long lifetime of the metastable states is due to the small value of the distribution function between the two maxima. When $\mathfrak{H}$ reaches $-\mathfrak{H}_c$, these metastable states can become completely unstable and beyond that, the behavior of the system depends on the atomic mechanism and cannot be deduced by statistical considerations alone. Two possible hysteresis loops are shown in Figs. 6 and 9. Note that, for this interpretation, we have introduced the concepts of quantum-mechanical transition and of time, both foreign to the Ising model.

So far, all the results can be derived from a knowledge of the partition function alone. In Sec. 8, we turn to the question of the two-spin correlation functions. It is then not sufficient to know the partition function only, and we use the Pfaffian method of calculating averages of spin products. For this purpose, we must be able to compute certain elements of the inverse of the matrix whose Pfaffian gives the partition function. For the same reason that the partition function is easily obtained, these inverse matrix elements are not hard to get, as shown in Sec. 7. In all cases with $T\neq T_c$, the correlation functions of two spins on the boundary row approach their limiting values exponentially. However, in the ferromagnetic case below T_c, the rate of the exponential fall-off depends on whether $|\mathfrak{H}|$ is above or below the value at which the analytic continuation of $\mathfrak{M}_1$ meets $\mathfrak{M}_1$ itself. Furthermore, for $T=T_c$ and $\mathfrak{H}=0$, this correlation function approaches zero as the inverse of the separation, instead of the inverse fourth root for the bulk case as shown in *I*. For $T=T_c$ but $\mathfrak{H}\neq 0$, it falls off as the inverse fourth power of the separation. The behavior for large separation but $N|1-T/T_c|$ fixed and of order 1 is also studied.

In Sec. 9, we make some remarks about the thermodynamic averages of spins not on the boundary row. A great deal of exact calculations can be carried out in a straightforward, though tedious, manner, but we make no attempt to do this systematically. We mention here only one result. The spontaneous magnetization on the row next to the boundary, in the ferromagnetic case, is found to be also proportional to $(1-T/T_c)^{1/2}$ as $T\to T_c-$. It is believed that this behavior holds for any fixed row from the boundary.

The results are summarized in Sec. 10.

2. FORMULATION OF THE PROBLEM

The system to be studied in this paper is a two-dimensional, rectangular Ising model with cyclic boundary conditions imposed in the horizontal direction only. The lattice has $2\mathfrak{M}$ rows and $2\mathfrak{N}$ columns and interacts with a magnetic field $\mathfrak{H}$ applied to one of the two boundary rows (defined to be the first row). We use German letters to denote quantities pertaining to the boundary. The Hamiltonian for this system is

$$\mathcal{E}=-E_1\sum_{j=1}^{2\mathfrak{M}}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j,k+1}-E_2\sum_{j=1}^{2\mathfrak{M}-1}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{j,k}\sigma_{j+1,k}-\mathfrak{H}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}, \tag{2.1}$$

where each σ is equal to $+1$ or -1, j and k label, respectively, the row and column of the lattice site with $k=\mathfrak{N}+1$ identified with $k=-\mathfrak{N}+1$, and $E_1(E_2)$ is the horizontal (vertical) interaction energy between neighboring spins. The first row does *not* interact with the $2\mathfrak{M}$th row. In (2.1), the magnetic moment factor for the spins on the first row has been absorbed in $\mathfrak{H}$. We shall be interested in the limit $\mathfrak{M}\to\infty$ and $\mathfrak{N}\to\infty$ where the cylinder becomes a semi-infinite half-plane; only in this limit will a phase transition occur.

With (2.1), the partition function is

$$\begin{aligned}Z&=\sum_{\sigma=\pm1}e^{-\beta\mathcal{E}}\\&=\sum_{\sigma=\pm1}\exp\Big[\sum_{j=1}^{2\mathfrak{M}}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\beta E_1\sigma_{j,k}\sigma_{j,k+1}+\sum_{j=1}^{2\mathfrak{M}-1}\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\beta E_2\sigma_{j,k}\sigma_{j+1,k}+\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\beta\mathfrak{H}\sigma_{1,k}\Big]\\&=(\cosh\beta E_1)^{4\mathfrak{N}\mathfrak{M}}(\cosh\beta E_2)^{2\mathfrak{N}(2\mathfrak{M}-1)}(\cosh\beta\mathfrak{H})^{2\mathfrak{N}}\\&\quad\times\sum_{\sigma=\pm1}\Big\{\Big[\prod_{j=1}^{2\mathfrak{M}}\prod_{k=-\mathfrak{N}+1}^{\mathfrak{N}}(1+z_1\sigma_{j,k}\sigma_{j,k+1})\Big]\Big[\prod_{j=1}^{2\mathfrak{M}-1}\prod_{k=-\mathfrak{N}+1}^{\mathfrak{N}}(1+z_2\sigma_{j,k}\sigma_{j+1,k})\Big]\Big[\prod_{k=-\mathfrak{N}+1}^{\mathfrak{N}}(1+z\sigma_{1,k})\Big]\Big\},\end{aligned} \tag{2.2}$$

where

$$z_1=\tanh\beta E_1,$$

$$z_2=\tanh\beta E_2,$$

and

$$z=\tanh\beta\mathfrak{H}. \tag{2.3}$$

If the sum over $\sigma=\pm1$ is carried out, the result is[7]

$$Z=(2\cosh\beta E_1)^{4\mathfrak{M}\mathfrak{N}}(\cosh\beta E_2)^{2\mathfrak{N}(2\mathfrak{M}-1)}(\cosh\beta\mathfrak{H})^{2\mathfrak{N}}\sum_{p,q,r}z_1^pz_2^qz^rN_{pqr}, \tag{2.4}$$

[7] R. B. Potts and J. C. Ward, Progr. Theoret. Phys. (Kyoto) 13, 38 (1955).

where N_{pqr} is the number of figures that can be drawn on the lattice with the following properties. First, each bond between nearest neighbors may be used, at most, once. Secondly, the figure contains p horizontal bonds and q vertical bonds. Thirdly, let e_{jk} be the number of bonds with the site (j, k) as one end; then, for $j>1$, e_{jk} is even, i.e., $e_{jk}=0$, 2, or 4. And lastly, r is the number of e_{1k} which is odd. An example with $p=12$, $q=14$, and $r=4$ is shown in Fig. 1(a).

We wish to express the sum in (2.4) in terms of an appropriate Pfaffian.[4,5] To do so, we first note that if z is zero, then in the sum it is sufficient to keep only the terms with N_{pq0}, which is the number of *closed* polygons with p horizontal bonds and q vertical bonds. The factor $z_1^p z_2^q$ is taken into account by associating a factor z_1 with each horizontal bond and a z_2 with each vertical bond, as shown in Fig. 2. This procedure may also be followed for the case of general z by adding a

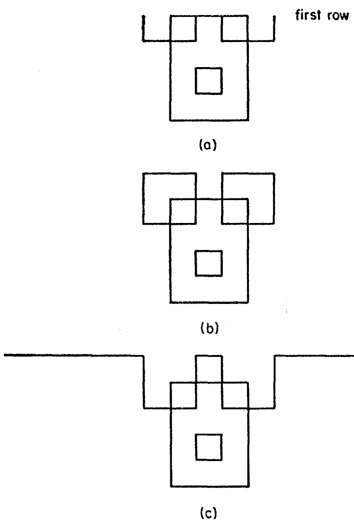

FIG. 1. An example of a figure with $p=12$, $q=14$, and $r=4$.

zeroth row of sites connected to the first row of sites by vertical bonds of weight z. The sites in this zeroth row are also connected to each other by bonds of weight 1 between nearest neighbors, as shown in Fig. 3(a). Each figure on the original lattice counted in N_{pqr} corresponds, because of the cyclic boundary condition in the horizontal direction, to *two* closed polygons on the lattice in Fig. 3(a). These polygons that correspond to the example of Fig. 1(a) are shown in Figs. 1(b) and 1(c). Each of the closed polygons has p horizontal bonds not including those on the zeroth row, and $p+r$ vertical bonds, of which r are between the zeroth row and the first row. That there are two closed polygons is clear from the example of Fig. 1; in case $r=0$, either all the bonds on the zeroth row are used or none is used. We have thus reduced the problem of evaluating (2.4) to that of finding the generating

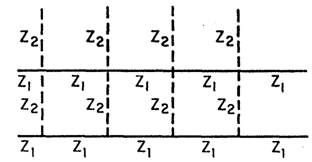

FIG. 2. Lattice with weights z_1 and z_2.

function for closed polygons on the lattice of Fig. 3(a). The solution is immediately given in terms of the Pfaffian for the dimer problem of Fig. 3(b). More explicitly,[5]

$$Z=\tfrac{1}{2}(2\cosh\beta E_1)^{4\mathfrak{M}\mathfrak{N}}(\cosh\beta E_2)^{2\mathfrak{N}(2\mathfrak{M}-1)}(\cosh\beta\mathfrak{H})^{2\mathfrak{N}}\,\mathrm{Pf}\mathfrak{A},\tag{2.5}$$

where the antisymmetrical matrix $\mathfrak{A}$ is given by

$$\mathfrak{A}(j,k;j,k)=\begin{array}{c} \\ R\\ L\\ U\\ D\end{array}\!\!\begin{array}{c}\begin{array}{cccc}R & L & U & D\end{array}\\ \begin{bmatrix}0 & 1 & -1 & -1\\ -1 & 0 & 1 & -1\\ 1 & -1 & 0 & 1\\ 1 & 1 & -1 & 0\end{bmatrix}\end{array}\tag{2.6a}$$

for $0\leq j\leq 2\mathfrak{M}$ and $-\mathfrak{N}+1\leq k\leq\mathfrak{N}$,

$$\mathfrak{A}(j,k;j,k+1)=-\mathfrak{A}^T(j,k+1;j,k)=\begin{bmatrix}0 & z_1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\tag{2.6b}$$

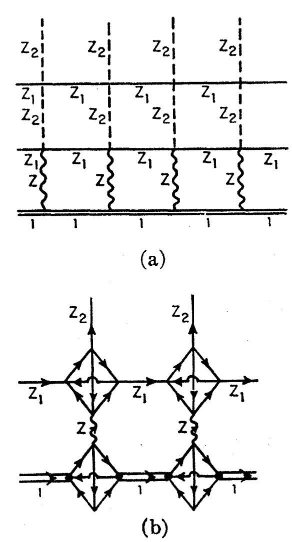

FIG. 3(a). Lattice representing a half-plane of Ising spins interacting with a magnetic field applied to the boundary row; (b) oriented half-plane lattice of Ising spins used to compute the matrix $\mathfrak{A}$.

for $1\le j\le 2\mathfrak{M}$ and $-\mathfrak{N}+1\le k\le \mathfrak{N}-1$,

$$\mathfrak{A}(0,k;0,k+1)=-\mathfrak{A}^T(0,k+1;0,k)=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \tag{2.6c}$$

for $-\mathfrak{N}+1\le k\le \mathfrak{N}-1$,

$$\mathfrak{A}(j,\mathfrak{N};j,-\mathfrak{N}+1) =-\mathfrak{A}^T(j,-\mathfrak{N}+1;j,\mathfrak{N})=-\mathfrak{A}(j,0;j,1) \tag{2.6d}$$

for $0\le j\le 2\mathfrak{M}$,

$$\mathfrak{A}(j,k;j+1,k)=-\mathfrak{A}^T(j+1,k;j,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z_2\\0&0&0&0\end{bmatrix} \tag{2.6e}$$

for $1\le j\le 2\mathfrak{M}$ and $-\mathfrak{N}+1\le k\le \mathfrak{N}$,

$$\mathfrak{A}(0,k;1,k)=-\mathfrak{A}^T(1,k;0,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z\\0&0&0&0\end{bmatrix} \tag{2.6f}$$

for $-\mathfrak{N}+1\le k\le \mathfrak{N}$, and all the other elements of $\mathfrak{A}$ are zero. In (2.6d), an extra minus sign is introduced in the weight for the bonds between the $(-\mathfrak{N}+1)$th column and the $\mathfrak{N}$th column; as is well known,[8] this is required to make the Pfaffian count correctly for a lattice with cyclic boundary conditions. Using tht connection between Pfaffians and determinants, we finally obtain

$$Z^2=\tfrac{1}{4}(2\cosh\beta E_1)^{8\mathfrak{M}\mathfrak{N}}(\cosh\beta E_2)^{4\mathfrak{N}(2\mathfrak{M}-1)} \times(\cosh\beta\mathfrak{H})^{4\mathfrak{N}}\det\mathfrak{A}. \tag{2.7}$$

In Appendix A, we give an alternative deviation which is more physical but mathematically less satisfactory.

We proceed to discuss the expectation values of various products of σ's. The simplest case is the magnetization on the Jth row for $J\ge 1$,

$$\mathfrak{M}_J=\lim_{\mathfrak{M},\mathfrak{N}\to\infty}\langle\sigma_{J,k}\rangle, \tag{2.8}$$

which is, of course, independent of k. More explicitly,

$$\mathfrak{M}_J=\lim_{\mathfrak{M},\mathfrak{N}\to\infty} Z^{-1}\sum_{\sigma=\pm1}\sigma_{J,0}e^{-\beta\mathcal{E}}. \tag{2.9}$$

[8] C. Domb, Advan. Phys. 9, 149 (1960).

By (2.2), we get in particular for $J=1$

$$\mathfrak{M}_1=\beta^{-1}\lim_{\mathfrak{M},\mathfrak{N}\to\infty}(2\mathfrak{N})^{-1}(\partial/\partial\mathfrak{H})\ln Z. \tag{2.10}$$

More generally, however, a knowledge of the free energy is insufficient to determine $\mathfrak{M}_J$ for $J>1$. To find $\mathfrak{M}_J$, note that

$$\sigma_{J,0}=\sigma_{1,0}(\sigma_{1,0}\sigma_{2,0})(\sigma_{2,0}\sigma_{3,0})\cdots(\sigma_{J-1,0}\sigma_{J,0}). \tag{2.11}$$

We therefore define a $8\mathfrak{N}(2\mathfrak{M}+1)\times 8\mathfrak{N}(2\mathfrak{M}+1)$ matrix $\delta^{(J)}$ by

$$\delta^{(J)}(j,0;j+1,0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z_2^{-1}-z_2\\0&0&0&0\end{bmatrix}$$

and

$$\delta^{(J)}(j+1,0;j,0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-(z_2^{-1}-z_2)&0\end{bmatrix} \tag{2.12}$$

for $1\le j\le J-1$,

$$\delta^{(J)}(0,0;1,0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z^{-1}-z\\0&0&0&0\end{bmatrix},$$

$$\delta^{(J)}(1,0;0,0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-(z^{-1}-z)&0\end{bmatrix}, \tag{2.13}$$

and all other elements of $\delta^{(J)}$ are zero. In terms of $\delta^{(J)}$, $\mathfrak{M}_J$ is given by[5,7]

$$\mathfrak{M}_J=zz_2^{J-1}\lim_{\mathfrak{M},\mathfrak{N}\to\infty}\mathrm{Pf}(\mathfrak{A}+\delta^{(J)})/\mathrm{Pf}(\mathfrak{A}) =\pm zz_2^{J-1}[\lim_{\mathfrak{M},\mathfrak{N}\to\infty}\det(1+\mathfrak{A}^{-1}\delta^{(J)})]^{1/2}. \tag{2.14}$$

A knowledge of $\mathfrak{A}^{-1}$ is therefore needed; this problem of finding the inverse of $\mathfrak{A}$ is studied in Sec. 7.

Spin–spin correlation functions are not much more complicated. Explicitly,

$$\langle\sigma_{J,0}\sigma_{J',N}\rangle=\pm z^2z_2^{J+J'-2}[\lim_{\mathfrak{M},\mathfrak{N}\to\infty}\det(1+\mathfrak{A}^{-1}\delta^{(J,J',N)})]^{1/2}, \tag{2.15}$$

where $\delta^{(J,J',N)}$ is defined, similar to Eqs. (2.12) and

(2.13), by

$$\delta^{(J,J',N)}(j,k;j+1,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z_2^{-1}-z_2\\0&0&0&0\end{bmatrix}$$

and

$$\delta^{(J,J',N)}(j+1,k;j,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-(z_2^{-1}-z_2)&0\end{bmatrix} \tag{2.16}$$

for $1\leq j\leq J-1$ when $k=0$ and $1\leq j\leq J'-1$ when $k=N$,

$$\delta^{(J,J',N)}(0,k;1,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z^{-1}-z\\0&0&0&0\end{bmatrix}$$

and

$$\delta^{(J,J',N)}(1,k;0,k)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-(z^{-1}-z)&0\end{bmatrix} \tag{2.17}$$

for $k=0$ or N, and all the other elements of $\delta^{(J,J',N)}$ are zero. Note that the number of nonvanishing elements of $\delta^{(J,J',N)}$ is independent of N.

Higher-order correlation functions can be easily written down in the same form.

3. PARTITION FUNCTION

This section is to be devoted to the evaluation of $\det\mathfrak{A}$, which appears on the right-hand side of (2.7). We first note that $\mathfrak{A}$ is nearly cyclic in the horizontal direction; accordingly,

$$\det\mathfrak{A}=\Pi_\theta \det\mathfrak{B}(\theta), \tag{3.1}$$

where the product is over the values

$$\theta=i\pi(2n-1)/(2\mathfrak{N}) \tag{3.2}$$

with $n=1, 2, 3, \cdots, 2\mathfrak{N}$, and $\mathfrak{B}(\theta)$ is a $4(2\mathfrak{N}+1)\times 4(2\mathfrak{N}+1)$ matrix defined by

$$\mathfrak{B}_{j,j}(\theta)=\begin{array}{c}\\R\\L\\U\\D\end{array}\begin{array}{c}\begin{array}{cccc}R&L&U&D\end{array}\\\begin{bmatrix}0&1+z_1e^{i\theta}&-1&-1\\-1-z_1e^{-i\theta}&0&1&-1\\1&-1&0&1\\1&1&-1&0\end{bmatrix}\end{array} \tag{3.3a}$$

for $1\leq j\leq 2\mathfrak{N}$,

$$\mathfrak{B}_{0,0}(\theta)=\begin{bmatrix}0&1+e^{i\theta}&-1&-1\\-1-e^{-i\theta}&0&1&-1\\1&-1&0&1\\1&1&-1&0\end{bmatrix}, \tag{3.3b}$$

$$\mathfrak{B}_{j,j+1}(\theta)=-\mathfrak{B}_{j+1,j}{}^T(\theta)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z_2\\0&0&0&0\end{bmatrix} \tag{3.3c}$$

for $1\leq j\leq 2\mathfrak{N}-1$,

$$\mathfrak{B}_{0,1}(\theta)=-\mathfrak{B}_{1,0}{}^T(\theta)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&z\\0&0&0&0\end{bmatrix}, \tag{3.3d}$$

and all the other matrix elements are zero. Since both $\delta^{(J)}$ and $\delta^{(J,J',N)}$ have only nonvanishing matrix elements for rows and columns labeled by U and D, it is convenient to eliminate all rows and columns labeled by R and L in $\mathfrak{B}(\theta)$. For this purpose, let $\mathfrak{T}(\theta)$ be the $4(2\mathfrak{N}+1)\times4(2\mathfrak{N}+1)$ matrix with

$$\mathfrak{T}_{j,j}(\theta)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\(1+z_1e^{i\theta})^{-1}&(1+z_1e^{-i\theta})^{-1}&1&0\\-(1+z_1e^{i\theta})^{-1}&(1+z_1e^{-i\theta})^{-1}&0&1\end{bmatrix} \tag{3.4a}$$

for $1\leq j\leq 2\mathfrak{N}$,

$$\mathfrak{T}_{0,0}(\theta)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\(1+e^{i\theta})^{-1}&(1+e^{-i\theta})^{-1}&1&0\\-(1+e^{i\theta})^{-1}&(1+e^{-i\theta})^{-1}&0&1\end{bmatrix}, \tag{3.4b}$$

and all other matrix elements are zero. Let

$$\mathfrak{B}'(\theta)=\mathfrak{T}(\theta)\mathfrak{B}(\theta); \tag{3.5}$$

then by (3.3) and (3.4), $\mathfrak{B}'(\theta)$ is given by

$$\mathfrak{B}_{j,j}'(\theta)=\begin{bmatrix} 0 & 1+z_1e^{i\theta} & -1 & -1 \\ -1-z_1e^{-i\theta} & 0 & 1 & -1 \\ 0 & 0 & 2iz_1\sin\theta\,|1+z_1e^{i\theta}|^{-2} & -(1-z_1^2)\,|1+z_1e^{i\theta}|^{-2} \\ 0 & 0 & (1-z_1^2)\,|1+z_1e^{i\theta}|^{-2} & -2iz_1\sin\theta\,|1+z_1e^{i\theta}|^{-2} \end{bmatrix} \qquad (3.6a)$$

for $1\leq j\leq 2\mathfrak{M}$,

$$\mathfrak{B}_{0,0}'(\theta)=\begin{bmatrix} 0 & 1+e^{i\theta} & -1 & -1 \\ -1-e^{-i\theta} & 0 & 1 & -1 \\ 0 & 0 & 2i\sin\theta\,|1+e^{i\theta}|^{-2} & 0 \\ 0 & 0 & 0 & -2i\sin\theta\,|1+e^{i\theta}|^{-2} \end{bmatrix}, \qquad (3.6b)$$

and all the other matrix elements are identical to those of $\mathfrak{B}(\theta)$. Because of (3.6), it is convenient to introduce the symbols

$$\mathfrak{a}=2iz_1\sin\theta\,|1+z_1e^{i\theta}|^{-2}, \qquad (3.7a)$$

$$\mathfrak{b}=(1-z_1^2)\,|1+z_1e^{i\theta}|^{-2}, \qquad (3.7b)$$

$$\mathfrak{c}=2i\sin\theta\,|1+e^{i\theta}|^{-2}, \qquad (3.7c)$$

and the $2(2\mathfrak{M}+1)\times 2(2\mathfrak{M}+1)$ matrix $\mathfrak{C}(\theta)$ defined by

$$\mathfrak{C}_{j,j}(\theta)=\begin{matrix} & \begin{matrix} D & U \end{matrix} \\ \begin{matrix} D \\ U \end{matrix} & \begin{bmatrix} -\mathfrak{a} & \mathfrak{b} \\ -\mathfrak{b} & \mathfrak{a} \end{bmatrix} \end{matrix} \qquad (3.8a)$$

for $1\leq j\leq 2\mathfrak{M}$,

$$\mathfrak{C}_{0,0}(\theta)=\begin{bmatrix} -\mathfrak{c} & 0 \\ 0 & \mathfrak{c} \end{bmatrix}, \qquad (3.8b)$$

$$\mathfrak{C}_{j,j+1}(\theta)=-\mathfrak{C}_{j+1,j}^{T}(\theta)=\begin{bmatrix} 0 & 0 \\ z_2 & 0 \end{bmatrix} \qquad (3.8c)$$

for $1\leq j\leq 2\mathfrak{M}-1$,

$$\mathfrak{C}_{0,1}(\theta)=-\mathfrak{C}_{1,0}^{T}(\theta)=\begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}, \qquad (3.8d)$$

and again all the other elements are zero. More explicitly, the matrix $\mathfrak{C}(\theta)$ is of the following form:

$$\begin{bmatrix} -\mathfrak{c} & 0 & & & & & & & \\ 0 & \mathfrak{c} & z & & & & & & \\ & -z & -\mathfrak{a} & \mathfrak{b} & & & & & \\ & & -\mathfrak{b} & \mathfrak{a} & z_2 & & & & \\ & & & -z_2 & -\mathfrak{a} & \mathfrak{b} & & & \\ & & & & -\mathfrak{b} & \mathfrak{a} & z_2 & & \\ & & & & & -z_2 & \ddots & & \\ & & & & & & \ddots & -\mathfrak{a} & \mathfrak{b} \\ & & & & & & & -\mathfrak{b} & \mathfrak{a} \end{bmatrix}, \qquad (3.9)$$

and $\det\mathfrak{A}$ is given by

$$\det\mathfrak{A}=\Pi_\theta[\,|1+e^{i\theta}|^2\,|1+z_1e^{i\theta}|^{4\mathfrak{M}}\det\mathfrak{C}(\theta)]. \qquad (3.10)$$

Let $\mathfrak{C}_n(\theta)$ be the determinant of the $2(n+1)\times 2(n+1)$ matrix of the form (3.9), and $\mathfrak{D}_n(\theta)$ be the corresponding $(2n+1)\times(2n+1)$ determinant with the last row and last column removed; then

$$\det\mathfrak{C}(\theta)=\mathfrak{C}_{2\mathfrak{M}}(\theta). \qquad (3.11)$$

The recurrence relations for $\mathfrak{C}_n(\theta)$ and $\mathfrak{D}_n(\theta)$ are

$$\begin{bmatrix} \mathfrak{C}_n(\theta) \\ z_2\mathfrak{D}_n(\theta) \end{bmatrix}=\begin{bmatrix} -\mathfrak{a}^2+\mathfrak{b}^2 & \mathfrak{a}z_2 \\ -\mathfrak{a}z_2 & z_2^2 \end{bmatrix}\begin{bmatrix} \mathfrak{C}_{n-1}(\theta) \\ z_2\mathfrak{D}_{n-1}(\theta) \end{bmatrix} \qquad (3.12a)$$

for $n>1$, and

$$\begin{bmatrix} \mathfrak{C}_1(\theta) \\ z_2\mathfrak{D}_1(\theta) \end{bmatrix} = \begin{bmatrix} -\mathfrak{a}^2+\mathfrak{b}^2 & \mathfrak{a}z_2 \\ -\mathfrak{a}z_2 & z_2{}^2 \end{bmatrix} \begin{bmatrix} \mathfrak{C}_0(\theta) \\ z^2 z_2{}^{-1}\mathfrak{D}_0(\theta) \end{bmatrix}, \quad (3.12b)$$

together with the boundary conditions

$$\mathfrak{C}_0(\theta) = -\mathfrak{c}^2 \quad \text{and} \quad \mathfrak{D}_0(\theta) = -\mathfrak{c}. \quad (3.13)$$

The 2×2 matrix that appears in (3.12) is Hermitian with the eigenvalues

$$\lambda = |1+z_1e^{i\theta}|^{-2}\, z_2(1-z_1{}^2)\alpha$$

and

$$\lambda' = |1+z_1e^{i\theta}|^{-2}\, z_2(1-z_1{}^2)\alpha^{-1}, \quad (3.14)$$

where α is the larger root[9] in magnitude of the quadratic equation

$$(1+z_1{}^2)(1+z_2{}^2) - z_1(1-z_2{}^2)(e^{i\theta}+e^{-i\theta}) - z_2(1-z_1{}^2)(\alpha+\alpha^{-1}) = 0. \quad (3.15)$$

More explicitly, α is given by

$$\alpha = \tfrac{1}{2}z_2{}^{-1}(1-z_1{}^2)^{-1} \times\{(1+z_1{}^2)(1+z_2{}^2) - z_1(1-z_2{}^2)(e^{i\theta}+e^{-i\theta}) + (1-z_2{}^2)[(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2{}^{-1}e^{i\theta})(1-\alpha_2{}^{-1}e^{-i\theta})]^{1/2}\}, \quad (3.16)$$

where

$$\alpha_1 = z_1(1-|z_2|)/(1+|z_2|)$$

and

$$\alpha_2 = z_1{}^{-1}(1-|z_2|)/(1+|z_2|). \quad (3.17)$$

The normalized eigenvector with the eigenvalue λ of (3.14) is

$$\begin{bmatrix} \mathfrak{v} \\ i\mathfrak{v}' \end{bmatrix}, \quad (3.18a)$$

while that with the eigenvalue λ' is

$$\begin{bmatrix} i\mathfrak{v}' \\ \mathfrak{v} \end{bmatrix}, \quad (3.18b)$$

where

$$\mathfrak{v} = \{\tfrac{1}{2}[1+(\lambda'-\lambda)^{-1}(z_2{}^2+\mathfrak{a}^2-\mathfrak{b}^2)]\}^{1/2}$$

and

$$\mathfrak{v}' = \{\tfrac{1}{2}[1-(\lambda'-\lambda)^{-1}(z_2{}^2+\mathfrak{a}^2-\mathfrak{b}^2)]\}^{1/2}\,\mathrm{sgn}(i\mathfrak{a}z_2). \quad (3.19)$$

Note that

$$\mathfrak{v}/\mathfrak{v}' = i(z_2{}^2-\lambda)/(\mathfrak{a}z_2) = i\mathfrak{a}z_2/(z_2{}^2-\lambda'). \quad (3.20)$$

With Eqs. (3.18)–(3.20), the equations (3.12) with the boundary condition (3.13) can be solved to give explicitly, for $n\geq0$,

$$\mathfrak{C}_n(\theta) = -\lambda^n\mathfrak{v}^2(\mathfrak{c}^2 - iz^2z_2{}^{-1}\mathfrak{c}\mathfrak{v}'/\mathfrak{v}) - \lambda'^n\mathfrak{v}'^2(\mathfrak{c}^2+iz^2z_2{}^{-1}\mathfrak{c}\mathfrak{v}/\mathfrak{v}'), \quad (3.21)$$

and for $n\geq1$,

$$z_2\mathfrak{D}_n(\theta) = -i\lambda^n\mathfrak{v}'^2(\mathfrak{c}^2\mathfrak{v}/\mathfrak{v}' - iz^2z_2{}^{-1}\mathfrak{c}) + i\lambda'^n\mathfrak{v}^2(\mathfrak{c}^2\mathfrak{v}'/\mathfrak{v}+iz^2z_2{}^{-1}\mathfrak{c}). \quad (3.22)$$

The substitution of (3.21) and (3.11) into (3.10) gives that

$$\det\mathfrak{A} = \Pi_\theta\{4\,|1+e^{i\theta}|^{-2}\sin^2\theta\,|1+z_1e^{i\theta}|^{4\mathfrak{N}}\lambda^{2\mathfrak{M}}[\mathfrak{v}^2(1-iz^2z_2{}^{-1}\mathfrak{c}^{-1}\mathfrak{v}'/\mathfrak{v}) + \alpha^{-4\mathfrak{M}}\mathfrak{v}'^2(1+iz^2z_2{}^{-1}\mathfrak{c}^{-1}\mathfrak{v}/\mathfrak{v}')]\}. \quad (3.23)$$

Since

$$\Pi_\theta\,|1+e^{i\theta}| = 2 \quad (3.24)$$

and

$$\Pi_\theta\,|2\sin\theta| = 4, \quad (3.25)$$

(3.23) can be simplified and the substitution into Eq. (2.7) gives

$$Z^2 = (2\cosh\beta E_1)^{8\mathfrak{M}\mathfrak{N}}(\cosh\beta E_2)^{4\mathfrak{N}(2\mathfrak{M}-1)}(\cosh\beta\mathfrak{H})^{4\mathfrak{N}} \times\Pi_\theta\{|1+z_1e^{i\theta}|^{4\mathfrak{M}}\lambda^{2\mathfrak{M}}[\mathfrak{v}^2(1-iz^2z_2{}^{-1}\mathfrak{c}^{-1}\mathfrak{v}'/\mathfrak{v}) + \alpha^{-4\mathfrak{M}}\mathfrak{v}'^2(1+iz^2z_2{}^{-1}\mathfrak{c}^{-1}\mathfrak{v}/\mathfrak{v}')]\}. \quad (3.26)$$

So far, the calculation is valid for any $\mathfrak{M}$ and $\mathfrak{N}$. We now take the limit of large $\mathfrak{M}$ and large $\mathfrak{N}$ for fixed $T\neq T_c$. We can therefore drop the term proportional to $\alpha^{-4\mathfrak{M}}$ in (3.26) and Z is given approximately by

$$-\beta^{-1}\ln Z \sim 4\mathfrak{M}\mathfrak{N}F + 4\mathfrak{N}\mathfrak{F}_0 + 2\mathfrak{N}\mathfrak{F}(\mathfrak{H}), \quad (3.27)$$

where

$$F = -\beta^{-1}\left\{\ln(2\cosh\beta E_1\cosh\beta E_2) + (4\pi)^{-1}\int_{-\pi}^{\pi} d\theta\,\ln[|1+z_1e^{i\theta}|^2\lambda(\theta)]\right\}, \quad (3.28)$$

[9] The α in this paper is not related to that of B. M. McCoy and T. T. Wu, Phys. Rev. **155**, 438 (1967). This paper is hereafter referred to as II.

and

$$2\mathfrak{F}_0+\mathfrak{F}(\mathfrak{H})=-\beta^{-1}\left[-\ln\cosh\beta E_2+\ln\cosh\beta\mathfrak{H}+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln[\mathfrak{v}^2(1-iz^2z_2^{-1}\mathfrak{c}^{-1}\mathfrak{v}'/\mathfrak{v})]\right]. \tag{3.29}$$

Physically, F is the bulk free energy per site, $\mathfrak{F}_0$ is the boundary free energy per boundary site in the absence of the magnetic field, and $\mathfrak{F}(\mathfrak{H})$ is the increase in boundary free energy per boundary site interacting with the magnetic field. Thus, $\mathfrak{F}_0$ is independent of $\mathfrak{H}$ and $\mathfrak{F}(0)$ is zero. Accordingly,

$$\mathfrak{F}_0=-\tfrac{1}{2}\beta^{-1}\left[-\ln\cosh\beta E_2+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln\mathfrak{v}^2\right] \tag{3.30}$$

and

$$\mathfrak{F}(\mathfrak{H})=-\beta^{-1}\left[\ln\cosh\beta\mathfrak{H}+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln(1-iz^2z_2^{-1}\mathfrak{c}^{-1}\mathfrak{v}'/\mathfrak{v})\right]. \tag{3.31}$$

By (3.19), (3.7), and (3.14), Eq. (3.30) is more explicitly

$$\mathfrak{F}_0=-\tfrac{1}{2}\beta^{-1}\left(-\ln\cosh\beta E_2+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln\tfrac{1}{2}\{1+z_2^{-1}(1-z_1^2)^{-1}(\alpha-\alpha^{-1})^{-1}[(1+z_1^2)(1-z_2^2)-2z_1(1+z_2^2)\cos\theta]\}\right). \tag{3.32}$$

Similarly, by (3.20), (3.14), and (3.15), Eq. (3.31) is

$$\mathfrak{F}(\mathfrak{H})=-\beta^{-1}\left(\ln\cosh\beta\mathfrak{H}+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln\{1-z^2z_1z_2^{-1}\,|\,1+e^{i\theta}\,|^2\,[z_2(1+z_1^2+2z_1\cos\theta)-(1-z_1^2)\alpha]^{-1}\}\right). \tag{3.33}$$

The free energy $\mathfrak{F}_0$ in the absence of a magnetic field is considered in detail in the next section, while the quantity $\mathfrak{F}(\mathfrak{H})$, or more precisely $\mathfrak{F}'(\mathfrak{H})$, is studied in Sec. 5.

4. BOUNDARY FREE ENERGY AND SPECIFIC HEAT ($\mathfrak{H}=0$)

In this section we discuss the thermodynamics of the boundary in the absence of a magnetic field; more specifically, we study the boundary free energy as given by (3.32) together with the boundary entropy and boundary specific heat, both of which are essentially derivatives of $\mathfrak{F}_0$ with respect to the temperature T. The interesting features are to be found in the vicinity of the critical temperature T_c: there the boundary entropy is unbounded while the boundary specific heat has a singularity of the form $(T_c-T)^{-1}$. These features are not possible for the corresponding bulk properties, and remind us very strongly that we are dealing with boundary effects. They are also closely connected with the large fluctuations at $T=T_c$, already mentioned in Sec. 8(G) of I.

Equation (3.32) can be simplified by using (3.16) and (3.17):

$$\begin{aligned}\mathfrak{F}_0=&-\tfrac{1}{2}\beta^{-1}\Big(-\ln\cosh\beta E_2+(4\pi)^{-1}\\ &\times\int_{-\pi}^{\pi}d\theta\,\ln\tfrac{1}{2}\{1+[1+\alpha_1\alpha_2^{-1}-(\alpha_1+\alpha_2^{-1})\cos\theta][(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2^{-1}e^{i\theta})(1-\alpha_2^{-1}e^{-i\theta})]^{-1/2}\}\Big)\\ =&-\tfrac{1}{2}\beta^{-1}\left\{-\ln\cosh\beta E_2+(4\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln\tfrac{1}{4}[2+\Phi(\theta)+\Phi^{-1}(\theta)]\right\},\end{aligned} \tag{4.1}$$

where

$$\Phi(\theta)=\left[\frac{(1-\alpha_1e^{i\theta})(1-\alpha_2^{-1}e^{-i\theta})}{(1-\alpha_1e^{-i\theta})(1-\alpha_2^{-1}e^{i\theta})}\right]^{1/2} \tag{4.2}$$

is defined to be positive, at $\theta=\pi$ if $z_1\geq0$ and at $\theta=0$ if $z_1\leq0$. Equation (4.2) is to be compared with (1.10) of I. Since

$$\Phi(\theta)\Phi(-\theta)=1, \tag{4.3}$$

it follows from (4.1) that

$$F_0=\tfrac{1}{2}\beta^{-1}\left\{\ln\cosh\beta E_2-(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,\ln\tfrac{1}{2}[1+\Phi(\theta)]\right\}. \tag{4.4}$$

Equation (4.4) is the desired result.

We begin with a qualitative discussion of $\mathfrak{F}_0$ as given by (4.4). First, as may be expected, $\mathfrak{F}_0$ is finite, non-negative and independent of the signs of E_1 and E_2. The behavior of $\mathfrak{F}_0$ in some simple limiting cases is as follows.

(a) $T\rightarrow 0$ for fixed E_1 and E_2. In this case,

$$z_1\rightarrow \operatorname{sgn}E_1, \tag{4.5}$$

$$z_2\rightarrow \operatorname{sgn}E_2, \tag{4.6}$$

$$\alpha_1\rightarrow 0, \tag{4.7}$$

$$\alpha_2\rightarrow 0, \tag{4.8}$$

$$\Phi(\theta)\rightarrow -e^{-i\theta}\operatorname{sgn}E_1, \tag{4.9}$$

and

$$\mathfrak{F}_0\rightarrow \tfrac{1}{2}\mid E_2\mid. \tag{4.10}$$

(b) $T\rightarrow\infty$ for fixed E_1 and E_2. In the case, $\beta\rightarrow 0$,

$$\Phi(\theta)=1+0(\beta), \tag{4.11}$$

and

$$\mathfrak{F}_0\rightarrow 0. \tag{4.12}$$

(c) $E_1\rightarrow\infty$ for fixed E_2 and T. Here $z_1\rightarrow 1$,

$$\alpha_1\sim\alpha_2<1, \tag{4.13}$$

$$\Phi(\theta)\rightarrow -e^{-i\theta}(1-\alpha_1 e^{i\theta})/(1-\alpha_1 e^{-i\theta}), \tag{4.14}$$

and

$$\mathfrak{F}_0\rightarrow\tfrac{1}{2}\beta^{-1}[\ln\cosh\beta E_2-\ln\tfrac{1}{2}(1+\alpha_1)]=\tfrac{1}{2}\mid E_2\mid. \tag{4.15}$$

(d) $E_2\rightarrow\infty$ for fixed E_1 and T. This case is very similar to (a); in particular, (4.6)–(4.10) hold.

(e) $E_1\rightarrow 0$ for fixed E_2 and T. In this case, $z_1\rightarrow 0$, $\alpha_1\rightarrow 0$, $\alpha_2^{-1}\rightarrow 0$,

$$\Phi(\theta)\rightarrow 1, \tag{4.16}$$

and

$$\mathfrak{F}_0\rightarrow\tfrac{1}{2}\beta^{-1}\ln\cosh\beta E_2. \tag{4.17}$$

This is the result for the one-dimensional Ising model, and may indeed be written down without calculation.

(f) $E_2\rightarrow 0$ for fixed E_1 and T. In this case, $z_2\rightarrow 0$,

$$\alpha_1=\alpha_2^{-1}=z_1, \tag{4.18}$$

and Eqs. (4.16) and (4.17) hold; i.e., $\mathfrak{F}_0\rightarrow 0$.

The rest of this section is devoted to an analysis of the behavior of $\mathfrak{F}_0$ when T is near T_c. Since there is no magnetic field, $\mathfrak{F}_0$ depends only on the magnitudes of E_1 and E_2, but not on their signs. Therefore, without loss of generality, we assume both E_1 and E_2 to be positive. With this convention, $\alpha_2=1$ when $T=T_c$. An inspection of (4.4) with (4.2) then indicates that the expansion of $\mathfrak{F}_0$ for α_2 near 1 may contain terms proportional to the following: 1, $(1-\alpha_2)\ln\mid 1-\alpha_2\mid$, $1-\alpha_2$, $(1-\alpha_2)^2\ln\mid 1-\alpha_2\mid$, $(1-\alpha_2)^2$, etc. We are only interested in the terms containing the logarithms, since they are responsible for the singularities in the boundary entropy

$$\mathfrak{S}=-\partial\mathfrak{F}_0/\partial T \tag{4.19}$$

and the specific heat

$$\mathfrak{c}_v=-T\partial^2\mathfrak{F}_0/\partial T^2. \tag{4.20}$$

The computation of these required terms is rather complicated. The first step is to change the variable of integration to

$$\omega=(e^{i\theta}-1)/(e^{i\theta}+1), \tag{4.21}$$

so that the path of integration is changed from the unit circle to the imaginary axis. The result is

$$\mathfrak{F}_0=\tfrac{1}{2}\beta^{-1}\left(\ln\cosh\beta E_2+i\pi^{-1}\int_{-i\infty}^{i\infty}d\omega(1-\omega^2)^{-1}\ln\tfrac{1}{2}\{1+[(\tau_1-\omega)(\tau_2-\omega)(\tau_1+\omega)^{-1}(\tau_2+\omega)^{-1}]^{1/2}\}\right), \tag{4.22}$$

where

$$\tau_i=(1-\alpha_i)/(1+\alpha_i) \tag{4.23}$$

for $i=1, 2$. In (4.22), the square root is equal to 1 as $\omega\rightarrow\pm i\infty$. It is convenient to redefine the square root by the value at $\omega=0$. Thus

$$\mathfrak{F}_0=\tfrac{1}{2}\beta^{-1}\left(\ln\cosh\beta E_2+i\pi^{-1}\int_{-i\infty}^{i\infty}d\omega(1-\omega^2)^{-1}\ln\tfrac{1}{2}\{1+[(\tau_1-\omega)(\tau+\omega)(\tau_1+\omega)^{-1}(\tau-\omega)^{-1}]^{1/2}\}\right) \tag{4.24a}$$

for $T>T_c$, and

$$\mathfrak{F}_0=\tfrac{1}{2}\beta^{-1}\left(\ln\cosh\beta E_2+i\pi^{-1}\int_{-i\infty}^{i\infty}d\omega(1-\omega^2)^{-1}\ln\tfrac{1}{2}\{1-[(\tau_1-\omega)(\tau-\omega)(\tau_1+\omega)^{-1}(\tau+\omega)^{-1}]^{1/2}\}\right) \tag{4.24b}$$

for $T<T_c$. In (4.24), the square roots are defined to be 1 at $\omega=0$, and

$$\tau=\tau_2\operatorname{sgn}(T_c-T). \tag{4.25}$$

The second step is to continue analytically in τ, taken to be a complex variable. Define disc$\mathfrak{F}_0$ by

$$\operatorname{disc}\mathfrak{F}_0=\mathfrak{F}_0(\tau e^{2\pi i})-\mathfrak{F}_0(\tau). \tag{4.26}$$

We consider the case $T>T_c$ first. Both $\mathfrak{F}_0(\tau)$ and $\mathfrak{F}_0(\tau e^{2\pi i})$ are given by (4.24a) with the contours of integration shown in Fig. 4(a) and Fig. 4(b), respectively. Accordingly,

$$\mathrm{disc}\mathfrak{F}_0=\tfrac{1}{2}i\beta^{-1}\pi^{-1}\int d\omega(1-\omega^2)^{-1}\ln\{1+[(\tau_1-\omega)(\tau+\omega)(\tau_1+\omega)^{-1}(\tau-\omega)^{-1}]^{1/2}\}, \tag{4.27}$$

where the contour of integration is shown in Fig. 4(c), which was first used by Pochammer[10] nearly a century ago. It follows immediately from (4.27) that

$$\mathrm{disc}\mathfrak{F}_0=i\beta^{-1}\pi^{-1}\times\int_{-\tau}^{\tau} d\omega(\ln\{1+[(\tau_1-\omega)(\tau+\omega)(\tau_1+\omega)^{-1}(\tau-\omega)^{-1}]^{1/2}\}-\ln|1-[(\tau_1-\omega)(\tau+\omega)(\tau_1+\omega)^{-1}(\tau-\omega)^{-1}]^{1/2}|). \tag{4.28}$$

In the form (4.28), it is straightforward to expand into a power series in τ; the two leading terms are

$$\mathrm{disc}\mathfrak{F}_0=i\beta^{-1}\pi^{-1}\tau\bigg(\int_{-1}^{1}dx\{\ln[1+(1+x)^{1/2}(1-x)^{-1/2}]-\ln|1-(1+x)^{1/2}(1-x)^{-1/2}|\}$$
$$-\tau_1^{-1}\tau\int_{-1}^{1}dx\{x(1+x)^{1/2}(1-x)^{-1/2}[1+(1+x)^{1/2}(1-x)^{-1/2}]^{-1}$$
$$+x(1+x)^{1/2}(1-x)^{-1/2}[1-(1+x)^{1/2}(1-x)^{-1/2}]^{-1}\}+O(\tau^2)\bigg). \tag{4.29}$$

The integrals on the right-hand side of (4.29) are easily evaluated; the first one is found to be π, while the second one is $-\frac{1}{2}\pi$. Therefore

$$\mathrm{disc}\mathfrak{F}_0=i\beta^{-1}[\tau+\tfrac{1}{2}\tau_1^{-1}\tau^2+O(\tau^3)] \tag{4.30}$$

for $T>T_c$. This implies that $\mathfrak{F}_0$ is, for small positive τ, of the form

$$\mathfrak{F}_0=\text{ Taylor series in }\tau+(2\pi\beta)^{-1}[\tau+\tfrac{1}{2}\tau_1^{-1}\tau^2+O(\tau^3)]\ln\tau. \tag{4.31}$$

To obtain the corresponding result for $T<T_c$, it is convenient to introduce the function

$$\mathfrak{F}_0^-=\tfrac{1}{2}\beta^{-1}\bigg(\ln\cosh\beta E_2+i\pi^{-1}\int_{-i\infty}^{i\infty}d\omega(1-\omega^2)^{-1}\ln\tfrac{1}{2}\{1+[(\tau_1-\omega)(\tau-\omega)(\tau_1+\omega)^{-1}(\tau+\omega)^{-1}]^{1/2}\}\bigg). \tag{4.32}$$

which differs from $\mathfrak{F}_0$ of (4.24b) only in the sign of the square root. It is easily verified that $\mathfrak{F}_0+\mathfrak{F}_0^-$ is analytic in τ for sufficiently small τ. Therefore, it follows from

$$\mathfrak{F}_0^-=\text{ Taylor series in }\tau+(2\pi\beta)^{-1}\times[\tau-\tfrac{1}{2}\tau_1^{-1}\tau^2+O(\tau^3)]\ln\tau \tag{4.33}$$

that

$$\mathfrak{F}_0=\text{ Taylor series in }\tau-(2\pi\beta)^{-1}\times[\tau-\tfrac{1}{2}\tau_1^{-1}\tau^2+O(\tau^3)]\ln\tau \tag{4.34}$$

for $T<T_c$. Note that the imaginary part of each of the logarithms in (4.24) and (4.32) has been taken to be less than π in magnitude. By Eq. (4.25), (4.31), and (4.34) can be combined in the form

$$\mathfrak{F}_0=\text{ Taylor series in }\tau_2-(2\pi\beta)^{-1}\times[\tau_2-\tfrac{1}{2}\tau_1^{-1}\tau_2^2+O(\tau_2^3)]\ln|\tau_2| \tag{4.35}$$

for both $T>T_c$ and $T<T_c$. Note that the Taylor series to be used in (4.35) is different for $T>T_c$ and for $T<T_c$. We shall return to this point later in this section.

It remains to substitute (4.35) into Eqs. (4.19) and (4.20). Let z_{1c} and z_{2c} be the values of z_1 and z_2 when $T=T_c$, so that

$$1-z_{1c}-z_{2c}-z_{1c}z_{2c}=0. \tag{4.36}$$

Then it is easily verified that, for T near T_c,

$$\tau_2=[(kT)^{-1}-(kT_c)^{-1}]\times\{s_1-E_1^2(1-z_{1c})(z_{1c}+z_{2c})(1-z_{2c})^{-2}\times[(kT)^{-1}-(kT_c)^{-1}]+O[(T-T_c)^2]\}, \tag{4.37}$$

where

$$s_1=(1-z_{2c})^{-1}[E_1(1-z_{1c})+E_2(1-z_{2c})]. \tag{4.38}$$

Accordingly, since at $T=T_c$

$$\tau_1=(1-z_{1c})/(z_{1c}+z_{2c}), \tag{4.39}$$

$$\beta^{-1}(\tau_2-\tfrac{1}{2}\tau_1^{-1}\tau_2^2)=(1-T/T_c)\times\{s_1-\tfrac{1}{2}s_2(1-T/T_c)+O[(T-T_c)^2]\}, \tag{4.40}$$

[10] See, for example, L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, England, 1966), pp. 22–23.

where

$$s_2=(kT_c)^{-1}(1-z_{2c})^{-2}(z_{1c}+z_{2c})\{3E_1^2(1-z_{1c})+2E_1E_2(1-z_{2c})+E_2^2(1-z_{2c})^2/(1-z_{1c})\}. \quad (4.41)$$

The results for the entropy and the specific heat are thus

$$\mathfrak{S}=-(2\pi T_c)^{-1}s_1\ln|1-T/T_c|+O(1), \quad (4.42)$$

and

$$\mathfrak{c}_v=-(2\pi)^{-1}[s_1(T-T_c)^{-1}+s_2T_c^{-1}\ln|1-T/T_c|]+O(1). \quad (4.43)$$

Note that s_1 is positive so that $\mathfrak{S}$ is unbounded from above for T near T_c. This and the singularity of $\mathfrak{c}_v$ are already discussed at the beginning of this section.

These singularities can be easily understood in the following way. Suppose, in an infinite, two-dimensional Ising lattice, we change the interaction between a pair of nearest-neighbor spins—an impurity bond. The change of free energy due to this impurity bond is easily expressed in terms of the two-spin correlation function $\langle\sigma_{0,0}\sigma_{0,1}\rangle$. Near the critical temperature, the resulting changes in entropy and specific heat exhibit precisely the kinds of singularities of (4.42) and (4.43).

We write down more explicity the singularities of $\mathfrak{S}$ and $\mathfrak{c}_v$ for the special case E_1 and E_2:

$$\mathfrak{S}=-k(2\pi)^{-1}[\ln(1+\sqrt{2})][\ln|1-T/T_c|+O(1)], \quad (4.44)$$

$$\mathfrak{c}_v=k(2\pi)^{-1}[\ln(1+\sqrt{2})]\{(1-T/T_c)^{-1}-3\sqrt{2}^{-1}[\ln(1+\sqrt{2})][\ln|1-T/T_c|]+O(1)\}. \quad (4.45)$$

Equation (4.42) does not quite tell the whole story. It should be supplemented by

$$\lim_{\delta T\to 0}[\mathfrak{S}(T_c+\delta T)-\mathfrak{S}(T_c-\delta T)]=-\tfrac{1}{2}s_1/T_c. \quad (4.46)$$

This "latent heat" is not understood by the authors.

In spite of the peculiarities of the boundary entropy exhibited in this section we will proceed to a discussion of the boundary magnetization and hysteresis.

5. MAGNETIZATION AND HYSTERESIS

Attention is next focused on the additional boundary free energy due to the presence of a magnetic field, as given by (3.33). More precisely, we shall consider the magnetization $\mathfrak{M}_1$ of the first row. The substitution of (3.33) and (3.27) into (2.10) gives that

$$\begin{aligned}\mathfrak{M}_1&=-\mathfrak{F}'(\mathfrak{H})\\&=z+(4\pi)^{-1}(1-z^2)(\partial/\partial z)\int_{-\pi}^{\pi}d\theta\ln\{1-z^2z_1z_2^{-1}|1+e^{i\theta}|^2[z_2(1+z_1^2+2z_1\cos\theta)-(1-z_1^2)\alpha]^{-1}\}\\&=z+(2\pi)^{-1}(1-z^2)zz_1\int_{-\pi}^{\pi}d\theta|1+e^{i\theta}|^2[z^2z_1|1+e^{i\theta}|^2-z_2^2(1+z_1^2+2z_1\cos\theta)+z_2(1-z_1^2)\alpha]^{-1}.\end{aligned} \quad (5.1)$$

Clearly, $\mathfrak{M}_1\to 1$ as $\mathfrak{H}\to\infty$.

It is useful to rewrite (5.1) in the following two ways. First, by (3.15),

$$\mathfrak{M}_1=z+(2\pi)^{-1}(1-z^2)z\int_{-\pi}^{\pi}d\theta[-z_2(1-z_1)\alpha+(1+z_1)][z_2(1-z_1)(1-z^2)\alpha-(1+z_1)(z_2^2-z^2)]^{-1}. \quad (5.2)$$

Thus the integrand is singular if and only if

$$\alpha=[(1+z_1)(z_2^2-z^2)]/[z_2(1-z_1)(1-z^2)]. \quad (5.3)$$

Alternatively, α as given by (3.16) may be substituted into (5.1) to give

$$\begin{aligned}\mathfrak{M}_1=z+(2\pi)^{-1}(1-z^2)zz_1\int_{-\pi}^{\pi}d\theta|1+e^{i\theta}|^2(z^2z_1|1+e^{i\theta}|^2-\tfrac{1}{4}z_1(1+|z_2|)^2\\\times\{(1+\alpha_1\alpha_2)(e^{i\theta}+e^{-i\theta})-2(\alpha_1+\alpha_2)-2[(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{i\theta})(1-\alpha_2e^{-i\theta})]^{1/2}\})^{-1}.\end{aligned} \quad (5.4)$$

At least when $T<T_c$, that is, $|\alpha_1|<1$ and $|\alpha_2|<1$, the last factor in (5.4) can be further factored to give

$$\mathfrak{M}_1=z+(2\pi)^{-1}(1-z^2)4z\int_{-\pi}^{\pi}d\theta|1+e^{i\theta}|^2(\mathfrak{s}_1\mathfrak{s}_2)^{-1}, \quad (5.5)$$

where

$$\mathfrak{s}_1=2z(1+e^{i\theta})-(1+|z_2|)\{[(1-\alpha_1e^{i\theta})(1-\alpha_2e^{i\theta})]^{1/2}-e^{i\theta}[(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{-i\theta})]^{1/2}\}, \quad (5.6)$$

and

$$\mathfrak{s}_2=2z(1+e^{-i\theta})-(1+|z_2|)\{[(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{-i\theta})]^{1/2}-e^{-i\theta}[(1-\alpha_1e^{i\theta})(1-\alpha_2e^{i\theta})]^{1/2}\}, \quad (5.7)$$

with both square roots defined to be positive at $\theta=0$ and π. The right-hand side of (5.5) may be expressed as a partial fraction

$$\mathfrak{M}_1=z+(2\pi)^{-1}(1-z^2)\int_{-\pi}^{\pi}d\theta[(1+e^{i\theta})\mathfrak{s}_1^{-1}+(1+e^{-i\theta})\mathfrak{s}_2^{-1}]. \tag{5.8}$$

This form is needed for purposes of analytical continuation.

We study in some detail the location of the singularity of the integrand as given by (5.3). Let $\mathfrak{r}$ be the value of $e^{\pm i\theta}$ such that (3.15) and (5.3) are both satisfied. Therefore

$$\begin{aligned}\mathfrak{r}+\mathfrak{r}^{-1}&=z_1^{-1}(1-z_2^2)^{-1}(1+z_1^2)(1+z_2^2)-(1+z_1)^2(z_2^2-z^2)[z_1(1-z_2^2)(1-z^2)]^{-1}\\&\quad-z_2^2(1-z_1)^2(1-z^2)[z_1(1-z_2^2)(z_2^2-z^2)]^{-1}\\&=2[(1-\alpha_1\alpha_2)^2-2(\alpha_1+\alpha_2)\alpha_3^2-\alpha_3^4]/[(1-\alpha_1\alpha_2)^2-2(1+\alpha_1\alpha_2)\alpha_3^2+\alpha_3^4],\end{aligned} \tag{5.9}$$

where

$$\alpha_3=2z/(1+|z_2|). \tag{5.10}$$

With the additional condition $|\mathfrak{r}|\leq 1$, (5.9) gives, with an appropriate choice of sign,

$$\begin{aligned}\mathfrak{r}&=[(1-\alpha_1\alpha_2)^2-2(1+\alpha_1\alpha_2)\alpha_3^2+\alpha_3^4]^{-1}\\&\quad\times([(1-\alpha_1\alpha_2)^2-2(\alpha_1+\alpha_2)\alpha_3^2-\alpha_3^4]\pm 2\{\alpha_3^2[\alpha_3^2-(1-\alpha_1)(1-\alpha_2)][(1+\alpha_1)(1+\alpha_2)\alpha_3^2-(1-\alpha_1\alpha_2)^2]\}^{1/2}).\end{aligned} \tag{5.11}$$

The qualitative motion of $\mathfrak{r}$ is of interest. In the $e^{i\theta}$ plane, α has four branch points at α_1, α_1^{-1}, α_2, and α_2^{-1}. We define the cut plane for $e^{i\theta}$ by joining these branch points pairwise along the real axis; thus the unit circle does not intersect the branch cuts unless $|\alpha_2|=1$. In this cut plane, $|\alpha|\geq 1$. Therefore, by (5.3), there is a pair of singular points at $\mathfrak{r}$ and $\mathfrak{r}^{-1}$ in the cut plane if and only if

$$|[(1+z_1)(z_2^2-z^2)]/[z_2(1-z_1)(1-z^2)]|\geq 1. \tag{5.12}$$

Since $|z|\leq 1$, (5.12) holds if and only if either

$$z^2\geq |z_2|(1-\alpha_1)/(1+\alpha_1) \tag{5.13}$$

or

$$T\leq T_c, \qquad E_1\geq 0,$$

and

$$z^2\leq |z_2|(1-\alpha_2)/(1+\alpha_2). \tag{5.14}$$

Accordingly, in the cut plane, $\mathfrak{r}$ is real; moreover,

$$0\leq \mathfrak{r}/\alpha_1\leq 1 \tag{5.15}$$

when (5.13) holds and

$$0\leq \alpha_2\leq \mathfrak{r}\leq 1 \tag{5.16}$$

when (5.14) holds. In (5.16), $\mathfrak{r}=1$ if $z=0$.

With this information on $\mathfrak{r}$, it is clear that $\mathfrak{M}_1$ is an analytic function of $\mathfrak{H}$ except when $T\leq T_c$ and $\mathfrak{H}=0$. We proceed to study the behavior of $\mathfrak{M}_1$ near $\mathfrak{H}=0$ and also the analytic continuation of $\mathfrak{M}_1$ as a function of $\mathfrak{H}$.

A. Spontaneous Magnetization

The boundary spontaneous magnetization is defined to be

$$\mathfrak{M}_1(0+)=\lim_{\mathfrak{H}\to 0+}\mathfrak{M}_1(\mathfrak{H}). \tag{5.17}$$

By (5.1), it is zero unless $\mathfrak{r}\to 1$ in this limit. That is by, (5.16), it is zero unless $T<T_c$ and $E_1>0$. We consider only this case. Expansion about $\theta=0$ gives

$$\begin{aligned}\alpha&\sim z_2(1-z_1)^{-1}(1+z_1)\\&\quad\times\{1+z_1(1-z_2^2)[z_2^2(1+z_1)^2-(1-z_1)^2]^{-1}\theta^2\}\end{aligned} \tag{5.18}$$

from (3.15) or (3.16), and hence

$$\begin{aligned}\mathfrak{M}_1(0+)&=\lim_{z\to 0}(2\pi)^{-1}z\int_{-\infty}^{\infty}d\theta\\&\quad\times\{z^2+z_1z_2^2[z_2^2(1+z_1)^2-(1-z_1)^2]^{-1}\theta^2\}^{-1}\\&=\tfrac{1}{2}z_1^{-1/2}|z_2|^{-1}[z_2^2(1+z_1)^2-(1-z_1)^2]^{1/2}.\end{aligned} \tag{5.19}$$

This is the desired result. In terms of E_1 and E_2, (5.19) is

$$\mathfrak{M}_1(0+)=\left[\frac{\cosh 2\beta E_2-\coth 2\beta E_1}{\cosh 2\beta E_2-1}\right]^{1/2}. \tag{5.20}$$

This vanishes at the critical temperature as $(T_c-T)^{1/2}$,

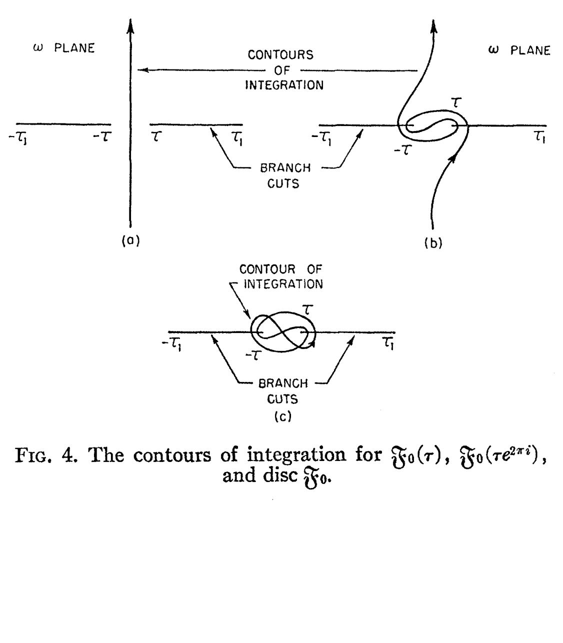

FIG. 4. The contours of integration for $\mathfrak{F}_0(\tau)$, $\mathfrak{F}_0(\tau e^{2\pi i})$, and disc $\mathfrak{F}_0$.

as compared with the eighth root for the bulk spontaneous magnetization of Yang.[2] As may be expected, the boundary spontaneous magnetization $\mathfrak{M}_1(0+)$ is less than the bulk value. The result is plotted in Fig. 5 for the case $E_1=E_2$. It is conjectured that, for $z_2>0$ and $\mathfrak{H}=0^+$, $\mathfrak{M}_J<\mathfrak{M}_{J+1}$, and as $J\to\infty$, $\mathfrak{M}_J$ approaches the bulk spontaneous magnetization. We shall return to this point in Sec. 9.

The boundary magnetic susceptibility at zero field can also be obtained from (5.1). Note that in (5.17) the limit $\mathfrak{H}\to 0+$ is taken after the thermodynamic limit $\mathfrak{N}\to\infty$. In Appendix B, we study the behavior of $\mathfrak{M}_1$ for small $\mathfrak{H}$ and large but finite $\mathfrak{N}$. It is shown that in this case $\mathfrak{M}_1$ is continuous but varies extremely rapidly for $\mathfrak{H}$ of the order of $[z_2^{-1}(1-z_1)(1+z_1)^{-1}]^{4\mathfrak{N}}$.

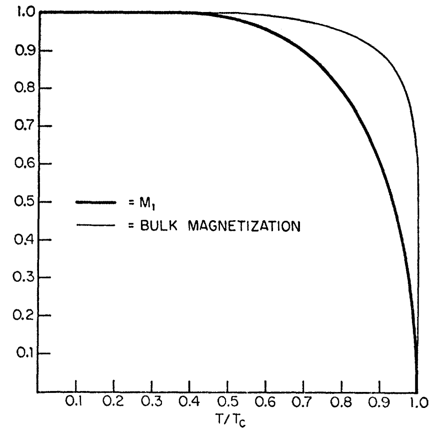

FIG. 5. Comparison of $\mathfrak{M}_1$ and bulk magnetization for $E_1=E_2$ as a function of temperature.

B. BEHAVIOR NEAR CRITICAL TEMPERATURE

We apply essentially the same procedure to study the behavior of $\mathfrak{M}_1$ when T is near T_c and $\mathfrak{H}$ is positive and small. We shall consider only the ferromagnetic case where $E_1>0$. The basic idea is still to expand about $\theta=0$, but the actual computation is somewhat less straightforward than that of spontaneous magnetization. Consider first $T<T_c$; we neglect throughout terms in $\mathfrak{M}_1$ of order z. Then it follows from (5.4) that

$$\mathfrak{M}_1\sim\mathfrak{M}_1^{(1)}+\mathfrak{M}_1^{(2)}, \tag{5.21}$$

where

$$\mathfrak{M}_1^{(1)}=(2\pi)^{-1}zz_1\int_{-\pi}^{\pi}d\theta\,|1+e^{i\theta}|^2\{\tfrac{1}{2}z_1(1+|z_2|)^2[(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{i\theta})(1-\alpha_2e^{-i\theta})]^{1/2}\}^{-1}, \tag{5.22}$$

and

$$\begin{aligned}\mathfrak{M}_1^{(2)}=(2\pi)^{-1}zz_1\int_{-\pi}^{\pi}d\theta\,|1+e^{i\theta}|^2[(z^2z_1|1+e^{i\theta}|^2-\tfrac{1}{4}z_1(1+|z_2|)^2\{(1+\alpha_1\alpha_2)(e^{i\theta}+e^{-i\theta})-2(\alpha_1+\alpha_2)\\ -2[(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{i\theta})(1-\alpha_2e^{-i\theta})]^{1/2}\})^{-1}\\ -(\tfrac{1}{2}z_1(1+|z_2|)^2[(1-\alpha_1e^{i\theta})(1-\alpha_1e^{-i\theta})(1-\alpha_2e^{i\theta})(1-\alpha_2e^{-i\theta})]^{1/2})^{-1}].\end{aligned} \tag{5.23}$$

These two parts are to be approximated differently. Since α_2 is close to 1,

$$\begin{aligned}\mathfrak{M}_1^{(1)}&\sim 8\pi^{-1}z(1+|z_2|)^{-2}\int_0^{\pi}d\theta[(1-\alpha_2e^{i\theta})(1-\alpha_2e^{-i\theta})]^{-1/2}\\ &=16\pi^{-1}z(1+|z_2|)^{-2}(1+\alpha_2)^{-1}K[2\alpha_2^{1/2}(1+\alpha_2)^{-1}]\sim-2\pi^{-1}zz_2^{-1}\ln(1-\alpha_2),\end{aligned} \tag{5.24}$$

where K denotes the complete elliptic integral of the first kind. In order to compute $\mathfrak{M}_1^{(2)}$ approximately, we expand all $e^{i\theta}$ into power series for small θ:

$$\begin{aligned}\mathfrak{M}_1^{(2)}\sim 2\pi^{-1}z\int_{-\infty}^{\infty}d\theta[(4z^2-\tfrac{1}{2}(1+|z_2|)^2\{(1-\alpha_1)(1-\alpha_2)-(1-\alpha_1)[(1-\alpha_2)^2+\theta^2]^{1/2}\})^{-1}\\ -(\tfrac{1}{2}(1+|z_2|)^2(1-\alpha_1)[(1-\alpha_2)^2+\theta^2]^{1/2})^{-1}].\end{aligned} \tag{5.25}$$

A change of variable reduces the right-hand side of (5.25) to

$$\mathfrak{M}_1^{(2)}\sim 2\pi^{-1}zz_2^{-1}(1-\mathfrak{p})\int_0^{\infty}d\theta(\mathfrak{p}-1+\cosh\theta)^{-1}, \tag{5.26}$$

where

$$\mathfrak{p}=2z^2|z_2|^{-1}(1-\alpha_2)^{-1}. \tag{5.27}$$

Note that $\mathfrak{p}$ can take any real positive value. The integral in (5.26) can be approximately evaluated

$$\begin{aligned}(1-\mathfrak{p})\int_0^{\infty}d\theta(\mathfrak{p}-1+\cosh\theta)^{-1}\\ =(2\mathfrak{p})^{-1/2}\pi-\ln[1+\tfrac{1}{2}|z_2|\mathfrak{p}]+O(1).\end{aligned} \tag{5.28}$$

In (5.28), the coefficient of $\mathfrak{p}$ in the logarithm is arbitrary; it has been chosen to make (5.29) below simple. The desired result follows immediately from

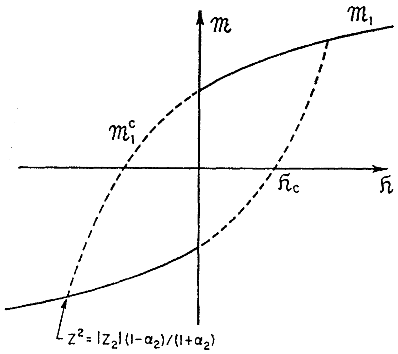

FIG. 6. Hysteresis loop for the magnetization on the first row. Solid curve gives $\mathfrak{M}_1$, while the dotted curve shows its analytic continuation.

Eqs. (5.21), (5.24), and (5.26)–(5.28):

$$\mathfrak{M}_1 \sim |z_2|^{-1/2}(1-\alpha_2)^{1/2}\,\mathrm{sgn}z - 2\pi^{-1}z\,|z_2|^{-2}\ln(1-\alpha_2+z^2) \tag{5.29}$$

for $T<T_c$. The computation is virtually identical in the case $T>T_c$. Equations (5.21)–(5.23) hold without modification, and (5.24) is also valid if α_2 is replaced by α_2^{-1}. So far as $\mathfrak{M}_1^{(2)}$ is concerned, the main change is the appearance of $1+\cosh\theta$ instead of $-1+\cosh\theta$ The result is

$$\mathfrak{M}_1 \sim -2\pi^{-1}z\,|z_2|^{-1}\ln(1-\alpha_2^{-1}+z^2) \tag{5.30}$$

for $T>T_c$.

As $z\to 0+$, (5.29) agrees with (5.19) and exhibits the square-root behavior explicitly. At $T=T_c$, it follows from either (5.29) or (5.30) that

$$\mathfrak{M}_1 \sim -4\pi^{-1}z\,|z_2|^{-1}\ln|z|. \tag{5.31}$$

Thus the boundary magnetic susceptibility is not finite at $T=T_c$. More generally, we get from (5.29) and (5.30) that

$$\partial\mathfrak{M}_1/\partial\mathfrak{H}\,|_{\mathfrak{H}=0} = -2\pi^{-1}\beta\coth\beta E_2\ln|1-\alpha_2| + O(1) \tag{5.32}$$

both above and below the critical temperature. In other words, the boundary magnetic susceptibility at zero field has a logarithmic singularity at the critical temperature. This is *qualitatively different* from the bulk magnetic susceptibility, as obtained by numerical computation by Baker.[6]

C. HYSTERESIS

We return once more to the ferromagnetic case below critical temperature, i.e., $T<T_c$ and $E_1>0$. As seen above, $\mathfrak{M}_1$ is an analytic function of $\mathfrak{H}$ for all $\mathfrak{H}\neq 0$, and $\mathfrak{M}_1$ is discontinuous at $\mathfrak{H}=0$. We discuss here the analytic continuation of $\mathfrak{M}_1$; since $\mathfrak{M}_1$ is odd, it is sufficient to consider the continuation of $\mathfrak{M}_1$ for $\mathfrak{H}>0$ to negative values of $\mathfrak{H}$. Let $\mathfrak{M}_1^c$, defined for some nonpositive $\mathfrak{H}$, be such that $\mathfrak{M}_1(\mathfrak{H})$ with $\mathfrak{H}>0$ and $\mathfrak{M}_1^c(\mathfrak{H})$ with $\mathfrak{H}\leq 0$ taken together be analytic at $\mathfrak{H}=0$. That this analytic continuation is possible can be most easily seen from (5.8), where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ each has at most one zero in the cut $e^{i\theta}$ plane. For $\mathfrak{H}$ small, $\mathfrak{s}_1$ has a zero outside the unit circle namely the $\mathfrak{r}$ of (5.11), while $\mathfrak{s}_2$ has a zero outside the unit circle, namely $\mathfrak{r}^{-1}$. After analytic continuation to negative small values of $\mathfrak{H}$,

$$\mathfrak{r}>1, \tag{5.33}$$

and, still as before,

$$\mathfrak{s}_1(\mathfrak{r}) = \mathfrak{s}_2(\mathfrak{r}^{-1}) = 0. \tag{5.34}$$

For $\mathfrak{H}<0$, the difference between $\mathfrak{M}_1$ and $\mathfrak{M}_1^c$ is due to the residues at $\mathfrak{r}$ and $\mathfrak{r}^{-1}$; more explicitly,

$$\mathfrak{M}_1^c(\mathfrak{H}) - \mathfrak{M}_1(\mathfrak{H}) = 2z(\mathfrak{r}^{-1}-\mathfrak{r})^{-1}(1-z^2)^{-1}z_1^{-1}(z_2^2-z^2)^{-2} \times[(1+z_1)^2(z_2^2-z^2)^2 - z_2^2(1-z_1)^2(1-z^2)^2]. \tag{5.35}$$

When $-\mathfrak{H}$ is small, the right-hand side of (5.35) is positive and decreases with decreasing $\mathfrak{H}$. It reaches zero, as seen from (5.12) and (5.14), at

$$z^2 = |z_2|(1-\alpha_2)/(1+\alpha_2). \tag{5.36}$$

The situation is thus as shown schematically in Fig. 6.

It is natural to interpret this figure as a hysteresis loop. From (5.36), this loop shrinks to the single point $\mathfrak{H}=\mathfrak{M}=0$ as $T\to T_c-$. As $T\to 0$, $|z_2|$ is close to 1, and hence by (3.17)

$$2\alpha_2 \sim 1-|z_2|. \tag{5.37}$$

Substitution into (5.36) then gives

$$|z| \sim |z_2|. \tag{5.38}$$

Thus, in this limit of zero temperature, the hysteresis loop becomes a square, as shown in Fig. 7. Note that the limit $T\to 0$ of the analytic continuation of $\mathfrak{M}_1(\mathfrak{H})$

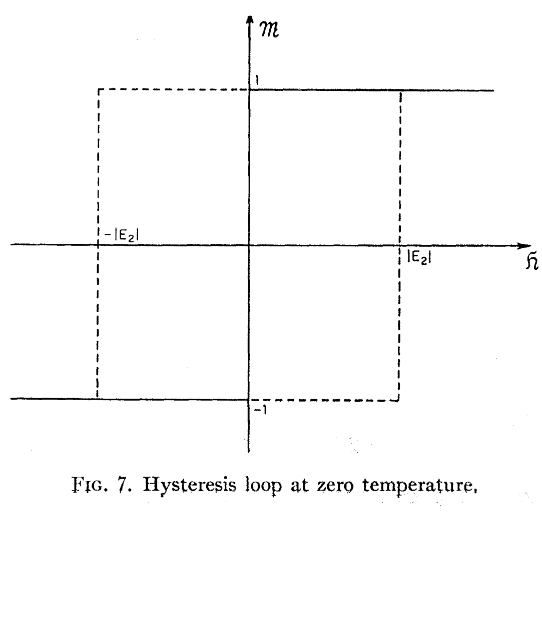

FIG. 7. Hysteresis loop at zero temperature.

is different from the analytic continuation of

$$\lim_{T\to 0}\mathfrak{M}_1(\mathfrak{H}).$$

6. DISTRIBUTION FUNCTION

We consider further in this section the ferromagnetic case where $T<T_c$ and $E_1>0$. Even though it is natural to identify the loop of Fig. 6 with hysteresis, the situation is actually much more involved. In order to get some more insight into the meaning of $\mathfrak{M}_1^c$ obtained by analytic continuation, we consider the distribution function in the limit of zero magnetic field for a given $\bar{\sigma}$ such that $\mathfrak{N}\bar{\sigma}$ is an integer less than $\mathfrak{N}$ (δ is the Kronecker delta).

$$\begin{aligned}&\langle\delta(\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}-2\mathfrak{N}\bar{\sigma})\rangle\\&=(4\mathfrak{N})^{-1}\sum_{k'=-2\mathfrak{N}+1}^{2\mathfrak{N}}\langle\exp[2\pi ik'(4\mathfrak{N})^{-1}(\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}-2\mathfrak{N}\bar{\sigma})]\rangle\\&=(4\mathfrak{N})^{-1}Z(0)^{-1}\sum_{k=-2\mathfrak{N}+1}^{2\mathfrak{N}}\exp(-\pi ik\bar{\sigma})Z(\tfrac{1}{2}\pi ik\beta^{-1}\mathfrak{N}^{-1}).\end{aligned}\tag{6.1}$$

It is important to note that this distribution function is *non-negative*. Let $\mathcal{M}$ and $\mathfrak{N}$ be very large; then by (3.27), which is applicable even to complex $\mathfrak{H}$,

$$\begin{aligned}&\langle\delta(\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}-2\mathfrak{N}\bar{\sigma})\rangle\\&\sim(4\mathfrak{N})^{-1}\sum_{k=-2\mathfrak{N}+1}^{2\mathfrak{N}}\exp\{-\pi ik\bar{\sigma}-2\mathfrak{N}\beta\mathfrak{F}(\tfrac{1}{2}\pi ik\beta^{-1}\mathfrak{N}^{-1})\}\\&\sim-i\beta(2\pi)^{-1}\int_{-i\pi/\beta}^{i\pi/\beta}d\xi\exp\{-2\mathfrak{N}\beta[\xi\bar{\sigma}+\mathfrak{F}(\xi)]\}.\end{aligned}\tag{6.2}$$

Therefore, the function $\mathfrak{W}(\bar{\sigma})$ defined by

$$\mathfrak{W}(\bar{\sigma})=\lim_{\mathfrak{N}\to\infty}(2\mathfrak{N})^{-1}\ln[\lim_{\mathcal{M}\to\infty}\langle\delta(\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}-2\mathfrak{N}\bar{\sigma})\rangle]\tag{6.3}$$

can be obtained from (6.2) by the method of steepest descent. First let

$$\bar{\sigma}\geq\mathfrak{M}_1(0+),\tag{6.4}$$

then

$$\mathfrak{W}(\bar{\sigma})=-\beta[\xi\bar{\sigma}+\mathfrak{F}(\xi)]_0,\tag{6.5}$$

where the right-hand side is evaluated at the point of steepest descent

$$\bar{\sigma}+\mathfrak{F}'(\xi)=0$$

or

$$\bar{\sigma}=\mathfrak{M}_1(\xi)\tag{6.6}$$

by (5.1). Therefore, if we identify $\bar{\sigma}$ with $\mathfrak{M}_1$ and ξ with $\mathfrak{H}$, (6.5) is

$$\mathfrak{W}=\mathfrak{W}(\mathfrak{M}_1)=-\beta(\mathfrak{H}\mathfrak{M}_1+\mathfrak{F}).\tag{6.7}$$

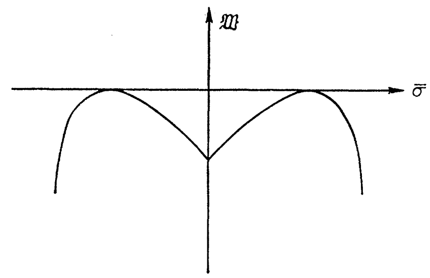

FIG. 8. Logarithm of the distribution function in the absence of a magnetic field.

Secondly, let

$$0\leq\bar{\sigma}\leq\mathfrak{M}_1(0+);\tag{6.8}$$

then the point of steepest descent is shifted to the analytic continuation of $\mathfrak{M}_1$. That is, (6.5) is still valid provided that we note that the point of steepest descent is now at

$$\bar{\sigma}=\mathfrak{M}_1^c(\xi).\tag{6.9}$$

Thus, for (6.8), (6.7) takes the form

$$\mathfrak{W}=\mathfrak{W}(\mathfrak{M}_1^c)=-\beta(\mathfrak{H}\mathfrak{M}_1^c+\mathfrak{F}^c),\tag{6.10}$$

where $\mathfrak{F}^c$ is the analytic continuation of $\mathfrak{F}$. For simplicity, we shall use (6.7), with $0\leq\mathfrak{M}_1\leq 1$, to mean both (6.7) and (6.10). In particular, differentiation with respect to $\mathfrak{M}_1$ gives

$$\partial\mathfrak{W}/\partial\mathfrak{M}_1=-\beta\mathfrak{H}.\tag{6.11}$$

Thus

$$\mathfrak{W}=\mathfrak{W}'=0\tag{6.12}$$

at $\mathfrak{H}=0$, i.e., $\mathfrak{M}_1=(0+)$. In other words, the distribution does have a maximum at $\bar{\sigma}=\mathfrak{M}_1(0+)$. The curve $\mathfrak{W}(\bar{\sigma})$ is sketched in Fig. 8. Note the discontinuity of $\mathfrak{W}'(\bar{\sigma})$ at $\bar{\sigma}=0$, as given by (6.11).

When a magnetic field is present, we can still define $\mathfrak{W}(\bar{\sigma},\mathfrak{H})$ through (6.3). This is very simply related to $\mathfrak{W}(\bar{\sigma})=\mathfrak{W}(\bar{\sigma},0)$ through

$$\mathfrak{W}(\bar{\sigma},\mathfrak{H})=\mathfrak{W}(\bar{\sigma})+\beta\mathfrak{H}\bar{\sigma}-\text{const},\tag{6.13}$$

where the constant, which is independent of $\bar{\sigma}$, is determined by the condition

$$\max_{-1\leq\bar{\sigma}\leq 1}\mathfrak{W}(\bar{\sigma},\mathfrak{H})=0.\tag{6.14}$$

With reference to Fig. 8, we see that, for $|\mathfrak{H}|$ not too large, $\mathfrak{W}(\bar{\sigma},\mathfrak{H})$ has two maxima, located at $\bar{\sigma}_r$ and $\bar{\sigma}_l$, say, with $\bar{\sigma}_r>\bar{\sigma}_l$. For $\mathfrak{H}>0$, the right maximum at $\bar{\sigma}_r$ is larger, while for $\mathfrak{H}<0$, the left maximum at $\bar{\sigma}_l$ is larger. For $\mathfrak{H}>0$, by (6.11) and (6.13), $\bar{\sigma}_r$ is located at the point where

$$\mathfrak{M}_1(\mathfrak{H})=\bar{\sigma}_r.\tag{6.15}$$

Similarly, for $\mathfrak{H}<0$ and $|\mathfrak{H}|$ sufficiently small, $\bar{\sigma}_r$ satisfies

$$\mathfrak{M}_1^c(\mathfrak{H})=\bar{\sigma}_r.\tag{6.16}$$

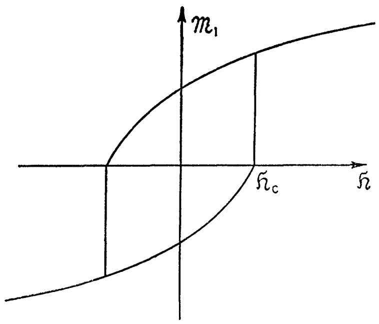

FIG. 9. Alternative hysteresis loop for the magnetization on the first row.

However, as seen from Fig. 8, the right maximum *disappears* after $\bar{\sigma}_r$ reaches 0. For values of $\mathfrak{H}$ such that $\mathfrak{M}_1^c(\mathfrak{H})\leq 0$, $\mathfrak{W}(\bar{\sigma},\mathfrak{H})$ has only one maximum. In other words, if and only if

$$\mathfrak{M}_1^c(\mathfrak{H})>0, \tag{6.17}$$

this analytic continuation is closely related to the secondary maximum of the distribution function.

The discussion in the Introduction about the hysteresis loop is based on the results of this section. In particular, in the absence of any mechanism to prevent the metastable state from becoming unstable at $|\mathfrak{H}|=\mathfrak{H}_c$, the hysteresis loop takes on the form shown in Fig. 9, instead of that of Fig. 5. Any loop intermediary between those shown in Fig. 6 and Fig. 9 is possible. However, as $T\to 0$, that of Fig. 7 remains.

We write down $\mathfrak{W}(\bar{\sigma})$ for T near T_c and $\bar{\sigma}$ small from (5.29), (5.30), and (6.7):

$$\mathfrak{W}(\bar{\sigma})\sim\pi^{-1}z^2\,|\,z_2\,|^{-1}\ln(|\,1-\alpha_2\,|+z^2), \tag{6.18}$$

with the parameter z determined from

$$|\,\bar{\sigma}\,|=|\,z_2\,|^{-1/2}(1-\alpha_2)^{1/2}-2\pi^{-1}z\,|\,z_2\,|^{-1}\ln(1-\alpha_2+z^2) \tag{6.19}$$

for $T\leq T_c$, and from

$$|\,\bar{\sigma}\,|=-2\pi^{-1}z\,|\,z_2\,|^{-1}\ln(\alpha_2-1+z^2) \tag{6.20}$$

for $T\geq T_c$. In particular, at $T=T_c$,

$$\mathfrak{W}(\bar{\sigma})\sim 2\pi^{-1}z^2\,|\,z_2\,|^{-1}\ln z, \tag{6.21}$$

with

$$|\,\bar{\sigma}\,|=-4\pi^{-1}z\,|\,z_2\,|^{-1}\ln z. \tag{6.22}$$

In other words, at $T=T_c$,

$$\mathfrak{W}(\bar{\sigma})\sim\pi\,|\,z_2\,|\,\bar{\sigma}^2/\ln\,|\,\bar{\sigma}\,|, \tag{6.23}$$

or, from (6.3), roughly

$$\langle\delta(\sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}}\sigma_{1,k}-2\mathfrak{N}\bar{\sigma})\rangle\sim\exp(2\pi\mathfrak{N}\,|\,z_2\,|\,\bar{\sigma}^2/\ln\,|\,\bar{\sigma}\,|) \tag{6.24}$$

for small $\bar{\sigma}$. Thus, even at the critical temperature, the distribution function does not deviate too much from a Gaussian.

Finally, we remark that, as seen from (6.3), the limit $\mathfrak{M}\to\infty$ has been taken before the limit $\mathfrak{N}\to\infty$. Some of the results of this stction depend on this order of taking limits in a way which will be made precise in a separate communication.

7. INVERSE MATRIX

To use the results of Sec. 2 to evaluate correlation functions and magnetizations in an arbitrary row, we need to evaluate the matrix elements of $\mathfrak{A}^{-1}$ in the subspace determined by $\delta^{(J,J',N)}$. In this subspace, the rows (columns) of $\mathfrak{A}^{-1}$ may only be labeled by U or D. We first note that because $\mathfrak{A}$ is nearly cyclic in the horizontal direction,

$$\mathfrak{A}^{-1}(j,k;j',k')=(1/2\mathfrak{N})\sum_{\theta}e^{i\theta(k-k')}[\mathfrak{B}^{-1}(\theta)]_{jj'}, \tag{7.1}$$

where $\mathfrak{B}(\theta)$ is defined by (3.3) and θ by (3.2).

We may easily find the elements of $\mathfrak{B}^{-1}$ in the U, D subspace by relating these elements to the elements of $\mathfrak{C}^{-1}$[$\mathfrak{C}$ given by (3.9)]. We first remark that if we rearrange the rows and columns of $\mathfrak{B}'$, $\mathfrak{B}$, and $\mathfrak{T}$ (as defined in Sec. 3) so that all R, L rows (columns) precede all U, D rows (columns) and call the resulting $4(2\mathfrak{M}+1)\times 4(2\mathfrak{M}+1)$ matrices,

$$\begin{bmatrix}\mathfrak{b}_{11}' & \mathfrak{b}_{12}'\\ 0 & \mathfrak{b}_{22}'\end{bmatrix},\quad\begin{bmatrix}\mathfrak{b}_{11} & \mathfrak{b}_{12}\\ \mathfrak{b}_{21} & \mathfrak{b}_{22}\end{bmatrix},\quad\text{and}\quad\begin{bmatrix}1 & 0\\ \mathfrak{t}_{21} & 1\end{bmatrix}, \tag{7.2}$$

where each entry is a $2(2\mathfrak{M}+1)\times 2(2\mathfrak{M}+1)$ matrix, then we may write, using Eq. (3.5),

$$\begin{bmatrix}\mathfrak{b}_{11} & \mathfrak{b}_{12}\\ \mathfrak{b}_{21} & \mathfrak{b}_{22}\end{bmatrix}^{-1}=\begin{bmatrix}\mathfrak{b}_{11}' & \mathfrak{b}_{12}'\\ 0 & \mathfrak{b}_{22}'\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ \mathfrak{t}_{21} & 1\end{bmatrix}$$

$$=\begin{bmatrix}\mathfrak{b}_{11}'^{-1} & -\mathfrak{b}_{11}'^{-1}\mathfrak{b}_{12}'\mathfrak{b}_{22}'^{-1}\\ 0 & \mathfrak{b}_{22}'^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ \mathfrak{t}_{21} & 1\end{bmatrix}. \tag{7.3}$$

The matrix $\mathfrak{C}$ is just $\mathfrak{b}_{22}'$ with U and D interchanged, so we have from (7.3) the relation

$$[\mathfrak{B}^{-1}]_{jl,j'l'}=[\mathfrak{C}^{-1}]_{jl,j'l'}\qquad l=U,D,\qquad l'=U,D. \tag{7.4}$$

We now compute $\mathfrak{C}^{-1}$ from the formula

$$[\mathfrak{C}^{-1}]_{jl,j'l'}=\text{cofactor }\mathfrak{C}_{j'l',jl}/\det\mathfrak{C}. \tag{7.5}$$

To evaluate these cofactors, we define the $2n\times 2n$ determinant $\bar{\mathfrak{C}}_n$ to be the determinant obtained from $\mathfrak{C}_n$ by striking out the first two rows and columns. Similarly, we define $\bar{\mathfrak{D}}_n$ to be the $(2n-1)\times(2n-1)$ determinant obtained from $\mathfrak{D}_n$ by striking out the first two rows and columns. We evaluate $\bar{\mathfrak{C}}_n$ and $\bar{\mathfrak{D}}_n$ exactly as we did in Sec. 3 and find

$$\bar{\mathfrak{C}}_n=\mathfrak{v}^2\lambda^n+\mathfrak{v}'^2\lambda'^n, \tag{7.6a}$$

$$\bar{\mathfrak{D}}_n=z_2^{-1}i\mathfrak{v}'\mathfrak{v}(\lambda^n-\lambda'^n). \tag{7.6b}$$

Because $\mathfrak{C}$ has only three nonvanishing diagonals, we find for $j\geq j'\geq 1$

$$[\mathfrak{B}^{-1}]_{jD,j'D}=-[\mathfrak{B}^{-1}]_{j'D,jD}{}^{*}$$
$$=-z_2{}^{j-j'}\mathfrak{h}^{j-j'}\bar{\mathfrak{D}}_{2\mathfrak{M}-j+1}\mathfrak{C}_{j'-1}/\mathfrak{C}_{2\mathfrak{M}}, \quad (7.7\text{a})$$

$$[\mathfrak{B}^{-1}]_{jU,j'U}=-[\mathfrak{B}^{-1}]_{j'U,jU}{}^{*}$$
$$=z_2{}^{j-j'}\mathfrak{h}^{j-j'}\bar{\mathfrak{C}}_{2\mathfrak{M}-j}\mathfrak{D}_{j'}/\mathfrak{C}_{2\mathfrak{M}}, \quad (7.7\text{b})$$

$$[\mathfrak{B}^{-1}]_{jU,j'D}=-[\mathfrak{B}^{-1}]_{j'D,jU}{}^{*}$$
$$=z_2{}^{j-j'}\mathfrak{h}^{j-j'+1}\bar{\mathfrak{C}}_{2\mathfrak{M}-j}\mathfrak{C}_{j'-1}/\mathfrak{C}_{2\mathfrak{M}}; \quad (7.7\text{c})$$

for $j>j'\geq 1$

$$[\mathfrak{B}^{-1}]_{jD,j'U}=-[\mathfrak{B}^{-1}]_{j'U,jD}{}^{*}$$
$$=-z_2{}^{j-j'}\mathfrak{h}^{j-j'-1}\bar{\mathfrak{D}}_{2\mathfrak{M}-j+1}\mathfrak{D}_{j'}/\mathfrak{C}_{2\mathfrak{M}}; \quad (7.7\text{d})$$

for $j>0$

$$[\mathfrak{B}^{-1}]_{jD,0U}=-[\mathfrak{B}^{-1}]_{0U,jD}{}^{*}$$
$$=z\mathfrak{c}\mathfrak{h}^{j-1}z_2{}^{j-1}\bar{\mathfrak{D}}_{2\mathfrak{M}-j+1}/\mathfrak{C}_{2\mathfrak{M}}, \quad (7.7\text{e})$$

$$[\mathfrak{B}^{-1}]_{jU,0U}=-[\mathfrak{B}^{-1}]_{0U,jU}{}^{*}$$
$$=-z\mathfrak{c}\mathfrak{h}^{j}z_2{}^{j-1}\bar{\mathfrak{C}}_{2\mathfrak{M}-j}/\mathfrak{C}_{2\mathfrak{M}}, \quad (7.7\text{f})$$

$$[\mathfrak{B}^{-1}]_{0U,0U}=-\mathfrak{c}\mathfrak{C}_{2\mathfrak{M}}/\bar{\mathfrak{C}}_{2\mathfrak{M}}, \quad (7.7\text{g})$$

$$[\mathfrak{B}^{-1}]_{jD,0D}=0, \quad (7.7\text{h})$$

$$[\mathfrak{B}^{-1}]_{0D,0D}=-\mathfrak{c}^{-1}, \quad (7.7\text{i})$$

and for all j

$$[\mathfrak{B}^{-1}]_{jU,0D}=0. \quad (7.7\text{j})$$

For fixed j, k, j' and k' as $\mathfrak{M}\rightarrow\infty$ and $\mathfrak{N}\rightarrow\infty$, we have for $j\geq j'\geq 1$

$$\mathfrak{A}^{-1}(j,k;j',k')_{DD}=-\mathfrak{A}^{-1}(j',k';j,k)_{DD}$$
$$=-(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}\alpha^{j'-j}z_1z_2{}^{-1}(1-z_1{}^2)^{-1}(e^{i\theta}-e^{-i\theta})(\alpha^{-1}-\alpha)^{-1}$$
$$\times\left[1+\alpha^{-2(j'-1)}(\mathfrak{v}'/\mathfrak{v})^2\left(\frac{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}+iz^2z_2{}^{-1}\mathfrak{v}/\mathfrak{v}'}{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2{}^{-1}\mathfrak{v}'/\mathfrak{v}}\right)\right], \quad (7.8\text{a})$$

$$\mathfrak{A}^{-1}(j,k;j',k')_{UU}=-\mathfrak{A}^{-1}(j',k';j,k)_{UU}$$
$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}\alpha^{j'-j}z_1z_2{}^{-1}(1-z_1{}^2)^{-1}(e^{i\theta}-e^{-i\theta})(\alpha^{-1}-\alpha)^{-1}$$
$$\times\left[1-\alpha^{-2j'}\left(\frac{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}+iz^2z_2{}^{-1}\mathfrak{v}/\mathfrak{v}'}{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2{}^{-1}\mathfrak{v}'/\mathfrak{v}}\right)\right], \quad (7.8\text{b})$$

$$\mathfrak{A}^{-1}(j,k;j',k')_{UD}=-\mathfrak{A}^{-1}(j',k';j,k)_{DU}$$
$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}\alpha^{j'-j}z_2{}^{-1}(1-z_1{}^2)^{-1}(\alpha^{-1}-\alpha)^{-1}[-1+z_1{}^2+\alpha^{-1}z_2\,|\,1+z_1e^{i\theta}\,|^2]$$
$$\times\left[1+\alpha^{-2(j'-1)}(\mathfrak{v}'/\mathfrak{v})^2\left(\frac{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}+iz^2z_2{}^{-1}\mathfrak{v}/\mathfrak{v}'}{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2{}^{-1}\mathfrak{v}'/\mathfrak{v}}\right)\right]; \quad (7.8\text{c})$$

for $j>j'\geq 1$

$$\mathfrak{A}^{-1}(j,k;j',k')_{DU}=-\mathfrak{A}^{-1}(j',k';j,k)_{UD}$$
$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}\alpha^{j'-j}z_2{}^{-1}(1-z_1{}^2)^{-1}(\alpha^{-1}-\alpha)^{-1}[1-z_1{}^2-z_2\alpha\,|\,1+z_1e^{i\theta}\,|^2]$$
$$\times\left[1-\alpha^{-2j'}\left(\frac{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}+iz^2z_2{}^{-1}\mathfrak{v}/\mathfrak{v}'}{(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2{}^{-1}\mathfrak{v}'/\mathfrak{v}}\right)\right]; \quad (7.8\text{d})$$

for $j\geq 1$

$$\mathfrak{A}^{-1}(j,k;0,k')_{DU}=-\mathfrak{A}^{-1}(0,k';j,k)_{UD}$$
$$=-(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}iz\alpha^{-j+1}[z_2(e^{i\theta}-1)(e^{i\theta}+1)^{-1}\mathfrak{v}/\mathfrak{v}'-iz^2]^{-1}, \quad (7.8\text{e})$$

$$\mathfrak{A}^{-1}(j,k;0,k')_{UU}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}zz_2^{-1}\alpha^{-j}[(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2^{-1}\mathfrak{v}'/\mathfrak{v}]^{-1}$$

$$=-\mathfrak{A}^{-1}(0,k';j,k)_{UU}, \tag{7.8f}$$

$$\mathfrak{A}^{-1}(0,k;0,k')_{UU}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}[(e^{i\theta}-1)(e^{i\theta}+1)^{-1}-iz^2z_2^{-1}\mathfrak{v}'/\mathfrak{v}]^{-1}, \tag{7.8g}$$

$$\mathfrak{A}^{-1}(j,k;0,k')_{DD}=0, \tag{7.8h}$$

$$\mathfrak{A}^{-1}(0,k;0,k')_{DD}=-(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\, e^{i\theta(k-k')}(e^{i\theta}+1)(e^{i\theta}-1)^{-1}; \tag{7.8i}$$

for all $j\geq 0$

$$\mathfrak{A}^{-1}(j,k;0,k')_{UD}=0. \tag{7.8j}$$

In the above, $\mathfrak{v}/\mathfrak{v}'$ is given by (3.20). We note in particular that

$$\mathfrak{A}^{-1}(j,k;j',k)_{DD}=\mathfrak{A}^{-1}(j,k;j',k)_{UU}=0, \tag{7.9}$$

and that as $j'\to\infty$, the elements of (7.8) approach the corresponding inverse elements of the bulk problem as given by Montroll, Potts, and Ward.[5]

In Appendix C we compute all elements of $\mathfrak{B}^{-1}$ as $\mathfrak{M}\to\infty$, using a Weiner–Hopf technique.

8. BOUNDARY SPIN-SPIN CORRELATION

We may now use the considerations of Sec. 2 to calculate

$$\mathfrak{S}_{J,J'}(N,\mathfrak{H})=\langle\sigma_{J,0}\sigma_{J',N}\rangle \tag{8.1}$$

for the special case $J=J'=1$. Define y to be the nonzero submatrix of $\delta^{(1,1,N)}$; then

$$y=\begin{array}{cc} & \begin{array}{cccc} 10 & 1N & 00 & 0N \\ D & D & U & U \end{array} \\ \begin{array}{c} 10\ D \\ 1N\ D \\ 00\ U \\ 0N\ U \end{array} & \left[\begin{array}{cccc} 0 & 0 & -(z^{-1}-z) & 0 \\ 0 & 0 & 0 & -(z^{-1}-z) \\ z^{-1}-z & 0 & 0 & 0 \\ 0 & z^{-1}-z & 0 & 0 \end{array}\right] \end{array}. \tag{8.2}$$

Define $\mathfrak{Q}$ to be the elements of $\mathfrak{A}^{-1}$ in the subspace defined by y; then

$$\mathfrak{Q}=\left[\begin{array}{cccc} 0 & \mathfrak{A}^{-1}(1,0;1,N)_{DD} & \mathfrak{A}^{-1}(1,0;0,0)_{DU} & \mathfrak{A}^{-1}(1,0;0,N)_{DU} \\ \mathfrak{A}^{-1}(1,N;1,0)_{DD} & 0 & \mathfrak{A}^{-1}(1,N;0,0)_{DU} & \mathfrak{A}^{-1}(1,N;0,N)_{DU} \\ \mathfrak{A}^{-1}(0,0;1,0)_{UD} & \mathfrak{A}^{-1}(0,0;1,N)_{UD} & 0 & \mathfrak{A}^{-1}(0,0;0,N)_{UU} \\ \mathfrak{A}^{-1}(0,N;1,0)_{UD} & \mathfrak{A}^{-1}(0,N;1,N)_{UD} & \mathfrak{A}^{-1}(0,N;0,0)_{UU} & 0 \end{array}\right]. \tag{8.3}$$

Then (2.15) may be re-expressed as the product of two Pfaffians:

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})&\\ &=\pm z^2\,\mathrm{Pf}(y^{-1}+\mathfrak{Q})\,\mathrm{Pf}(y)\\ &=\pm(1-z^2)^2\{[\mathfrak{A}^{-1}(1,0;0,0)_{DU}-(z^{-1}-z)^{-1}]^2-\mathfrak{A}^{-1}(1,0;1,N)_{DD}\mathfrak{A}^{-1}(0,0;0,N)_{UU}-[\mathfrak{A}^{-1}(1,N;0,0)_{DU}]^2\}.\end{aligned} \tag{8.4}$$

Using the explicit forms of the matrix elements of $\mathfrak{A}^{-1}$ found in the last section, we have

$$\pm\mathfrak{S}_{1,1}(N,\mathfrak{H})=\left[\frac{z(1-z^2)}{2\pi}z_1\int_{-\pi}^{\pi}d\theta\,|1+e^{i\theta}|^2\,(z_2^2|1+z_1e^{i\theta}|^2-z^2z_1|1+e^{i\theta}|^2-z_2(1-z_1^2)\alpha)^{-1}-z\right]^2$$

$$-(1-z^2)^2\left\{\left[\frac{z}{2\pi}\int_{-\pi}^{\pi}d\theta\,e^{iN\theta}\,|1+e^{i\theta}|^2\,z_1(z_2^2|1+z_1e^{i\theta}|^2-z^2z_1|1+e^{i\theta}|^2-z_2(1-z_1^2)\alpha)^{-1}\right]^2\right.$$

$$-\left[(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,e^{iN\theta}z_1(e^{i\theta}-1)(e^{-i\theta}+1)(z_2^2|1+z_1e^{i\theta}|^2-z^2z_1|1+e^{i\theta}|^2-z_2(1-z_1^2)\alpha)^{-1}\right]$$

$$\times(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\,e^{iN\theta}(e^{i\theta}+1)(e^{i\theta}-1)^{-1}$$

$$\left.\times[1+z_1z^2|1+e^{i\theta}|^2\,(z_2^2|1+z_1e^{i\theta}|^2-z^2z_1|1+e^{i\theta}|^2-z_2(1-z_1^2)\alpha)^{-1}]\right\},\tag{8.5}$$

where the first term is recognized as $\mathfrak{M}_1^2$. If $\mathfrak{H}\to\infty$, the right-hand side of (8.5) goes to 1; so, since we know that $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ must go to 1 when $\mathfrak{H}\to\infty$, the plus sign must be chosen in (8.5) when $\mathfrak{H}$ is large. Because $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ is a continuous function of $\mathfrak{H}$, this consideration determines the correct sign in (8.5) for all $\mathfrak{H}$ unless there is an $\mathfrak{H}$ for which $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ vanishes. This only occurs at fixed N when the lattice is antiferromagnetic and $T<T_c$. In this case, when $\mathfrak{H}$ is small, $\mathfrak{S}_{1,1}$ tends to alternate in sign. The plus sign in (8.5) still holds but now we will determine it by continuity from $T=0$, where $\mathfrak{S}_{1,1}(N,\mathfrak{H})=(-1)^N$ as is explicitly shown later.

Before considering asymptotic expansions, it is instructive to consider a number of simple limiting cases. To do this in a systematic fashion and also because it clearly exhibits the several types of exponential behavior as $N\to\infty$, we will shift the contours of integration of the integrals with a term $e^{iN\theta}$ in the integrand from the unit circle to the contour Γ which goes around the branch cuts of α inside the unit circle. In doing this, we pick up contributions from the poles at $e^{i\theta}=1,\,-1$, and $\mathfrak{r}$. The form of $\mathfrak{S}_{1,1}$ now depends on whether or not $\mathfrak{r}$ is in the cut $e^{i\theta}$-plane as determined by (5.13) and (5.14). We also rationalize the denominators in (8.5) and obtain the following forms for the correlation where $\zeta=e^{i\theta}$ and we use the following notation:

$$\Xi_1=4z_2(1-z_1^2)(2\pi i)^{-1}\int_\Gamma d\zeta(\zeta^2-1)^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}\zeta^{N-1}(\mathfrak{r}^{-1}\zeta-1)\alpha^{-1},$$

$$\Xi_2=z_2(1-z_1^2)(2\pi i)^{-1}\int_\Gamma d\zeta(\zeta-1)(\zeta+1)^{-1}\zeta^{N-1}(\zeta-\mathfrak{r})^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}\alpha^{-1},$$

$$\Xi_3=4z_2^2(z_1^{-1}-z_1)^2z^2(z_2^2-z^2)^{-2}\left\{\left[(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N+1}(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}\alpha^{-1}\right]\right.$$

$$\times\left[(2\pi i)^{-1}\int_\Gamma d\zeta'\zeta'^{N-1}(\zeta'^2-1)^{-1}(\zeta'-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta'^{-1}-1)^{-1}\alpha^{-1}\right]$$

$$\left.-\left[(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^N(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}\alpha^{-1}\right]^2\right\},$$

$$\Xi_4=z_2(1-z_1^2)(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}(\zeta+1)(\zeta-1)^{-1}\alpha^{-1};$$

if $T>T_c$ and $z^2\geq|z_2|(1-\alpha_1)/(1+\alpha_1)$, then

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})=\mathfrak{M}_1^2+\mathfrak{r}^Nz^2[(1+z_1)^2(z^2-z_2^2)^2-z_2^2(1-z_1)^2(1-z^2)^2](1-z^2)^{-1}z_1^{-2}(z^2-z_2^2)^{-3}(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}$$
$$\times[(1-z_1)^2-z_2^2(1+z_1)^2-\Xi_1]+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}\Xi_2+\Xi_3;\tag{8.6}$$

if $T<T_c$, $E_1>0$, and either $z^2\geq|z_2|(1-\alpha_1)/(1+\alpha_1)$ or $z^2\leq|z_2|(1-\alpha_2)/(1+\alpha_2)$, then

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})=\mathfrak{M}_1^2-\mathfrak{r}^Nz^2[(1+z_1)^2(z^2-z_2^2)^2-z_2^2(1-z_1)^2(1-z^2)^2](1-z^2)^{-1}(z^2-z_2^2)^{-3}(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}z_1^{-2}\Xi_1+\Xi_3;\tag{8.7}$$

if $T>T_c$ and $z^2 \leq |z_2|(1-\alpha_1)/(1+\alpha_1)$, then

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})=\mathfrak{M}_1^2+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}\Xi_2+\Xi_3; \tag{8.8}$$

if $T<T_c$, $E_1>0$, and $|z_2|(1-\alpha_2)/(1+\alpha_2)\leq z^2\leq |z_2|(1-\alpha_1)/(1+\alpha_1)$, then

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})=\mathfrak{M}_1^2+\Xi_3; \tag{8.9}$$

if $T<T_c$, $E_1<0$, and $z^2\geq |z_2|(1-\alpha_1)/(1+\alpha_1)$, then

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})=&\mathfrak{M}_1^2-z^2(-1)^N(z_2^2-z^2)^{-2}z_1^{-2}(\mathfrak{r}+1)^{-1}(\mathfrak{r}^{-1}+1)^{-1}[z_2^2(1-z_1)^2-(1+z_1)^2]\\&\times\{(\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}[(1-z_1)^2-z_2^2(1+z_1)^2-(1-z^2)^{-1}(z_2^2-z^2)^{-1}\mathfrak{r}^N\\&\times[(z_2^2-z^2)^2(1+z_1)^2-z_2^2(1-z^2)^2(1-z_1)^2]]+\Xi_4\}\\&+\mathfrak{r}^Nz^2[(1+z_1)^2(z^2-z_2)^2-z_2^2(1-z_1)^2(1-z^2)^2]\\&\times(1-z^2)^{-1}z_1^{-2}(z^2-z_2^2)^{-3}(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}[(1-z_1)^2-z_2^2(1+z_1)^2-\Xi_1]\\&+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}\Xi_2+\Xi_3;\end{aligned} \tag{8.10}$$

and if $T<T_c$, $E_1<0$, and $z^2<|z_2|(1-\alpha_1)/(1+\alpha_1)$, then

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})=&\mathfrak{M}_1^2-z^2(-1)^N(z_2^2-z^2)^{-2}z_1^{-2}(\mathfrak{r}+1)^{-1}(\mathfrak{r}^{-1}+1)^{-1}[z_2^2(1-z_1)^2-(1+z_1)^2]\\&\times\{(\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}[(1-z_1)^2-z_2^2(1+z_1)^2]+\Xi_4\}\\&+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}\Xi_2+\Xi_3,\end{aligned} \tag{8.11}$$

where it is convenient to note that

$$(\mathfrak{r}-1)(\mathfrak{r}^{-1}-1)z_1(1-z^2)(z^2-z_2^2)=-z^2[(1-z_1)^2-z_2^2(1+z_1)^2+4z_1z^2] \tag{8.12a}$$

and

$$(\mathfrak{r}+1)(\mathfrak{r}^{-1}+1)z_1(1-z^2)(z^2-z_2^2)=z^2(1+z_1)^2-z^2z_2^2(1-z_1)^2-4z_1z_2^2. \tag{8.12b}$$

We now consider several limiting cases.

(i) $E_1\to\infty$. In this case, $T<T_c$ and $\alpha_1=\alpha_2>0$. Therefore, (8.7) holds and, using (5.1) for $\mathfrak{M}_1$, we easily see that

$$\lim_{E_1\to\infty}\mathfrak{S}_{1,1}(N,\mathfrak{H})=1. \tag{8.13}$$

(ii) $E_1\to-\infty$. In this case, $T<T_c$ and $\alpha_1=-(1-|z_2|)(1+|z_2|)^{-1}$. Therefore, (8.11) always holds and we have

$$\lim_{E_1\to-\infty}\mathfrak{S}_{1,1}(N,\mathfrak{H})=(-1)^N. \tag{8.14}$$

(iii) $E_2\to\pm\infty$. In this case, $T<T_c$, $\alpha_1=\alpha_2=0$, and $\mathfrak{r}=(1-|z|)(1+|z|)^{-1}$. If $E_1>0$, (8.7) holds and

$$\lim_{E_2\pm\infty}\mathfrak{S}_{1,1}(N,\mathfrak{H})=1. \tag{8.15a}$$

If $E_1<0$, (8.11) holds and

$$\lim_{E_2\to\pm\infty}\mathfrak{S}_{1,1}(N,\mathfrak{H})=(-1)^N. \tag{8.15b}$$

(iv) $E_1\to 0$. For this limit, it is easier to use (8.5) directly to see that

$$\lim_{E_1\to 0}\mathfrak{S}_{1,1}(N,\mathfrak{H})=z^2. \tag{8.16}$$

(v) $T\to 0$. In this limit, $\mathfrak{r}\to 0$, $\alpha_1\sim\alpha_2=0$. If $E_1>0$, then (8.7) holds, none of the integrals contributes and $\mathfrak{S}_{1,1}(N,\mathfrak{H})=1$. If $E_1<0$, then if $|\mathfrak{H}|>2|E_1|+E_2$, then $\lim_{T\to 0}(z^2-z_2^2)\mathfrak{r}^{-1}=\infty$ and $\mathfrak{S}_{1,1}(N,\mathfrak{H})=1$. If $|\mathfrak{H}|<2|E_1|+E_2$, then $\lim_{T\to 0}(z^2-z_2^2)\mathfrak{r}^{-1}=4$ and $\mathfrak{S}_{1,1}(N,\mathfrak{H})=(-1)^N$. If $|\mathfrak{H}|=2|E_1|+E_2$, then $\lim_{T\to 0}(z^2-z_2^2)\mathfrak{r}^{-1}=8$ and $\mathfrak{S}_{1,1}(N,\mathfrak{H})=\frac{1}{4}(1+(-1)^N)$.

(vi) $E_2\to 0$. In this limit, we have reduced the vertical bond strength to zero, $T>T_c$, $\alpha_1=\alpha_2^{-1}=z_1$ and (8.6) holds. The integrals vanish and we have

$$\lim_{E_2\to 0}\mathfrak{S}_{1,1}(N,\mathfrak{H})=\frac{(1-z^2)(1+z_1)^2}{4z^2z_1+(1-z_1)^2}\left[\frac{z^2}{1-z^2}+\mathfrak{r}^N\left(\frac{1-z_1}{1+z_1}\right)^2\right]. \tag{8.17}$$

This is the spin–spin correlation function for the one-dimensional Ising model. It agrees with the one-dimensional calculation of Sec. 5 of II if we note that

$$\lim_{E_2\to 0} \mathfrak{r} = \lambda_-/\lambda_+, \tag{8.18}$$

where λ_+, λ_- are defined by (5.9) of II.

(vii) $\mathfrak{H}\to 0$. The behavior of $\mathfrak{M}_1(\mathfrak{H})$ in this limit has already been obtained in (5.19). Therefore,

(a) if $T>T_c$, Eq. (8.8) holds and

$$\mathfrak{S}_{1,1}(N,0) = -z_2^{-1}(z_1^{-1}-z_1)(2\pi i)^{-1}\int_\Gamma d\zeta\, \zeta^N(\zeta^2-1)^{-1}\alpha^{-1}; \tag{8.19a}$$

(b) if $T<T_c$ and $E_1>0$, (8.7) holds and

$$\mathfrak{S}_{1,1}(N,0) = \tfrac{1}{4}z_2^{-2}z_1^{-1}[z_2^2(1+z_1)^2-(1-z_1)^2]-z_2^{-1}(z_1^{-1}-z_1)(2\pi i)^{-1}\int_\Gamma d\zeta\, \zeta^N(\zeta^2-1)^{-1}\alpha^{-1}; \tag{8.19b}$$

and

(c) if $T<T_c$ and $E_1<0$, (811) holds and

$$\mathfrak{S}_{1,1}(N,0) = -(-1)^N\tfrac{1}{4}z_1^{-1}z_2^{-2}[(1-z_1)^2z_2^2-(1+z_1)^2]-z_2^{-1}(z_1^{-1}-z_1)(2\pi i)^{-1}\int_\Gamma d\zeta\, \zeta^N(\zeta^2-1)^{-1}\alpha^{-1}. \tag{8.19c}$$

We now turn to the question of the behavior of $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ for large N. There are many special cases and we make no claim to completeness. We first consider the regions in which

$$N\,|\,1-T/T_c\,|\gg 1 \tag{8.20}$$

and

$$N\,|\,1-z^2(1+\alpha_2)(1-\alpha_2)^{-1}\,|\,z_2\,|^{-1}\,|\gg 1. \tag{8.21}$$

In this region, as the expressions (8.6)–(8.11) show, the correlation function approaches its limiting value exponentially rapidly. We will compute the asymptotic series multiplying the exponential for the several regions (8.6)–(8.11) and explicitly exhibit the first few terms. The method used closely follows Sec. 3 of I. We then will consider the region where T is near T_c but where (8.21) still holds by assuming that N is such that $N\,|\,1-T/T_c\,|$ is *fixed* and of order 1. In this case, the correlation functions do not approach their limiting value exponentially but only as an inverse power of N. The coefficients of the first few powers of N will be evaluated as functions of $N\,|\,1-T/T_c\,|$. We next examine the case where $\mathfrak{H}=0$ and $N\,|\,1-T/T_c\,|$ is of order 1. Here, we obtain approximations to the simpler expressions (8.19). Finally, we consider the case where $T=T_c$ and Nz^2 is fixed and of order 1.

A. $T>T_c$, $N\,|\,1-T_c/T\,|\gg 1$

From (5.15), when $\mathfrak{r}$ is in the cut $e^{i\theta}$ plane, $|\,\mathfrak{r}\,|\le|\,\alpha_1\,|$. Furthermore, $0\le|\,\alpha_1\,|\le|\,\alpha_2^{-1}\,|\le 1$, so that each integral in (8.6) and (8.8) is of order α_2^{-N}. Thus, for all values of $\mathfrak{H}$, we have

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H}) \doteqdot{}& \mathfrak{M}_1^2+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}\\ &\times z_1^{-2}(1-z_2^2)(2\pi)^{-1}\int_{\alpha_1}^{\alpha_2^{-1}} d\zeta\,\zeta^{N-1}(\zeta-1)(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}\\ &\times[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2},\end{aligned} \tag{8.22}$$

where $\doteqdot$ is defined as in I. We now call $\zeta_1=\alpha_2\zeta$ and write

$$\begin{aligned}(2\pi)^{-1}&\int_{\alpha_1}^{\alpha_2^{-1}} d\zeta\,\zeta^{N-1}(\zeta-1)(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}\\ &=\tfrac{1}{2}\alpha_2^{-N}\pi^{-1}\int_{\alpha_1\alpha_2}^{1} d\zeta_1(\alpha_2^{-1}\zeta_1-1)(\alpha_2^{-1}\zeta_1+1)^{-1}(\alpha_2^{-1}\zeta_1-\mathfrak{r})^{-1}(\alpha_2\mathfrak{r}^{-1}-\zeta_1)^{-1}\zeta_1^{N}\\ &\quad\times[(1-\alpha_1\alpha_2^{-1}\zeta_1)(1-\alpha_1\alpha_2\zeta_1^{-1})(1-\alpha_2^{-2}\zeta_1)(\zeta_1^{-1}-1)]^{1/2}.\end{aligned} \tag{8.23}$$

Define as in I

$$x_1=(1-\alpha_1/\alpha_2)^{-1}(1+\alpha_1/\alpha_2), \tag{8.24}$$

$$x_2=(1-\alpha_1\alpha_2)^{-1}(1+\alpha_1\alpha_2), \tag{8.25}$$

$$x_3=(\alpha_2^2-1)^{-1}(\alpha_2^2+1). \tag{8.26}$$

We further define from (5.23)

$$x_4=(\alpha_2-1)^{-1}(\alpha_2+1)=-\tau_2^{-1}, \tag{8.27}$$

$$x_5=(1-\alpha_2\mathfrak{r})^{-1}(1+\alpha_2\mathfrak{r}), \tag{8.28}$$

and

$$x_6=(1-\alpha_2\mathfrak{r}^{-1})^{-1}(1+\alpha_2\mathfrak{r}^{-1}). \tag{8.29}$$

Then we may write the right-hand side of (8.23) as

$$\tfrac{1}{2}\alpha_2^{-N-1}\pi^{-1}(x_1+1)^{-1/2}(x_2+1)^{-1/2}(x_3+1)^{-1/2}x_4^{-1}(x_5-1)(x_6-1)\int_{\alpha_1\alpha_2}^{1} d\zeta_1\zeta_1^{N-1}(1+\zeta_1)^{-1/2}(1-\zeta_1)^{1/2}\mathfrak{A}_>\left(\frac{1-\zeta_1}{1+\zeta_1}\right), \tag{8.30}$$

where

$$\mathfrak{A}_>(\eta)=(1+x_4\eta)(1+x_4^{-1}\eta)^{-1}(1-x_5\eta)^{-1}(1-x_6\eta)^{-1}[(1+x_1\eta)(1-x_2\eta)(1+x_3\eta)]^{1/2}. \tag{8.31}$$

We will be able to reduce many of our asymptotic expressions to forms similar to (8.30). It is thus convenient to consider the following generalization:

$$\int^1 d\zeta_1\zeta_1^{N'}(1+\zeta_1)^{-p}(1-\zeta_1)^{q}\mathfrak{R}\left(\frac{1-\zeta_1}{1+\zeta_1}\right). \tag{8.32}$$

We expand $\mathfrak{R}(\eta)$ in a power series as

$$\mathfrak{R}(\eta)=\sum_{n=0}^{\infty}\mathfrak{R}_n\eta^n, \tag{8.33}$$

which we substitute into (8.32). The lower limit of integration in (8.32) if it is not 1, may always be extended to zero without altering the asymptotic series. Integrating term by term, we obtain

$$\int^1 d\zeta_1\zeta_1^{N'}(1+\zeta_1)^{-p}(1-\zeta_1)^{q}\mathfrak{R}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)$$
$$\doteq\sum_{n=0}^{\infty}\mathfrak{R}_n\Gamma(N'+1)\Gamma(n+q+1)[\Gamma(n+q+2+N')]^{-1}2^{-n-p}F(n+p,n+q+1;N+n+q+2;\tfrac{1}{2}). \tag{8.34}$$

In (8.34), the sum over n is to be interpreted in the sense of an asymptotic series and we have used Euler's integral representation of the hypergeometric function F.[11] We may now rearrange the series to obtain the result

$$\int^1 d\zeta_1\zeta_1^{N'}(1+\zeta_1)^{-p}(1-\zeta_1)^{q}\mathfrak{R}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)$$
$$\doteq N'!\sum_{m=0}^{\infty}2^{-m-p}\Gamma(m+p)\Gamma(m+q+1)[\Gamma(N'+m+q+2)]^{-1}\sum_{n=0}^{m}\mathfrak{R}_n[\Gamma(p+n)(m-n)!]^{-1}. \tag{8.35}$$

In the present case, we define $\mathfrak{A}_{n>}$ by

$$\mathfrak{A}_>(\eta)=\sum_{n=0}^{\infty}\mathfrak{A}_{n>}\eta^n, \tag{8.36}$$

where the first few terms are

$$\mathfrak{A}_{0>}=1, \tag{8.37}$$

$$\mathfrak{A}_{1>}=x_4-x_4^{-1}+x_5+x_6+\tfrac{1}{2}(x_1-x_2+x_3), \tag{8.38}$$

$$\mathfrak{A}_{2>}=x_4^{-2}+x_5^2+x_6^2-\tfrac{1}{8}(x_1^2+x_2^2+x_3^2)-1+(x_4-x_4^{-1})(x_5+x_6)+x_5x_6$$
$$+\tfrac{1}{2}(x_4-x_4^{-1}+x_5+x_6)(x_1-x_2+x_3)-\tfrac{1}{4}(x_1x_2-x_1x_3+x_2x_3). \tag{8.39}$$

We now may specialize (8.35) to (8.30) and obtain

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteq\mathfrak{M}_1^2+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}(1-z_2^2)$$
$$\times\alpha_2^{-N-1}\pi^{-1}(x_1+1)^{-1/2}(x_2+1)^{-1/2}(x_3+1)^{-1/2}x_4^{-1}(x_5-1)(x_6-1)$$
$$\times2^{-3/2}(N-1)!\sum_{m=0}^{\infty}[\Gamma(N+\tfrac{3}{2}+m)]^{-1}\Gamma(m+\tfrac{1}{2})\Gamma(m+\tfrac{3}{2})2^{-m}\sum_{n=0}^{m}\mathfrak{A}_{n>}[(m-n)!\Gamma(n+\tfrac{1}{2})]^{-1}. \tag{8.40}$$

[11] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 59.

This is the desired asymptotic expansion for $T>T_c$. For completeness, we write down the first three terms as $N\to\infty$,

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}(1-z_2^2)$$
$$\times\alpha_2^{-N-1}\pi^{-1/2}(x_1+1)^{-1/2}(x_2+1)^{-1/2}(x_3+1)^{-1/2}x_4^{-1}(x_5-1)(x_6-1)$$
$$\times2^{-5/2}N^{-3/2}\{1+(3/4N)\mathfrak{A}_{1>}+(5/32N^2)[6\mathfrak{A}_{2>}-1]+O(N^{-3})\}. \tag{8.41}$$

For this asymptotic series to be valid, we must have $N\gg\mathfrak{A}_{1>}$ which implies the restrictions

$$N\gg x_3 \tag{8.42}$$

and

$$N\gg|x_5+x_6|=|2(1-\alpha_2^2)(1-\alpha_2\mathfrak{r})^{-1}(1-\alpha_2\mathfrak{r}^{-1})^{-1}|. \tag{8.43}$$

For all values of $\mathfrak{H}$, both of these requirements are satisfied if N is much larger than $(T/T_c-1)^{-1}$. We may therefore let $\mathfrak{H}\to0$ and find that the leading term multiplying the exponential is $N^{-3/2}$, which is to be compared with the $N^{-1/2}$ behavior of the spin–spin correlation function for two spins in the same row at $H=0$ in the bulk case above T_c which was found in I.

B. $T<T_c$, $E_1>0$, $N[1-T/T_c]\gg1$

When $T<T_c$ and $E_1\geq0$, there are two cases. If $z^2<|z^2|(1-\alpha_2)/(1+\alpha_2)$, then (8.7) holds. By (5.16), we see that $\mathfrak{r}$ is real and $0\leq\alpha_2\leq\mathfrak{r}\leq1$. Therefore, we retain the leading exponential terms to find

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteqdot\mathfrak{M}_1^2-4\mathfrak{r}^N(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}(1-z^2)^{-1}(z^2-z_2^2)^{-3}z^2(z_1^{-2}-1)z_2[(1+z_1)^2(z^2-z_2^2)^2-z_2^2(1-z_1)^2(1-z^2)^2]$$
$$\times(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N-1}(\zeta^2-1)^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}(\zeta\mathfrak{r}^{-1}-1)\alpha^{-1}. \tag{8.44}$$

We proceed as in the previous case to obtain an asymptotic expansion to the integral

$$(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N-1}(\zeta^2-1)^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}(\zeta\mathfrak{r}^{-1}-1)\alpha^{-1}$$
$$=\frac{-1}{2\pi}z_2^{-1}(1-z_1^2)^{-1}(1-z_2^2)\alpha_2^Nx_4^{-1}(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-x_3-1)^{-1/2}(x_4-1)^2(x_5-1)(x_6+1)^{-1}$$
$$\times\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta_1^{N-1}(1-\zeta_1)^{1/2}(1+\zeta_1)^{-1/2}\mathfrak{A}_<^{(1)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right), \tag{8.45}$$

where $\zeta_1=\alpha_2^{-1}\zeta$ and

$$\mathfrak{A}_<^{(1)}(\eta)=[1-x_4\eta]^{-1}[1-x_4^{-1}\eta]^{-1}[1+x_5\eta]^{-1}[1+x_6\eta]\{[1+x_2\eta][1-x_1\eta][1-x_3\eta]\}^{1/2}. \tag{8.46}$$

Expand $\mathfrak{A}_<^{(1)}(\eta)$ as

$$\mathfrak{A}_<^{(1)}(\eta)=\sum_{n=0}^\infty\mathfrak{A}_{n<}^{(1)}\eta^n, \tag{8.47}$$

where the first few terms are

$$\mathfrak{A}_{0<}^{(1)}=1, \tag{8.48}$$

$$\mathfrak{A}_{1<}^{(1)}=\tfrac12(x_2-x_1-x_3)+x_4+x_4^{-1}-x_5+x_6, \tag{8.49}$$

$$\mathfrak{A}_{2<}^{(1)}=x_4^2+x_4^{-2}+x_5^2-\tfrac18(x_2^2+x_1^2+x_3^2)+1+(x_4+x_4^{-1})(x_6-x_5)$$
$$-x_5x_6+(x_4+x_4^{-1}-x_5+x_6)\tfrac12(x_2-x_1-x_3)-\tfrac14(x_1x_2+x_3x_3-x_1x_3). \tag{8.50}$$

We then use (8.35) to obtain

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteqdot\mathfrak{M}_1^2+\pi^{-1}\alpha_2^N\mathfrak{r}^N(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}(1-z^2)^{-1}(z^2-z_2^2)^{-3}z^2z_1^{-2}(1-z_2^2)$$
$$\times[(1+z_1)^2(z^2-z_2^2)^2-z_2^2(1-z_1)^2(1-z^2)^2](x_1+1)^{-1/2}(x_2+1)^{-1/2}$$
$$\times(-x_3-1)^{-1/2}(1-x_4)^2(x_5-1)(x_6+1)^{-1}2^{1/2}x_4^{-1}(N-1)!$$
$$\times\sum_{m=0}^\infty2^{-m}\Gamma(m+\tfrac12)\Gamma(m+\tfrac32)[\Gamma(N+\tfrac32+m)]^{-1}\sum_{n=0}^m\mathfrak{A}_{n<}^{(1)}[\Gamma(\tfrac12+n)(m-n)!]^{-1}. \tag{8.51}$$

Explicitly, for large N, this becomes

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\alpha_2^N\mathfrak{r}^N(\mathfrak{r}^{-1}-\mathfrak{r})^{-2}(1-z^2)^{-1}(z^2-z_2^2)^{-3}z^2z_1^{-2}(1-z_2^2)$$
$$\times[(1+z_1)^2(z^2-z_2^2)^2-z_2^2(1-z_1)^2(1-z^2)^2](x_1+1)^{-1/2}(x_2+1)^{-1/2}$$
$$\times(-x_3-1)^{-1/2}(1-x_4)^2(x_5-1)(x_6+1)^{-1}2^{-1/2}\pi^{-1/2}N^{-3/2}$$
$$\times\{1+(3/4N)\mathfrak{A}_{1<}^{(1)}+(5/32N^2)(6\mathfrak{A}_{2<}^{(1)}-1)+O(N^{-3})\}. \tag{8.52}$$

For this asymptotic expansion to be valid, we must have

$$N\gg|x_3|$$

and

$$N\gg x_6, \tag{8.53}$$

which holds if (8.20) and (8.21) are obeyed.

We may let $\mathfrak{H}\to 0$ without violating (8.53) and find that the leading term in the series multiplying the exponential is $N^{-3/2}$, exactly as it was for $T>T_c$ at $\mathfrak{H}=0$. This is to be contrasted with the N^{-2} behavior that the analogous term in the bulk correlation function below T_c at $H=0$ exhibits, as was derived in I.

If $z^2>|z_2|(1-\alpha_2)/(1+\alpha_2)$, then (8.7) or (8.9) holds. In either case, the terms of leading exponential order are given by

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteqdot\mathfrak{M}_1^2+4z_2^2(z_1^{-1}-z_1)^2z^2(z_2^2-z^2)^{-2}$$
$$\times\left\{\left[(2\pi i)^{-2}\int_\Gamma d\zeta\,\zeta^{N+1}(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}\alpha^{-1}\right]\right.$$
$$\times\left[(2\pi i)^{-1}\int_\Gamma d\zeta'\zeta'^{N-1}(\zeta'^2-1)^{-1}(\zeta'-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta'^{-1}-1)^{-1}\alpha^{-1}\right]$$
$$\left.-\left[(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^N(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}\alpha^{-1}\right]^2\right\}. \tag{8.54}$$

The three integrals in (8.54) differ only in the power of ζ in the integrand and clearly all have the same leading order term. To display the cancellation that occurs, we first write

$$(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^N(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}\alpha^{-1}$$
$$=-\alpha_2^{N+2}(1-z_2^2)z_2^{-1}(1-z_1^2)^{-1}x_4^{-1}(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-x_3-1)^{-1/2}$$
$$\times(x_4-1)^2(x_5+1)(x_6+1)(2\pi)^{-1}\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta_1^N(\zeta_1+1)^{-5/2}(1-\zeta_1)^{1/2}\mathfrak{A}_<^{(2)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right), \tag{8.55}$$

where

$$\mathfrak{A}_<^{(2)}(\eta)=[1-x_4\eta]^{-1}[1-x_4^{-1}\eta]^{-1}[1+x_5\eta]^{-1}[1+x_6\eta]^{-1}\{[1+x_2\eta][1-x_1\eta][1-x_3\eta]\}^{1/2}. \tag{8.56}$$

We now write

$$\zeta^{N+1}=[1-(1-\zeta)]\zeta^N \tag{8.57}$$

and

$$\zeta^{N-1}=\zeta^N[1-(1-\zeta^{-1})]. \tag{8.58}$$

Using (8.55), (8.57), and (8.58), we are able to write (8.54) as

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteqdot\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}z^2(z_2^2-z^2)^{-2}(1-z_2^2)^2\alpha_2^{2N+4}x_4^{-2}(x_1+1)^{-1}(x_2+1)^{-1}(-x_3-1)^{-1}(x_4-1)^4(x_5+1)^2(x_6+1)^2$$
$$\times\left\{\left[\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta_1^{N-1}(\zeta_1+1)^{-5/2}(1-\zeta_1)^{5/2}\mathfrak{A}_<^{(2)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)\right]\left[\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta^N(\zeta_1+1)^{-5/2}(1-\zeta)^{1/2}\mathfrak{A}_<^{(2)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)\right]\right.$$
$$\left.-\left[\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta_1^{N-1}(\zeta_1+1)^{-5/2}(1-\zeta_1)^{3/2}\mathfrak{A}_<^{(2)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)\right]\left[\int_{\alpha_1/\alpha_2}^1 d\zeta_1\zeta^N(\zeta+1)^{-5/2}(1-\zeta)^{3/2}\mathfrak{A}_<^{(2)}\left(\frac{1-\zeta_1}{1+\zeta_1}\right)\right]\right\}. \tag{8.59}$$

We now expand $\mathfrak{A}_<^{(2)}(\eta)$ as

$$\mathfrak{A}_<^{(2)}(\eta)=\sum_{n=0}^{\infty}\mathfrak{A}_n^{(2)}\eta^n, \tag{8.60}$$

where the first few coefficients are

$$\mathfrak{A}_{0<}^{(2)}=1, \tag{8.61}$$

$$\mathfrak{A}_{1<}^{(2)}=x_4+x_4^{-1}-x_5-x_6+\tfrac{1}{2}(x_2-x_1-x_3), \tag{8.62}$$

and

$$\begin{aligned}\mathfrak{A}_{2<}^{(2)}=&x_4^2+x_4^{-2}+x_5^2+x_6^2-\tfrac{1}{8}(x_1^2+x_2^2+x_3^2)+1-(x_4+x_4^{-1})(x_5+x_6)+x_5x_6\\&+(x_4+x_4^{-1}-x_5-x_6)\tfrac{1}{2}(x_2-x_1-x_3)-\tfrac{1}{4}(x_1x_2+x_2x_3-x_1x_3).\end{aligned} \tag{8.63}$$

We may now apply (8.35) and obtain

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim&\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(z_2^2-z^2)^{-2}z^2(1-z_2^2)^2\alpha_2^{2N+4}x_4^{-2}(x_1+1)^{-1}(x_2+1)^{-1}(-x_3-1)^{-1}(x_4-1)^4(x_5+1)^2(x_6+1)^2\\&\times\{[(N-1)!\sum_{m=0}^{\infty}2^{-m-5/2}\Gamma(m+\tfrac{5}{2})\Gamma(m+\tfrac{7}{2})[\Gamma(N+m+\tfrac{7}{2})]^{-1}\sum_{n=0}^{m}\mathfrak{A}_{n<}^{(2)}[\Gamma(\tfrac{5}{2}+n)(m-n)!]^{-1}]\\&\times[N!\sum_{m=0}^{\infty}2^{-m-5/2}\Gamma(m+\tfrac{5}{2})\Gamma(m+\tfrac{3}{2})[\Gamma(N+m+\tfrac{5}{2})]^{-1}\sum_{n=0}^{m}\mathfrak{A}_{n<}^{(2)}[\Gamma(\tfrac{5}{2}+n)(m-n)!]^{-1}]\\&-[(N-1)!\sum_{m=0}^{\infty}2^{-m-5/2}\Gamma(m+\tfrac{5}{2})\Gamma(m+\tfrac{5}{2})[\Gamma(N+m+\tfrac{5}{2})]^{-1}\sum_{n=0}^{m}\mathfrak{A}_{n<}^{(2)}[\Gamma(\tfrac{5}{2}+n)(m-n)!]^{-1}]\\&\times[N!\sum_{m=0}^{\infty}2^{-m-5/2}\Gamma(m+\tfrac{5}{2})\Gamma(m+\tfrac{5}{2})[\Gamma(N+m+\tfrac{7}{2})]^{-1}\sum_{n=0}^{m}\mathfrak{A}_{n<}^{(2)}[\Gamma(\tfrac{5}{2}+n)(m-n)!]^{-1}]\}.\end{aligned} \tag{8.64}$$

For large N, the first two terms of this expansion explicitly are

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim&\mathfrak{M}_1^2+z^2(1-z_2^2)^2\alpha_2^{2N+4}z_1^{-2}(z_2^2-z^2)^{-2}x_4^{-2}(x_1+1)^{-1}(x_2+1)^{-1}(-x_3-1)^{-1}(x_4-1)^4(x_5+1)^2(x_6+1)^2\\&\times\pi^{-1}2^{-9}N^{-5}3\{2+5N^{-1}\mathfrak{A}_{1<}^{(2)}+O(N^{-2})\}.\end{aligned} \tag{8.65}$$

C. $T<T_c$, $E_1<0$

In this case, (8.10) or (8.11) holds, depending on the strength of $\mathfrak{H}$. For both cases, the terms of leading exponential order are given by

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteq&\mathfrak{M}_1^2-z^2(-1)^N(z_2^2-z^2)^{-2}z_1^{-2}(\mathfrak{r}+1)^{-1}(\mathfrak{r}^{-1}+1)^{-1}[z_2^2(1-z_1)^2-(1+z_1)^2]\\&\times\left\{(\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}[(1-z_1)^2-z_2^2(1+z_1)^2]\right.\\&\left.+z_2(1-z_1^2)(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}(\zeta+1)(\zeta-1)^{-1}\alpha^{-1}\right\}\\&+z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_2(z_1^{-2}-1)\\&\times(2\pi i)^{-1}\int_\Gamma d\zeta\,\zeta^{N-1}(\zeta-1)(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\zeta^{-1}\mathfrak{r}^{-1}-1)^{-1}\alpha^{-1}.\end{aligned} \tag{8.66}$$

We obtain asymptotic expansions to the two integrals exactly as in the ferromagnetic cases, to obtain

$$\begin{aligned}\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteq&\mathfrak{M}_1^2-z^2(-1)^N(z_2^2-z^2)^{-2}z_1^{-2}(\mathfrak{r}+1)^{-1}(\mathfrak{r}^{-1}+1)^{-1}[z_2^2(1-z_1)^2-(1+z_1)^2]\\&\times\{(\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}[(1-z_1)^2-z_2^2(1+z_1)^2]\\&+\alpha_2^{N+1}(1-z_2^2)(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-1-x_3)^{-1/2}x_4(x_5+1)(x_6+1)\\&\times2^{-3/2}\pi^{-1}(N-1)!\sum_{m=0}^{\infty}2^{-m}\Gamma(m+\tfrac{1}{2})\Gamma(m+\tfrac{3}{2})[\Gamma(N+\tfrac{3}{2}+m)]^{-1}\sum_{n=0}^{m}\mathfrak{B}_{n<}^{(1)}[\Gamma(\tfrac{1}{2}+n)(m-n)!]^{-1}\}\\&+\alpha_2^{N+1}z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}(1-z_2^2)\\&\times(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-1-x_3)^{-1/2}x_4^{-1}(x_5+1)(x_6+1)2^{-3/2}\pi^{-1}\\&\times(N-1)!\sum_{m=0}^{\infty}2^{-m}\Gamma(m+\tfrac{1}{2})\Gamma(m+\tfrac{3}{2})[\Gamma(N+\tfrac{3}{2}+m)]^{-1}\sum_{n=0}^{m}\mathfrak{B}_{n<}^{(2)}[\Gamma(\tfrac{1}{2}+n)(m-n)!]^{-1},\end{aligned} \tag{8.67}$$

where

$$\mathfrak{B}_<^{(1)}(\eta)=[1-x_4^{-1}\eta][1-x_4\eta]^{-1}[1+x_5\eta]^{-1}[1+x_6\eta]^{-1}\{[1+x_2\eta][1-x_1\eta][1-x_3\eta]\}^{1/2}$$
$$=\sum_{n=0}^{\infty}\mathfrak{B}_{n<}^{(1)}\eta^n \tag{8.68a}$$

and

$$\mathfrak{B}_<^{(2)}(\eta)=[1-x_4\eta][1-x_4^{-1}\eta]^{-1}[1+x_5\eta]^{-1}[1+x_6\eta]^{-1}\{[1+x_2\eta][1-x_1\eta][1-x_3\eta]\}^{1/2}$$
$$=\sum_{n=0}^{\infty}\mathfrak{B}_{n<}^{(2)}\eta^n. \tag{8.68b}$$

In particular,

$$\mathfrak{B}_{0<}^{(1)}=1, \tag{8.69a}$$
$$\mathfrak{B}_{1<}^{(1)}=-x_4^{-1}+x_4-x_5-x_6+\tfrac{1}{2}(x_2-x_1-x_3), \tag{8.69b}$$
$$\mathfrak{B}_{2<}^{(1)}=x_4^2+x_5^2+x_6^2-\tfrac{1}{8}(x_2^2+x_1^2+x_3^2)-1-(x_4-x_4^{-1})(x_5+x_6)$$
$$+x_5x_6+\tfrac{1}{2}(x_4-x_4^{-1}-x_5-x_6)(x_2-x_1-x_3)-\tfrac{1}{4}(x_1x_2+x_2x_3-x_1x_3), \tag{8.69c}$$
$$\mathfrak{B}_{0<}^{(2)}=1, \tag{8.69d}$$
$$\mathfrak{B}_{1<}^{(2)}=x_4^{-1}-x_4-x_5-x_6+\tfrac{1}{2}(x_2-x_1-x_3), \tag{8.69e}$$

and

$$\mathfrak{B}_{2<}^{(2)}=x_4^{-2}+x_5^2+x_6^2-\tfrac{1}{8}(x_2^2+x_1^2+x_3^2)-1+(x_4-x_4^{-1})(x_5+x_6)$$
$$+x_5x_6+(x_4^{-1}-x_4-x_5-x_6)\tfrac{1}{2}(x_2-x_1-x_3)-\tfrac{1}{4}(x_1x_2+x_2x_3-x_1x_3). \tag{8.69f}$$

The two series to be expanded in (8.67) are each of the form of the series expanded in (8.40), so we immediately find that the first three terms of the asymptotic series are

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2-z^2(-1)^N(z_2^2-z^2)^{-2}z_1^{-2}(\mathfrak{r}+1)^{-1}(\mathfrak{r}^{-1}+1)^{-1}[z_2^2(1-z_1)^2-(1+z_1)^2]$$
$$\times\{(\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}[(1-z_1)^2-z_2^2(1+z_1)^2]$$
$$+\alpha_2^{N+1}(1-z_2^2)(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-1-x_3)^{-1/2}x_4(x_5+1)(x_6+1)\pi^{-1/2}2^{-5/2}$$
$$\times N^{-3/2}[1+(3/4N)\mathfrak{B}_{1<}^{(1)}+(5/32N^2)(6\mathfrak{B}_{2<}^{(1)}-1)]\}$$
$$+\alpha_2^{N+1}z^2[(1-z_1)^2-z_2^2(1+z_1)^2](\mathfrak{r}-1)^{-1}(\mathfrak{r}^{-1}-1)^{-1}(z_2^2-z^2)^{-2}z_1^{-2}(1-z_2^2)$$
$$\times(x_1+1)^{-1/2}(x_2+1)^{-1/2}(-1-x_3)^{-1/2}x_4^{-1}(x_5+1)(x_6+1)\pi^{-1/2}2^{-5/2}$$
$$\times N^{-3/2}[1+(3/4N)\mathfrak{B}_{1<}^{(2)}+(5/32N^2)(6\mathfrak{B}_{2<}^{(2)}-1)]. \tag{8.70}$$

This series is valid under the restriction (8.20).

D. T near T_c

All of the asymptotic series found so far are valid only when $N\gg|1-T/T_c|^{-1}$. We now consider the limit that $T\rightarrow T_c(\alpha_2\rightarrow1)$ such that $N|1-T/T_c|$ remains fixed and of order 1. We first consider $T<T_c$ and $E_1>0$. Then (8.9) or (8.7) holds. We consider only the case that $z^2\geq|z_2|(1-\alpha_2)/(1+\alpha_2)$. Then the term involving $\mathfrak{r}^N$ is exponentially small in (8.7) compared with the other terms and may be dropped. Therefore, both (8.7) and (8.9) reduce to (8.54) which may be written using (8.57) and (8.58) as

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\doteqdot\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}$$
$$\times\left\{\left[\int_{\alpha_1}^{\alpha_2}d\zeta\,\zeta^N(\zeta^2-1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}\right]\right.$$
$$\times\left[\int_{\alpha_1}^{\alpha_2}d\zeta\,\zeta^{N-1}(\zeta-1)(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}\right]$$
$$-\left[\int_{\alpha_1}^{\alpha_2}d\zeta\,\zeta^N(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}\right]$$
$$\left.\times\left[\int_{\alpha_1}^{\alpha_2}d\zeta\,\zeta^{N-1}(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}\right]\right\}. \tag{8.71}$$

All integrals in this expression are of the form

$$\int_{\alpha_1}^{\alpha_2} d\zeta\, \zeta^N(\zeta+1)^{-1}(\zeta-1)^n(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}\zeta^{-1}-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}. \tag{8.72}$$

To approximate this integral, make the change of variables

$$\xi=(1-\zeta)/(1-\alpha_2). \tag{8.73}$$

Then (8.72) becomes

$$\alpha_2^{-1}(\alpha_2-1)^{n+2}\int_1^{(1-\alpha_1)(1-\alpha_2)^{-1}} d\xi[1-(1-\alpha_2)\xi]^N\xi^n\tfrac{1}{2}(1-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-1)^{-1}(1-\alpha_1)$$
$$\times[1-(1-\alpha_2)\tfrac{1}{2}\xi]^{-1}[1-\xi(1-\alpha_2)(1-\mathfrak{r})^{-1}]^{-1}[1+\xi(1-\alpha_2)(\mathfrak{r}^{-1}-1)^{-1}]^{-1}$$
$$\times[(-1+\xi)(1+\alpha_2\xi)(1+\xi\alpha_1(1-\alpha_2)(1-\alpha_1)^{-1})(1-\xi(1-\alpha_2)(1-\alpha_1)^{-1})]^{1/2}. \tag{8.74}$$

Define

$$t=(1-\alpha_2)N \tag{8.75}$$

which is the fixed quantity of order 1. Then, correct to terms of second order, and we have

$$[1-(1-\alpha_2)\xi]^N\sim e^{-t\xi}[1-\tfrac{1}{2}t^2\xi^2N^{-1}]. \tag{8.76}$$

If we require $|1-\mathfrak{r}|>1-\alpha_2$, we may expand the rest of the integrand as a power series in $(1-\alpha_2)$ to obtain

$$\alpha_2^{-1}(\alpha_2-1)^{n+2}\tfrac{1}{2}(1-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-1)^{-1}(1-\alpha_1)\int_1^{(1-\alpha_1)(1-\alpha_2)^{-1}} d\xi\, \xi^n e^{-t\xi}(\xi^2-1)^{1/2}$$
$$\times\{1+(1-\alpha_2)\xi-\tfrac{1}{2}t^2\xi^2N^{-1}+\tfrac{1}{2}(\alpha_2-1)\xi(\xi+1)^{-1}+O((\alpha_2-1)^2)\}. \tag{8.77}$$

We may replace the upper limit by infinity without changing the asymptotic expansion. The integrals may then be evaluated as Bessel functions. In particular, when $n=0$, (8.72) has the expansion

$$\alpha_2^{-1}(1-\alpha_2)^2\tfrac{1}{2}(1-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-1)^{-1}(1-\alpha_1)\{K_1(t)t^{-1}-\tfrac{1}{2}N^{-1}[K_2(t)+K_1(t)+tK_0(t)]\}. \tag{8.78}$$

When $n=-1$, (8.72) has the expression

$$-(\alpha_2^{-1}-1)\tfrac{1}{2}(1-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-1)^{-1}(1-\alpha_1)\left\{\int_t^\infty d\xi K_1(\xi)\xi^{-1}-\tfrac{1}{2}N^{-1}tK_1(t)\right\}. \tag{8.79}$$

In the above, K_n is the modified Bessel function of the third kind of order n.[12] When we replace N by $N-1$, we obtain an additional correction term of order $(1-\alpha_2)$. Combining these expressions, we find that the first term of the asymptotic expansion for $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ is

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-\alpha_1)^2$$
$$\times N^{-4}\left\{K_2(t)t^3\int_t^\infty d\xi K_1(\xi)\xi^{-1}-t^2K_1^2(t)+O(N^{-1})\right\}. \tag{8.80}$$

The integral in (8.80) may be expressed in terms of modified Struve functions.[13] The term $O(N^{-1})$ is given in Appendix D. When $t=0$, this reduces to what one may simply obtain by letting $\alpha_2=1$ in (8.71); namely,

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\tfrac{1}{4}\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-z_1^2)^2(N^{-4}+O(N^{-6})). \tag{8.81}$$

For (8.80) and (8.81) to hold, we need $N>|1-\mathfrak{r}|^{-1}$.

We next consider T near T_c but $T>T_c$ and z not near zero. Then (8.6) or (8.8) holds. The terms with $\mathfrak{r}^N$ are exponentially small and may be neglected. Using

$$(1-z_1)^2-z_2^2(1+z_1)^2=(1-z_1)^2(1-\alpha_2^{-2}), \tag{8.82}$$

we see that the remaining terms in (8.6) or (8.8) have a leading order of N^{-4} and all must be retained. Define

$$t'=N(1-\alpha_2^{-1}). \tag{8.83}$$

In terms of this variable and the change of variable

$$\xi'=(1-\zeta)(1-\alpha_2^{-1})^{-1}, \tag{8.84}$$

[12] See Ref. 11, Vol. II, chap. 7.
[13] See Ref. 11, Vol. II, p. 89.

the calculation is almost identical with the $T<T_c$ case. In particular, (8.77) holds if we omit the first factor of α_2^{-1} and replace α_2 by α_2^{-1}.

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+(1/2\pi)z^2(1-z_2^2)z_1^{-2}(z_2^2-z^2)^{-2}(\mathfrak{r}-1)^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-z_1)^2(1-\alpha_1)N^{-4}t'^3K_2(t')$$
$$+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(\mathfrak{r}-1)^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-\alpha_1)^2N^{-4}\left[t'^3K_2(t')\int_{t'}^{\infty}d\xi K_1(\xi)\xi^{-1}-t'^2K_1^2(t')\right]. \tag{8.85}$$

When $T=T_c$ (and $t'=0$), the second term in (8.85) vanishes and we obtain the same limit as we attained from below T_c. We note that in both these cases the leading order term is N^{-4}. This is to be compared with the result to be obtained when $\mathfrak{H}=0$. Note also that Appendix D shows that in (8.81) and (8.85) the $O(N^{-5})$ terms vanish at $T=T_c$.

When $\mathfrak{H}=0$, the correlation functions reduce to the expressions (8.19). We expand these for the ferromagnetic case near T_c. When $T>T_c$, we have

$$\mathfrak{S}_{1,1}(N,0)=-(2\pi)^{-1}z_2^{-2}(1-z_2^2)z_1^{-1}\int_{\alpha_1}^{\alpha_2^{-1}}d\zeta\,\zeta^N(\zeta^2-1)^{-1}[(1-\alpha_1\zeta)(1-\alpha_1\zeta^{-1})(1-\alpha_2^{-1}\zeta)(\alpha_2^{-1}\zeta^{-1}-1)]^{1/2}. \tag{8.86}$$

Use the substitution (8.84) to reduce (8.86) to an integral similar to (8.72) with $n=-1$. Then,

$$\mathfrak{S}_{1,1}(N,0)=(4\pi)^{-1}(1-z_2^2)z_1^{-1}z_2^{-2}(1-\alpha_1)N^{-1}\left\{t'\int_{t'}^{\infty}d\xi K_1(\xi)\xi^{-1}-N^{-1}\tfrac{1}{2}t'^2K_1(t')+O(N^{-2})\right\}. \tag{8.87}$$

When $T<T_c$, an analogous calculation gives

$$\mathfrak{S}_{1,1}(N,0)=\tfrac{1}{4}z_2^{-2}z_1^{-1}[z_2^2(1+z_1)^2-(1-z_1)^2]+(4\pi)^{-1}z_2^{-2}(1-z_2^2)z_1^{-1}(1-\alpha_1)N^{-1}$$
$$\times\left\{t\int_t^{\infty}d\xi\,\xi^{-1}K_1(\xi)+N^{-1}t^2\left[\int_t^{\infty}d\xi\,\xi^{-1}K_1(\xi)-\tfrac{1}{2}K_1(t)\right]+O(N^{-2})\right\}. \tag{8.88}$$

In particular, if $T=T_c$,

$$\mathfrak{S}_{1,1}(N,0)=(4\pi)^{-1}(1-z_2^2)(1-z_1^2)z_1^{-1}z_2^{-2}N^{-1}+O(N^{-3}); \tag{8.89}$$

so when $\mathfrak{H}=0$, the correlation functions near T_c fall off much slower than when $\mathfrak{H}\neq0$. This should also be contrasted with the bulk correlation functions where, at T_c, S_N is proportional to $N^{-1/4}$ as shown in I.

Our final remark about the spin-spin correlation functions will be to find the asymptotic behavior when $T=T_c$, $E_1>0$, and z is near zero, such that

$$u=Nz^2 \tag{8.90}$$

is a constant of order 1. We may approach this case from either (8.8) or (8.9). In either case, we have only the product of integrals term (8.71), which with $\alpha_2=1$ specializes to

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})=\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)z^2(z_2^2-z^2)^{-2}$$
$$\times\left\{\left[\int_{z_1}^{1}d\zeta\,\zeta^N(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-\zeta)^{-1}[(1-z_1\zeta)(\zeta-z_1)]^{1/2}\right]\right.$$
$$\times\left[\int_{z_1}^{1}d\zeta\,\zeta^{N-1}(\zeta-1)^2(\zeta+1)^{-1}(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-\zeta)^{-1}[(1-z_1\zeta)(\zeta-z_1)]^{1/2}\right]$$
$$-\left[\int_{z_1}^{1}d\zeta\,\zeta^N(\zeta+1)^{-1}(1-\zeta)(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-\zeta)^{-1}[(1-z_1\zeta)(\zeta-z_1)]^{1/2}\right]$$
$$\left.\times\left[\int_{z_1}^{1}d\zeta\,\zeta^{N-1}(\zeta+1)^{-1}(1-\zeta)(\zeta-\mathfrak{r})^{-1}(\mathfrak{r}^{-1}-\zeta)^{-1}[(1-z_1\zeta)(\zeta-z_1)]^{1/2}\right]\right\}. \tag{8.91}$$

When $\mathfrak{H}$ is near zero,

$$\mathfrak{r}\approx1+2z^2z_2^{-1}i. \tag{8.92}$$

We obtain the leading asymptotic term if we approximate all of the integrand that varies slowly at $\zeta=1$ by its value there. Therefore,

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}(1-z_1^2)^2\tfrac{1}{4}\left\{\left[\int_{z_1}^1 d\zeta\,\zeta^N(\zeta-1-2z^2z_2^{-1}i)^{-1}(1-2z^2z_2^{-1}i-\zeta)^{-1}\right]\right.$$
$$\times\left[\int_{z_1}^1 d\zeta\,\zeta^N(\zeta-1)^2(\zeta-1-2z^2z_2^{-1}i)^{-1}(1-2z^2z_2^{-1}i-\zeta)^{-1}\right]$$
$$\left.-\left[\int_{z_1}^1 d\zeta\,\zeta^N(1-\zeta)(\zeta-1-2z^2z_2^{-1}i)^{-1}(1-2z^2z_2^{-1}i-\zeta)^{-1}\right]^2\right\}. \tag{8.93}$$

We make the change of variables

$$\xi=z^{-2}(1-\zeta) \tag{8.94}$$

and use (8.90) to obtain

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}z^2(1-z_2^2)^2z_2^{-4}(1-z_1^2)^2\tfrac{1}{4}\left\{\left[z^{-2}\int_0^{z^{-2}(1-z_1)}d\xi\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]\right.$$
$$\left.\times\left[z^2\int_0^{z^{-2}(1-z_1)}d\xi\,\xi^2\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]-\left[\int_0^{z^{-2}(1-z_1)}d\xi\,\xi\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]^2\right\}. \tag{8.95}$$

We may replace the upper limit by infinity to obtain

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z_2^{-4}(1-z_1^2)^2\tfrac{1}{4}z^2$$
$$\times\left\{\left[\int_0^\infty d\xi\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]\left[u^{-1}-4z_2^{-2}\int_0^\infty d\xi\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]-\left[\int_0^\infty d\xi\,\xi\,e^{-\xi u}(\xi^2+4z_2^{-2})^{-1}\right]^2\right\} \tag{8.96}$$
$$=\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z_2^{-4}(1-z_1^2)^2\tfrac{1}{4}N^{-1}$$
$$\times\left\{\int_0^\infty d\xi\,e^{-\xi u}(\xi^2+z_2^{-2})^{-1}-4z_2^{-2}u\left[\int_0^\infty d\xi\,e^{-\xi u}(\xi^2+z_2^{-2})^{-1}\right]^2-u\left[\int_0^\infty d\xi\,\xi\,e^{-\xi u}(\xi^2+z_2^{-2})^{-1}\right]^2\right\}. \tag{8.97}$$

This is the desired result. If now for fixed N we let $\mathfrak{H}\to 0$, then $u=0$ and this specializes to

$$\mathfrak{S}_{1,1}(N,0)\sim\pi^{-1}z_1^{-2}(1-z_2^2)^2\,|z_2|^{-3}\,(1-z_1^2)^2\tfrac{1}{16}N^{-1}, \tag{8.98}$$

which, if we note that at T_c

$$|z_1|^{-1}\,|z_2|^{-1}\,(1-z_2^2)\,(1-z_1^2)=4, \tag{8.99}$$

is exactly the result (8.89) previously obtained for $\mathfrak{S}_{1,1}(N,0)$ at $T=T_c$.

9. MAGNETIZATION IN INTERIOR ROWS

We obtain a formula for the magnetization in any interior row by combining the formalism of Sec. 2 with the inverse elements of Sec. 7. Define $y^{(J)}$ to be the nonzero submatrix of $\delta^{(J)}$. Explicitly, from (2.12) and (2.13),

$$y^{(J)}=\begin{array}{cc|cccccccc}
 & & 10 & 20 & \cdots & J0 & 00 & 10 & \cdots & J-10\\
 & & D & D & & D & U & U & & U\\
\hline
10 & D & 0 & 0 & \cdots & 0 & -(z^{-1}-z) & 0 & \cdots & 0\\
20 & D & 0 & 0 & \cdots & 0 & 0 & -(z_2^{-1}-z_2) & \cdots & 0\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
J0 & D & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -(z_2^{-1}-z_2)\\
00 & U & z^{-1}-z & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
10 & U & 0 & z_2^{-1}-z_2 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
\cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot\\
J-10 & U & 0 & 0 & \cdots & z_2^{-1}-z_2 & 0 & 0 & \cdots & 0
\end{array}. \tag{9.1}$$

Calling $\mathfrak{Q}^{(J)}$ the nonzero submatrix of $\mathfrak{A}^{-1}$ in the subspace defined by $y^{(J)}$, we have from (2.14),

$$\mathfrak{M}_J = \pm z z_2^{J-1} \operatorname{Pf}[y^{(J)}] \operatorname{Pf}[y^{(J)-1} + \mathfrak{Q}^{(J)}], \tag{9.2}$$

where the sign ($\pm$) is chosen to make $\mathfrak{M}_J$ have the same sign as does $\mathfrak{H} z_2^{J-1}$. We write this out explicitly as shown in Eq. (9.3) on page 467.

We may simplify this if we note from Eqs. (7.8a), (7.8b), (7.8f), and (7.8g) that, because the integrands are all odd functions of θ when $k=k'$,

$$\mathfrak{A}^{-1}(j,k;j',k)_{DD} = \mathfrak{A}^{-1}(j,k;j',k)_{UU} = 0. \tag{9.4}$$

Using this, the Pfaffian in (9.3) reduces to a determinant and we have

$$\mathfrak{M}_J = \pm (1-z^2)(1-z_2^2)^{J-1}$$

$$\times \det \begin{bmatrix} \mathfrak{A}^{-1}(1,0;0,0)_{DU} + (z^{-1}-z)^{-1} & \mathfrak{A}^{-1}(1,0;1,0)_{DU} & \cdots & \mathfrak{A}^{-1}(1,0;J-1,0)_{DU} \\ \mathfrak{A}^{-1}(2,0;0,0)_{DU} & \mathfrak{A}^{-1}(2,0;1,0)_{DU} + (z_2^{-1}-z_2)^{-1} & \cdots & \mathfrak{A}^{-1}(2,0;J-1,0)_{DU} \\ \vdots & \vdots & \cdots & \vdots \\ \mathfrak{A}^{-1}(J,0;0,0)_{DU} & \mathfrak{A}^{-1}(J,0;1,0)_{DU} & \cdots & \mathfrak{A}^{-1}(J,0;J-1,0)_{DU} + (z_2^{-1}-z_2)^{-1} \end{bmatrix}. \tag{9.5}$$

The question of spontaneous magnetization and hysteresis may be dealt with just as in Sec. 5. We see from (7.8) that as $\mathfrak{H}\to 0$, all of the matrix elements are continuous except those in the first column which may be written as

$$\mathfrak{A}^{-1}(j,0;0,0)_{DU} = (2\pi)^{-1} z z_1 \int_{-\pi}^{\pi} d\theta\, |1+e^{i\theta}|^2 \alpha^{-j+1} [z^2 z_1 |1+e^{i\theta}|^2 - z_2^2(1+z_1^2+2z_1\cos\theta) + z_2(1-z_1^2)\alpha]^{-1}. \tag{9.6}$$

The singularities of the integrand of (9.6) are exactly the same as those of (5.1). Therefore, the discussion of Sec. 5 applies. In particular, if $T>T_c$, (9.6) will vanish as $\mathfrak{H}\to 0$; but if $T<T_c$ and $E_1>0$, (9.6) is discontinuous as $\mathfrak{H}\to 0$ and, following (5.19), has the limit

$$\lim_{\mathfrak{H}\to 0^+} \mathfrak{A}^{-1}(j,0;0,0)_{DU} = \tfrac{1}{2} z_1^{-1/2} |z_2|^{-1} [z_2^2(1+z_1)^2 - (1-z_1)^2]^{1/2} \alpha(1)^{-j+1}, \tag{9.7}$$

where

$$\alpha(1) = z_2(1+z_1)(1-z_1)^{-1}.$$

Furthermore, the factor of α^{-j+1} in the numerator of (9.6) does not affect the factorization of the denominator made for $j=1$ in (5.8). Therefore, these matrix elements, and hence $\mathfrak{M}_J$ itself, may be analytically continued through $\mathfrak{H}=0$ just as $\mathfrak{M}_1$ was and the same sort of hysteresis behavior is observed in all rows.

While (9.5) may be used to compute M_J for any value of J, we content ourselves here with evaluating the spontaneous magnetization in the second row. This is

$$\lim_{\mathfrak{H}\to 0^+} M_2 = \pm (1-z_2^2) \lim_{\mathfrak{H}\to 0^+} [A^{-1}(1,0;0,0)_{UD}(A^{-1}(2,0;1,0)_{DU} + (z_2^{-1}-z_2)^{-1}) - \mathfrak{A}^{-1}(2,0;0,0)_{DU} \mathfrak{A}^{-1}(1,0;1,0)_{DU}]. \tag{9.8}$$

Using (9.7) and (7.8), we find

$$\mathfrak{M}_2(0^+) = \pm (1-z_2^2) \tfrac{1}{2} z_1^{-1/2} |z_2|^{-1} [z_2^2(1+z_1)^2 - (1-z_1)^2]^{1/2} \left\{ \frac{-z_2}{2\pi} \int_{-\pi}^{\pi} d\theta \bar{\mathfrak{D}}_2 \mathfrak{M}_{-1} \mathfrak{D}_1 / \mathfrak{C}_2 \mathfrak{M} + (z_2^{-1}-z_2)^{-1} + z_2^{-2}(1-z_1)(1+z_1)^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \alpha^{-1} \right\}, \tag{9.9}$$

which reduces to

$$\mathfrak{M}_2(0^+) = \tfrac{1}{2} z_1^{-1/2} |z_2|^{-1} [z_2^2(1+z_1)^2 - (1-z_1)^2]^{1/2} \left\{ z_2^{-1} - \frac{z_1(1-z_2^2)}{z_2^2(1-z_1^2)} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \alpha^{-1}(e^{i\theta}-1)(e^{-i\theta}-1) \right\}. \tag{9.10}$$

$$\mathfrak{M}_J = \pm(1-z^2)(1-z_2^2)^{J-1}$$

$$\times \mathrm{Pf}\begin{bmatrix}
\mathfrak{A}^{-1}(1,0;1,0)_{DD} & \mathfrak{A}^{-1}(1,0;2,0)_{DD} & \cdots & \mathfrak{A}^{-1}(1,0;J,0)_{DD} & \begin{matrix}\mathfrak{A}^{-1}(1,0;0,0)_{DU}\\ +(z^{-1}-z)^{-1}\end{matrix} & \mathfrak{A}^{-1}(1,0;1,0)_{DU} & \cdots & \mathfrak{A}^{-1}(1,0;J-1,0)_{DU} \\
\mathfrak{A}^{-1}(2,0;1,0)_{DD} & \mathfrak{A}^{-1}(2,0;2,0)_{DD} & \cdots & \mathfrak{A}^{-1}(2,0;J,0)_{DD} & \mathfrak{A}^{-1}(2,0;0,0)_{DU} & \begin{matrix}\mathfrak{A}^{-1}(2,0;1,0)_{DU}\\ +(z_2^{-1}-z_2)^{-1}\end{matrix} & \cdots & \mathfrak{A}^{-1}(2,0;J-1,0)_{DU} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathfrak{A}^{-1}(J,0;1,0)_{DD} & \mathfrak{A}^{-1}(J,0;2,0)_{DD} & \cdots & \mathfrak{A}^{-1}(J,0;J,0)_{DD} & \mathfrak{A}^{-1}(J,0;0,0)_{DU} & \mathfrak{A}^{-1}(J,0;1,0)_{DU} & \cdots & \begin{matrix}\mathfrak{A}^{-1}(J,0;J-1,1,0)_{DU}\\ +(z_2^{-1}-z_2)^{-1}\end{matrix} \\
\begin{matrix}\mathfrak{A}^{-1}(0,0;1,0)_{UD}\\ -(z^{-1}-z)^{-1}\end{matrix} & \mathfrak{A}^{-1}(0,0;2,0)_{UD} & \cdots & \mathfrak{A}^{-1}(0,0;J,0)_{UD} & \mathfrak{A}^{-1}(0,0;0,0)_{UU} & \mathfrak{A}^{-1}(0,0;1,0)_{UU} & \cdots & \mathfrak{A}^{-1}(0,0;J-1,0)_{UU} \\
\mathfrak{A}^{-1}(1,0;1,0)_{UD} & \begin{matrix}\mathfrak{A}^{-1}(1,0;2,0)_{UD}\\ -(z_2^{-1}-z_2)^{-1}\end{matrix} & \cdots & \mathfrak{A}^{-1}(1,0;J,0)_{UD} & \mathfrak{A}^{-1}(1,0;0,0)_{UU} & \mathfrak{A}^{-1}(1,0;1,0)_{UU} & \cdots & \mathfrak{A}^{-1}(1,0;J-1,0)_{UU} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathfrak{A}^{-1}(J-1,0;1,0)_{UD} & \mathfrak{A}^{-1}(J-1,0;2,0)_{UD} & \cdots & \begin{matrix}\mathfrak{A}^{-1}(J-1,0;J,0)_{UD}\\ -(z_2^{-1}-z_2)^{-1}\end{matrix} & \mathfrak{A}^{-1}(J-1,0;0,0)_{UU} & \mathfrak{A}^{-1}(J-1,0;1,0)_{UU} & \cdots & \mathfrak{A}^{-1}(J-1,0;J-1,0)_{UU}
\end{bmatrix}. \tag{9.3}$$

As $T \to T_c$, the brackets remains finite and nonzero so the magnetization in the second row goes to zero as $T \to T_c$ as a square root, just as $\mathfrak{M}_1$ does. We conjecture that all $\mathfrak{M}_J$ go to zero as $(1-T/T_c)^{1/2}$ when $T \to T_c-$ for any fixed J. We furthermore conjecture that (for $z_2 > 0$), at $\mathfrak{H}=0$, $\mathfrak{M}_J$ as given by (9.5) monotonically approaches the spontaneous magnetization of the bulk Ising model. The specific manner in which this limit is obtained is an open question.

10. SUMMARY

In conclusion, we wish to briefly summarize the quantities of physical interest calculated in the text. The partition function (3.26) has been computed and both bulk and boundary terms in tht fret energy (3.27) have been identified. The boundary entropy has been found to be logarithmically divergent (4.42) at T_c, and the boundary specific heat diverges at T_c as a single pole plus a logarithm (4.43). The boundary magnetization for any $\mathfrak{H}$ is computed in (5.1), and below T_c the spontaneous magnetization (5.19) is found to behave near T_c as $(1-T/T_c)^{1/2}$. The behavior of the boundary magnetization for small $\mathfrak{H}$ near but below T_c is given by (5.29), while near but above T_c it is given by (5.30). The logarithmic singularity in the zero field susceptibility at T_c is given by (5.32). We conclude Sec. 5 with an analytic continuation of $\mathfrak{M}_1$ (5.35) which we tentatively ascribe to a hysteresis phenomenon. This interpretation is sharpened in Sec. 6, where we compute $\mathfrak{W}$, the limit as $\mathfrak{N} \to \infty$ of $(2\mathfrak{N})^{-1}$ times the logarithm of the probability distribution function of the average boundary spin $\bar{\sigma}$. This function, at $\mathfrak{H}=0$, is shown to have maxima at $\pm \mathfrak{M}_1(0^+)$. When we turn on the magnetic field $\mathfrak{H}$, $\mathfrak{W}$ as given by (6.13) still has two maxima if $\mathfrak{H}$ is small. These maxima are at the values of $\bar{\sigma}$ given by the two branches of the hysteresis curve. However, when one of these branches passes through zero magnetization, the smaller maximum loses its identity and merges with the larger maximum as may be seen from Fig. 8. When T is near T_c and $\bar{\sigma}$ is small, the function $\mathfrak{W}$ is explicitly given by (6.18). In particular, when $T=T_c$, the probability distribution function is given by (6.24) which is not quite a Gaussian. In Sec. 8, some asymptotic limits of the correlation function of two spins on the boundary row (8.5) are derived. In particular, for $N\,|\,1-T/T_c\,| \gg 1$, we have the cases: (a) $T > T_c$ where (8.41) holds; (b) $T < T_c$, $E_1 > 0$, and $z^2 < |\,z_2\,|\,(1-\alpha_2)(1+\alpha_2)^{-1}$ where (8.52) holds, $T < T_c$, $E_1 > 0$ and $z_2 > |\,z_2\,|\,(1-\alpha_2)(1+\alpha_2)^{-1}$ where (8.65) holds; and (c) $T < T_c$, $E_1 < 0$ where (8.70) holds. When $N\,|\,1-T/T_c\,|$ is of order 1, T is near T_c, and $\mathfrak{H}$ is away from zero, the asymptotic expansions are given by (8.80) if $T < T_c$ and by (8.85) if $T > T_c$. When $\mathfrak{H}=0$, $E_1 > 0$, and T is near T_c, the asymptotic expansions are given by (8.87) if $T > T_c$ and by (8.88) if $T < T_c$. If $T=T_c$ and Nz^2 is of order 1, the correlation function is asymptotically given by (8.97). Lastly, we have derived a general formula for the magnetization of any interior row (9.5) and explicitly evaluate the magnetization in the second row at $\mathfrak{H}=0$ in (9.10).

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APPENDIX A

A more physical, but mathematically less satisfying, way of computing the partition function (2.2) is to modify the initial Hamiltonian by adding a term,

$$-E_3 \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{0,k}\sigma_{0,k+1}, \tag{A1}$$

and letting $E_3 \to \infty$. This means that we have added a zeroth row of spins to our lattice. This row has infinite strength bonds, so all the $\sigma_{0,k}$ will have the same value (defined to be $+1$). When any $\sigma_{0,k}$ is different from 1, (A1) becomes infinitely larger than its ground state. We may now write the magnetic field term in (2.1) as

$$-\mathfrak{H} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{1,k}\sigma_{0,k} \tag{A2}$$

and so consider the following modified Hamiltonian,

$$\begin{aligned}\mathcal{E}' = &-E_1 \sum_{j=1}^{2\mathfrak{M}} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{j,k}\sigma_{j,k+1} - E_2 \sum_{j=1}^{2\mathfrak{M}-1} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{j,k}\sigma_{j+1,k} \\ &-\mathfrak{H} \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{1,k}\sigma_{0,k} - E_3 \sum_{k=-\mathfrak{N}+1}^{\mathfrak{N}} \sigma_{0,k}\sigma_{0,k+1},\end{aligned} \tag{A3}$$

in the limit $E_3 \to \infty$. In this Hamiltonian, all interactions are nearest neighbor and the problem of computing the partition function for (A3) is obviously solvable by Pfaffians as

$$\begin{aligned}Z' = &(2\cosh\beta E_3)^{2\mathfrak{N}} (2\cosh\beta E_1)^{4\mathfrak{M}\mathfrak{N}} \\ &\times (\cosh\beta E_2)^{2\mathfrak{N}(2\mathfrak{M}-1)} (\cosh\beta\mathfrak{H})^{2\mathfrak{N}} \operatorname{Pf}\mathfrak{A},\end{aligned} \tag{A4}$$

where $\mathfrak{A}$ is given by (2.6) and we must take the $E_3 \to \infty$ limit. This partition function is the same as (2.5) except for a factor of 2 and the (infinite) factor containing E_3 whose contribution to the free energy is

$$-\beta^{-1} \ln \lim_{E_3\to\infty} (2\cosh\beta E_3)^{2\mathfrak{N}} = -\lim_{E_3\to\infty} 2\mathfrak{N}E_3, \tag{A5}$$

which is exactly the free energy one expects for $2\mathfrak{N}$ infinite strength bonds. The partition function (A4) does not contain the factor of $\frac{1}{2}$ which (2.5) does because in (A4) we have summed over both $\sigma_{0,k}=1$ and $\sigma_{0,k}=-1$. Because these extra contributions to the partition function are independent of both $\mathfrak{H}$ and T, the modified Hamiltonian will give the same results as the original Hamiltonian if we use it to compute correlation functions. From this point of view, the magnetization in the Jth row is viewed as a spin–spin correlation of $\sigma_{J,0}$ with $\sigma_{0,0}$. The spin–spin correlation $\mathfrak{S}_{J',J}(N)$ is viewed as the four-spin correlation $\langle \sigma_{J',0}\sigma_{0,0}\sigma_{0,N}\sigma_{J;N} \rangle$.

APPENDIX B

When the number of rows in our lattice is large but not infinite, we expect a behavior very much like the half-plane case. However, a phase transition occurs and spontaneous magnetization occurs only in the half-plane case. To make precise the manner in which the finite strip approaches the half-plane as $\mathfrak{M}\to\infty$, we calculate the boundary magnetization for large but finite $\mathfrak{M}$. The formalism of Sec. 2 applies, and from (2.14) we may immediately write

$$\mathfrak{M}_1=z+(1-z^2)\mathfrak{A}^{-1}(1,0;0,0)_{DU}. \tag{B1}$$

Using (7.7e), this becomes

$$\mathfrak{M}_1=z+(1-z^2)(2\pi)^{-1}z\int_{-\pi}^{\pi}d\theta\,\mathfrak{C}\bar{\mathfrak{D}}_{2\mathfrak{M}}/\mathfrak{C}_{2\mathfrak{M}}$$

$$=z+(1-z^2)(2\pi)^{-1}zz_1\int_{-\pi}^{\pi}d\theta(1-\alpha^{-4\mathfrak{M}})\,|1+e^{i\theta}|^2\{z^2z_1\,|1+e^{i\theta}|^2-z_2^2\,|1+z_1e^{i\theta}|^2+z_2(1-z_1^2)\alpha$$

$$+\alpha^{-4\mathfrak{M}}[z_2^2\,|1+z_1e^{i\theta}|^2-z_2(1-z_1^2)\alpha^{-1}-z^2z_1\,|1+e^{i\theta}|^2]\}^{-1}. \tag{B2}$$

If we let $\mathfrak{M}\to\infty$, we recover (5.1). In that case, when $T<T_c$, $E_1>0$ and $z\to 0$, the integral multiplying z diverges as z^{-1} and spontaneous magnetization occurs. If we keep $\mathfrak{M}$ finite, however, when $T<T_c$, $E_1>0$ and $z\to 0$, the integral multiplying z is finite so $\mathfrak{M}_1\to 0$. When z is not zero, then when $\mathfrak{M}$ is large enough so that

$$z^2\gg\tfrac{1}{4}z_1^{-1}[z_2^{-1}(1-z_1)(1+z_1)^{-1}]^{4\mathfrak{M}}[z_2^2(1+z_1)^2-(1-z_1)^2], \tag{B3}$$

the terms proportional to $\alpha^{-4\mathfrak{M}}$ may be neglected and the boundary magnetization for the finite strip becomes identical with the boundary magnetization of the half-plane. Only very near $\mathfrak{H}=0$ will the boundary magnetization of the strip be sensibly different from the boundary magnetization of the half-plane.

We are interested in seeing in detail how spontaneous magnetization arises when $T<T_c$ as $\mathfrak{M}\to\infty$. We may compute this behavior from (B2) for large $\mathfrak{M}$ by expanding the integrand about $\theta=0$ and keeping the lowest-order terms. Using (5.18), we obtain

$$\mathfrak{M}_1\sim z+(1-z^2)z[z_2^2(1+z_1)^2-(1-z_1)^2]z_1^{-1}z_2^{-2}(2\pi)^{-1}\int_{-\infty}^{\infty}d\theta\{[z_2^2(1+z_1)^2-(1-z_1)^2]z_2^2z_1^{-1}$$

$$(z^2+\tfrac{1}{4}z_1^{-1}[z_2(1+z_1)(1-z_1)^{-1}]^{-4\mathfrak{M}}[z_2^2(1+z_1)^2-(1-z_1)^2])+\theta^2\}^{-1}, \tag{B4}$$

so that when z is small and $\mathfrak{M}$ is large,

$$\mathfrak{M}_1\sim z+\tfrac{1}{2}(1-z^2)z_1^{-1/2}\,|z_2|^{-1}[z_2^2(1+z_1)^2-(1-z_1)^2]^{1/2}z$$

$$\times\{z^2+\tfrac{1}{4}z_1^{-1}[z_2(1+z_1)(1-z_1)^{-1}]^{-4\mathfrak{M}}[z_2^2(1+z_1)^2-(1-z_1)^2]\}^{-1/2}. \tag{B5}$$

If $\mathfrak{M}\to\infty$ and then $z\to 0$, $\mathfrak{M}_1$ clearly goes to the value of the spontaneous magnetization given by (5.19). On the other hand, if $\mathfrak{M}$ is finite and $z\to 0$, $\mathfrak{M}_1$ does vanish. From (B5) we find that the susceptibility at zero field for a large finite strip is

$$\partial\mathfrak{M}_1/\partial\mathfrak{H}\,|_{\mathfrak{H}=0}=\beta\{1+|z_2|^{-1}[z_2(1+z_1)(1-z_1)^{-1}]^{2\mathfrak{M}}\}, \tag{B6}$$

which becomes exponentially large as $\mathfrak{M}\to\infty$.

APPENDIX C

The matrix $\mathfrak{B}$ was partially inverted in Sec. 7. That procedure can obviously be extended to give all inverse elements. Here, we wish to present an alternative method of inverting $\mathfrak{B}$ when $\mathfrak{M}\to\infty$, using the Wiener-Hopf technique. We start from the definition

$$\sum_{l=0}^{\infty}\mathfrak{B}_{jl}[\mathfrak{B}^{-1}]_{lj'}=I\delta_{jj'}, \tag{C1}$$

where I is the 4×4 identity matrix. If $j\geq 1$ and $l\geq 1$, then $\mathfrak{B}_{j,l}=B_{j-l}$ depends on $j-l$ alone. Therefore, if $j\geq 2$ and j' fixed, (C1) forms a set of coupled Weiner-Hopf sum equations similar to those studied in I. These will be solved subject to the restrictions imposed by the $j=0$ and 1 equations. To solve the Weiner-Hopf part of the problem, define for $j\leq 1$

$$Y_{j,j'}=\sum_{l=1}^{\infty}B_{j-l}[\mathfrak{B}^{-1}]_{lj'}. \tag{C2}$$

Furthermore, define the Fourier transforms when $|\xi|=1$:

$$B(\xi)=\sum_{l=-\infty}^{\infty}B_l\xi^l, \tag{C3}$$

$$B^{-1}(\xi)_{j'}=\sum_{l=1}^{\infty}[\mathfrak{B}^{-1}]_{lj'}\xi^{l-1}, \tag{C4}$$

and

$$Y(\xi)_{j'}=\sum_{l=1}^{-\infty}Y_{l,j'}\xi^{-l+1}. \tag{C5}$$

Both $B^{-1}(\xi)_{j'}$ and $Y(\xi)_{j'}$ are (in the notation of I) + functions. We explicitly have

$$B(\xi)=\begin{bmatrix} 0 & 1+z_1e^{i\theta} & -1 & -1 \\ -1-z_1e^{-i\theta} & 0 & 1 & -1 \\ 1 & -1 & 0 & 1+z_2\xi^{-1} \\ 1 & 1 & -1-z_2\xi & 0 \end{bmatrix}. \tag{C6}$$

If we multiply (C1) for $j\geq 2$ by ξ^{j-1} and sum on j, we obtain

$$B(\xi)B^{-1}(\xi)_{j'}=Y(\xi^{-1})_{j'}+\xi^{j'-1}(1-\delta_{j',1}) \tag{C7}$$

for $j'\geq 1$. Assume for the moment that, as in II, we may find a factorization of $B(\xi)$ in the form

$$B(\xi)=C^{-1}[Q(\xi^{-1})]^{-1}K^{-1}[P(\xi)]^{-1}D^{-1}, \tag{C8}$$

where C, K, and D are constant matrices and

$$[Q(\xi)]^{-1}=[P(\xi)]^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \xi-\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C9}$$

We may choose P and Q of this form because det $B(\xi)=0$ only when $\xi=\alpha$, α^{-1}. Clearly, $P(\xi)$ is analytic and nonsingular for $|\xi|\leq 1$, while $Q(\xi^{-1})$ is analytic and nonsingular for $|\xi|\geq 1$. Using this factorization, we may write (C7) for $|\xi|=1$ as

$$K^{-1}[P(\xi)]^{-1}D^{-1}B^{-1}(\xi)_{j'}-[Q(\xi^{-1})C\xi^{j'-1}(1-\delta_{j',1})]_+-[Q(\xi^{-1})CY(\xi^{-1})_{j'}]_+$$
$$=[Q(\xi^{-1})C\xi^{j'-1}(1-\delta_{j',1})]_-+[Q(\xi^{-1})CY(\xi^{-1})_{j'}]_-. \tag{C10}$$

The left-hand side is analytic for $|\xi|\leq 1$. The right-hand side is analytic for $|\xi|\geq 1$ and goes to zero as $\xi\to\infty$. Therefore, both sides are separately equal to zero. Now

$$[Q(\xi^{-1})CY(\xi^{-1})_{j'}]_+=Q(0)CY(0)_{j'} \tag{C11}$$

and

$$[Q(\xi^{-1})\xi^{j'-1}]_+=\begin{bmatrix} \xi^{j'-1} & 0 & 0 & 0 \\ 0 & \xi^{j'-1} & 0 & 0 \\ 0 & 0 & \xi^{j'-1}[1-(\alpha^{-1}\xi^{-1})^{j'}]/(\xi^{-1}-\alpha) & 0 \\ 0 & 0 & 0 & \xi^{j'-1} \end{bmatrix}, \tag{C12}$$

so we have

$$B^{-1}(\xi)_{j'}=DP(\xi)K\left\{Q(0)CY(0)_{j'}+\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1-(\alpha^{-1}\xi^{-1})^{j'}]/(\xi^{-1}-\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}C\xi^{j'-1}(1-\delta_{j',1})\right\}. \tag{C13}$$

Therefore, for $j\geq 1, j'\geq 1$,

$$[\mathfrak{B}^{-1}]_{jj'}=(2\pi i)^{-1}\oint d\xi\,\xi^{-j}[B(\xi)]^{-1}C^{-1}[Q(\xi^{-1})]^{-1}\{Q(0)CY(0)_{j'}+[Q(\xi^{-1})\xi^{j'-1}]_+C(1-\delta_{j',1})\}, \tag{C14}$$

where the integral is on the path $|\xi|=1$.

The matrices D and C are easily obtained if we note that the factorization (C8) defines $C(D)$ to be that matrix which, when multiplying $B(\xi)$ on the left (right), gives a common factor of $\xi^{-1}-\alpha$ $(\xi-\alpha)$ in the third row (column). Such a matrix has the form,

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ C_1 & C_2 & C_3 & C_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}=D^{*T}, \tag{C15}$$

where the second equation comes from the skew-Hermiticity of $B(\xi)$. The resulting set of homogeneous equations has the solutions

$$C_1/C_3 = -z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1}[z_2\alpha(1+z_1e^{-i\theta})-(1-z_1e^{-i\theta})], \tag{C16}$$

$$C_2/C_3 = z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1}[z_2\alpha(1+z_1e^{i\theta})-(1-z_1e^{i\theta})], \tag{C17}$$

$$C_4/C_3 = z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[z_2\alpha(1+z_1e^{i\theta})(1+z_1e^{-i\theta})-1+z_1^2]. \tag{C18}$$

To determine $Y(0)_{j'}$, we must make use of the $j=0$ and 1 equations of (C1) which, so far, have not been used. There are

$$\mathfrak{B}_{00}[\mathfrak{B}^{-1}]_{0j'}+\mathfrak{B}_{01}[\mathfrak{B}^{-1}]_{1j'}=I\delta_{0j'} \tag{C19}$$

and

$$\mathfrak{B}_{10}[\mathfrak{B}^{-1}]_{0j'}+\mathfrak{B}_{11}[\mathfrak{B}^{-1}]_{1j'}+\mathfrak{B}_{12}[\mathfrak{B}^{-1}]_{2j'}=I\delta_{1j'}. \tag{C20}$$

Considering only $j'\geq 1$, these may be combined as

$$[\mathfrak{B}_{11}-\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}][\mathfrak{B}^{-1}]_{1j'}+\mathfrak{B}_{12}[\mathfrak{B}^{-1}]_{2j'}=I\delta_{1j'}, \tag{C21}$$

into which we substitute (C14) and solve for $Y(0)_{j'}$. Consider first $j'=1$. Then we obtain

$$(2\pi i)^{-1}\int d\xi\{[\mathfrak{B}_{11}-\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}]\xi^{-1}+\mathfrak{B}_{12}\xi^{-2}\} \times[B(\xi)]^{-1}C^{-1}[Q(\xi^{-1})]^{-1}Q(0)CY(0)_1=I. \tag{C22}$$

We simplify the evaluation of this if we note that from its definition (C2) if $\mathfrak{H}=0$, because $\mathfrak{B}_{10}$ vanishes then, then $Y(0)_j=I\delta_{0,j}$. Thus, setting $\mathfrak{H}=0$ in (C22), we obtain the identity

$$(2\pi i)^{-1}\oint d\xi[\mathfrak{B}_{11}\xi^{-1}+\mathfrak{B}_{12}\xi^{-2}][B(\xi)]^{-1} \times C^{-1}[Q(\xi^{-1})]^{-1}Q(0)C=I. \tag{C23}$$

If we now call

$$(2\pi i)^{-1}\oint d\xi\xi^{-1}[B(\xi)]^{-1}C^{-1}[Q(\xi^{-1})]^{-1} \times Q(0)C=[Q(\xi^{-1})CB(\xi)]_{\xi=0}^{-1}Q(0)C=R, \tag{C24}$$

we find

$$R^{-1}=\begin{bmatrix} 0 & 1+z_1e^{i\theta} & -1 & -1 \\ -1-z_1e^{-i\theta} & 0 & 1 & -1 \\ 1 & -1 & -C_4C_3^{-1}z_2\alpha^{-1} & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \tag{C25}$$

and as shown in Eq. (C26).

$$R=\begin{bmatrix} z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[-1+\alpha^{-1}z_2^{-1}(1-z_1^2)^{-1}\,|\,1-z_1e^{i\theta}\,|^2] & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[1-\alpha^{-1}z_2^{-1}(1-z_1^2)^{-1}(1-z_1e^{i\theta})^2] & z_2^{-1}(1-z_1^2)^{-1}\alpha^{-1}(1-z_1e^{i\theta}) & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[1+z_1e^{i\theta}-\alpha^{-1}z_2^{-1}(1-z_1e^{i\theta})] \\ z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[1-\alpha^{-1}z_2^{-1}(1-z_1^2)^{-1}(1-z_1e^{-i\theta})^2] & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[-1+\alpha^{-1}z_2^{-1}(1-z_1^2)^{-1}\,|\,1-z_1e^{i\theta}\,|^2] & -z_2^{-1}(1-z_1^2)^{-1}\alpha^{-1}(1-z_1e^{-i\theta}) & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[-1-z_1e^{-i\theta}+\alpha^{-1}z_2^{-1}(1-z_1e^{-i\theta})] \\ -z_2^{-1}(1-z_1^2)^{-1}\alpha^{-1}(1-z_1e^{-i\theta}) & z_2^{-1}(1-z_1^2)\alpha^{-1}(1-z_1e^{i\theta}) & -z_2^{-1}(1-z_1^2)^{-1}\alpha^{-1}z_1(e^{i\theta}-e^{-i\theta}) & z_2^{-1}\alpha^{-1} \\ z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[1+z_1e^{-i\theta}-\alpha^{-1}z_2^{-1}(1-z_1e^{-i\theta})] & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[-1-z_1e^{i\theta}+\alpha^{-1}z_2^{-1}(1-z_1e^{i\theta})] & -z_2^{-1}\alpha^{-1} & z_1^{-1}(e^{i\theta}-e^{-i\theta})^{-1} \times[-\,|\,1+z_1e^{i\theta}\,|^2+\alpha^{-1}z_2^{-1}(1-z_1^2)] \end{bmatrix}. \tag{C26}$$

Therefore,

$$Y(0)_1=[1-\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}R]^{-1}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \rho R_{DR}(1-\rho R_{DD})^{-1} & \rho R_{DL}(1-\rho R_{DD})^{-1} & \rho R_{DU}(1-\rho R_{DD})^{-1} & (1-\rho R_{DD})^{-1} \end{bmatrix}, \tag{C27}$$

where

$$\rho=-z^2(e^{i\theta}+1)(e^{i\theta}-1)^{-1}, \tag{C28}$$

so that

$$[B^{-1}]_{j1}=(2\pi i)^{-1}\oint d\xi\,\xi^{-j}[B(\xi)]^{-1}C^{-1}[Q(\xi^{-1})]^{-1}Q(0)C[1-\mathfrak{B}_{10}[B_{00}]^{-1}\mathfrak{B}_{10}R]^{-1}. \tag{C29}$$

When $j'\geq 2$, we proceed in the same fashion. Inserting (C14) into (C21), using the requirement that $Y(0)_j$ vanish at $\mathfrak{H}=0$ and the identity (C23), we obtain

$$Y(0)_{j'}=[1-\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}R]^{-1}\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}(2\pi i)^{-1}\oint d\xi\,\xi^{-1}[B(\xi)]^{-1}C^{-1}[Q(\xi^{-1})]^{-1}[Q(\xi^{-1})\xi^{j'-1}]_+C$$

$$=[1-\mathfrak{B}_{10}[\mathfrak{B}_{10}]^{-1}\mathfrak{B}_{01}R]^{-1}\mathfrak{B}_{10}[\mathfrak{B}_{00}]^{-1}\mathfrak{B}_{01}RCQ^{-1}(0)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^{-j} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}C, \tag{C30}$$

which is to be used in (C14).

To explicitly exhibit $\mathfrak{B}^{-1}$, define $\bar{R}$ by Eq. (C31) as shown on page 473.

Then we have for $j\geq 1$,

$$[\mathfrak{B}^{-1}]_{jj}=-z_2^{-1}(1-z_1^2)^{-1}(\alpha^{-1}-\alpha)^{-1}\left\{\bar{R}^{(0)}-\alpha^{-2j}\bar{R}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_1C_3^{-1} & C_2C_3^{-1} & 1 & C_4C_3^{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}$$

$$+\alpha^{2(1-j)}\rho(1-\rho R_{DD})^{-1}\begin{bmatrix} R_{RD}R_{DR} & R_{RD}R_{DL} & R_{RD}R_{DU} & R_{RD}R_{DD} \\ R_{LD}R_{DR} & R_{LD}R_{DL} & R_{LD}R_{DU} & R_{LD}R_{DD} \\ R_{UD}R_{DR} & R_{UD}R_{DL} & R_{UD}R_{DU} & R_{UD}R_{DD} \\ R_{DD}R_{DR} & R_{DD}R_{DL} & R_{DD}R_{DU} & R_{DD}R_{DD} \end{bmatrix}, \tag{C33}$$

$$\bar{R} = -z_2(1-z_1^2)\alpha(\alpha^{-1}-\alpha) \lim_{\xi\to\alpha^{-1}} (\xi-\alpha^{-1})[B(\xi^{-1})]^{-1}$$

$$= \begin{bmatrix} z_2(\alpha^{-1}-\alpha) & -(1-z_1^2)^{-1}[z_2^2(1+z_1e^{i\theta})^2-(1-z_1e^{i\theta})^2] & 1-z_1e^{i\theta}-z_2\alpha(1+z_1e^{i\theta}) & 1-z_1e^{i\theta}-z_2\alpha^{-1}(1+z_1e^{i\theta}) \\ (1-z_1^2)^{-1}[z_2^2(1+z_1e^{-i\theta})^2-(1-z_1e^{-i\theta})^2] & -z_2(\alpha^{-1}-\alpha) & -1+z_1e^{-i\theta}+z_2\alpha(1+z_1e^{-i\theta}) & 1-z_1e^{-i\theta}-z_2\alpha^{-1}(1+z_1e^{-i\theta}) \\ -1+z_1e^{-i\theta}+z_2\alpha^{-1}(1+z_1e^{-i\theta}) & 1-z_1e^{i\theta}-z_2\alpha^{-1}(1+z_1e^{i\theta}) & -z_1(e^{i\theta}-e^{-i\theta}) & 1-z_1^2-z_2\alpha^{-1}\,|\,1+z_1e^{i\theta}\,|^2 \\ -1+z_1e^{-i\theta}+z_2\alpha(1+z_1e^{-i\theta}) & -1+z_1e^{i\theta}+z_2\alpha(1+z_1e^{i\theta}) & -1+z_1^2+z_2\alpha\,|\,1+z_1e^{i\theta}\,|^2 & z_1(e^{i\theta}-e^{-i\theta}) \end{bmatrix} \quad \text{(C31)}$$

and define $\bar{R}^{(0)}$ by

$$\bar{R}^{(0)} = -z_2(1-z_1^2)(\alpha^{-1}-\alpha)(2\pi i)^{-1}\oint d\xi\, \xi^{-1}[B(\xi^{-1})]^{-1}$$

$$= \begin{bmatrix} 0 & \begin{matrix} -z_1z_2(\alpha^{-1}-\alpha)e^{i\theta}-(1-z_1^2)^{-1} \\ \times[z_2^2(1+z_1e^{i\theta})^2-(1-z_1e^{i\theta})^2] \end{matrix} & 1-z_1e^{i\theta}-z_2\alpha^{-1}(1+z_1e^{i\theta}) & 1-z_1e^{i\theta}-z_2\alpha^{-1}(1+z_1e^{i\theta}) \\ \begin{matrix} z_1z_2(\alpha^{-1}-\alpha)e^{-i\theta}+(1-z_1^2)^{-1} \\ \times[z_2^2(1+z_1e^{-i\theta})^2-(1-z_1e^{-i\theta})^2] \end{matrix} & 0 & -1+z_1e^{-i\theta}+z_2\alpha^{-1}(1+z_1e^{-i\theta}) & 1-z_1e^{-i\theta}-z_2\alpha^{-1}(1+z_1e^{-i\theta}) \\ -1+z_1e^{-i\theta}+z_2\alpha^{-1}(1+z_1e^{-i\theta}) & 1-z_1e^{i\theta}-z_2\alpha^{-1}(1+z_1e^{i\theta}) & -z_1(e^{i\theta}-e^{-i\theta}) & 1-z_1^2-z_2\alpha^{-1}\,|\,1+z_1e^{i\theta}\,|^2 \\ -1+z_1e^{-i\theta}+z_2\alpha^{-1}(1+z_1e^{-i\theta}) & -1+z_1e^{i\theta}+z_2\alpha^{-1}(1+z_1e^{i\theta}) & -1+z_1^2+z_2\alpha^{-1}\,|\,1+z_1e^{i\theta}\,|^2 & z_1(e^{i\theta}-e^{-i\theta}) \end{bmatrix}. \quad \text{(C32)}$$

and for $j>j'\geq 1$,

$$[\mathfrak{B}^{-1}]_{jj'}=-[\mathfrak{B}^{-1}]_{j'j}{}^{*T}$$

$$=-z_2^{-1}(1-z_1^2)^{-1}(\alpha^{-1}-\alpha)^{-1}\alpha^{j'-j}\bar{R}\left\{I-\alpha^{-2j'}\begin{bmatrix}0&0&0&0\\0&0&0&0\\C_1C_3^{-1}&C_2C_3^{-1}&1&C_4C_3^{-1}\\0&0&0&0\end{bmatrix}\right\}$$

$$+\alpha^{2-j-j'}\rho(1-\rho R_{DD})^{-1}\begin{bmatrix}R_{RD}R_{DR}&R_{RD}R_{DL}&R_{RD}R_{DU}&R_{RD}R_{DD}\\R_{LD}R_{DR}&R_{LD}R_{DL}&R_{LD}R_{DU}&R_{LD}R_{DD}\\R_{UD}R_{DR}&R_{UD}R_{DL}&R_{UD}R_{DU}&R_{UD}R_{DD}\\R_{DD}R_{DR}&R_{DD}R_{DL}&R_{DD}R_{DU}&R_{DD}R_{DD}\end{bmatrix}. \quad (C34)$$

To obtain $[\mathfrak{B}^{-1}]_{0j}$, we use

$$\mathfrak{B}_{00}[\mathfrak{B}^{-1}]_{0j}+\mathfrak{B}_{01}[\mathfrak{B}^{-1}]_{1j}=I\delta_{0j} \quad (C35)$$

to obtain

$$[\mathfrak{B}^{-1}]_{0j}=\delta_{0j}\,|\,e^{i\theta}-1\,|^{-2}\begin{bmatrix}0&1-e^{i\theta}&1-e^{i\theta}&1-e^{i\theta}\\-1+e^{-i\theta}&0&-1+e^{-i\theta}&1-e^{-i\theta}\\-1+e^{-i\theta}&1-e^{i\theta}&e^{-i\theta}-e^{i\theta}&0\\-1+e^{-i\theta}&-1+e^{i\theta}&0&e^{i\theta}-e^{-i\theta}\end{bmatrix}$$

$$-z\begin{bmatrix}(1-e^{-i\theta})^{-1}[\mathfrak{B}^{-1}]_{1D,jR}&(1-e^{-i\theta})^{-1}[\mathfrak{B}^{-1}]_{1D,jL}&(1-e^{-i\theta})^{-1}[\mathfrak{B}^{-1}]_{1D,jL}&(1-e^{-i\theta})^{-1}[\mathfrak{B}^{-1}]_{1D,jD}\\(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jR}&(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jL}&(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jU}&(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jD}\\(e^{i\theta}+1)(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jR}&(e^{i\theta}+1)(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jL}&(e^{i\theta}+1)(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jU}&(e^{i\theta}+1)(e^{i\theta}-1)^{-1}[\mathfrak{B}^{-1}]_{1D,jD}\\0&0&0&0\end{bmatrix} \quad (C36)$$

where, for $j\geq 1$, $\mathfrak{B}^{-1}$ is obtained from (C33) or (C34) and for $j=0$, $\mathfrak{B}^{-1}$ is obtained from (C36) with $j=1$. If we note the relation,

$$R_{DD}=-iz_2^{-1}\mathfrak{v}'/\mathfrak{v}=-z_2^{-1}\alpha^{-1}C_4C_3^{-1}, \quad (C37)$$

it will readily be seen that in the D, U subspace (C33), (C34), and (C36) specialize to (7.7). We also remark that $\mathfrak{B}$ may also be inverted by solving a set of coupled difference equations.

APPENDIX D

We give here the higher order terms in the asymptotic expansion of $\mathfrak{S}_{1,1}(N,\mathfrak{H})$ for T near T_c and H away from zero that are mentioned in the text. When $T<T_c$, we follow the procedure of Sec. 8D and retain the first two orders in N^{-1} in the expansion of the integrals in (8.71). This gives

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1{}^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-\alpha_1)^2\alpha_2^{-2}(1-\alpha_2)^4$$

$$\left\{\left[\int_t^\infty d\xi\,\xi^{-1}K_1(\xi)-\tfrac{1}{2}N^{-1}tK_1(t)\right][t^{-1}K_2(t)-\tfrac{1}{2}N^{-1}(K_2(t)+tK_1(t))]\right.$$

$$\left.-[t^{-1}K_1(t)-\tfrac{1}{2}N^{-1}[K_2(t)+K_1(t)+tK_0(t)]][t^{-1}K_1(t)\tfrac{1}{2}N^{-1}[K_2(t)-K_1(t)-tK_0(t)]]\right\}. \quad (D1)$$

Multiplying this out and using the recurrence relations for modified Bessel functions, we find

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}$$
$$\times(1-\alpha_1)^2N^{-4}\Big\{t^3K_2(t)\int_t^\infty d\xi\,\xi^{-1}K_1(\xi)-t^2K_1^2(t)+N^{-1}t^4\Big[\tfrac{1}{2}(3K_2(t)-tK_1(t))\int_t^\infty d\xi\,\xi^{-1}K_1(\xi)$$
$$-\tfrac{1}{2}K_1(t)(4t^{-1}K_1(t)-K_0(t))\Big]+O(N^{-2})\Big\}. \quad \text{(D2)}$$

When $T>T_c$, an analogous calculation gives the more accurate version of (8.85) of

$$\mathfrak{S}_{1,1}(N,\mathfrak{H})\sim\mathfrak{M}_1^2+(2\pi)^{-1}z^2(1-z_1)^2(\mathfrak{r}-1)^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-z_2^2)(z_2^2-z^2)^{-2}(1-\alpha_1)z_1^{-2}N^{-4}$$
$$\times\{t'^3K_2(t')-N^{-1}t'^4[K_2(t')+\tfrac{1}{2}t'K_1(t')]+O(N^{-2})\}$$
$$+\pi^{-2}z_1^{-2}(1-z_2^2)^2z^2(z_2^2-z^2)^{-2}\tfrac{1}{4}(1-\mathfrak{r})^{-2}(\mathfrak{r}^{-1}-1)^{-2}(1-\alpha_1)^2N^{-4}$$
$$\times\Big\{t'^3K_2(t')\int_{t'}^\infty d\xi\,\xi^{-1}K_1(\xi)-t'^2K_1^2(t')+N^{-1}t'^4$$
$$\times\Big[-\tfrac{1}{2}(t'K_1(t')+K_2(t'))\int_{t'}^\infty d\xi\,\xi^{-1}K_1(\xi)+\tfrac{1}{2}K_1(t')K_0(t')\Big]+O(N^{-2})\Big\}. \quad \text{(D3)}$$

By inspecting these equations, we see that as $t\to0$ the terms of order N^{-4} remain finite while the terms of order N^{-5} vanish. This vanishing of the next leading order term at $T=T_c$ has already been seen in the bulk problem as presented in I where, while the leading term in S_N is proportional to $N^{-1/4}$, there is no $N^{-5/4}$ term and the next nonvanishing term is of order $N^{-9/4}$.

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Critical Temperatures of Anisotropic Ising Lattices.* I. Lower Bounds

CHI-YUAN WENG AND ROBERT B. GRIFFITHS†
Department of Physics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
AND
MICHAEL E. FISHER
Baker Laboratory, Cornell University, Ithaca, New York
(Received 28 April 1967)

The critical or Curie temperature of the anisotropic rectangular Ising ferromagnet is known from Onsager's exact solution to vanish asymptotically as

$$kT_c/2J_x\sim[\ln(1/\eta)-\ln\ln(1/\eta)]^{-1}+\cdots,$$

when $\eta=J_y/J_x$, the ratio of exchange energies for bonds parallel to the y and x axes, approaches zero. An extension of the Peierls argument yields a simple interpretation of this slow decrease and provides, from first principles, a rigorous lower bound of precisely the same asymptotic form. For the anisotropic simple cubic lattice, a lower bound, also of this asymptotic form, is established in terms of $\eta=(J_y+J_z)/J_x$.

I. INTRODUCTION

THE problem of the Ising ferromagnet of spin $\frac{1}{2}$ with nearest-neighbor interaction has been studied extensively. It is well known that the one-dimensional model in the presence of an external magnetic field and various two-dimensional models in zero field are exactly soluble.[1] In particular, the spontaneous magnetization below the critical point has been calculated for both square and "rectangular" lattices.[2,3]

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† Alfred P. Sloan Research Fellow.

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