Spin Waves in Paramagnetic Fermi Gases

L. L. VAN ZANDT
Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts*
(Received 15 September 1966; revised manuscript received 12 July 1967)

The nature of spin-type zero sound, "paramagnetic spin waves" (PSW's), in free-electron gases is studied by means of the time-dependent Hartree-Fock approximation. The importance of attractive interactions is emphasized. A short-wavelength cutoff in the spectrum at microwave or low infrared frequencies is found. The effect of the application of magnetic fields is studied. The frequencies are shifted by the Larmor precession. Some branches of the spectrum are effectively eliminated; others are strengthened and split. Spin wave fronts travel in helices about magnetic field lines. PSW's are damped by electron collisions with crystal imperfections. The relaxation time is identical with the relaxation time as determined by the dc conductivity. Electron-electron collision damping is also studied and found to be negligible by comparison. The observation of Landau-damped modes is considered and conditions where this is possible are indicated. A boundary condition is found: A surface must be at an antinode of the spin density. A microwave, "thin"-film resonance experiment is suggested. The strength of the resonance lines is considered. An expression for power dissipation at PSW resonance is derived.

I. INTRODUCTION

THE problem of the low-lying excitations of a gas of interacting Fermions has been discussed by Landau.[1] Distinct from well-known excitations—quasiparticles, ordinary or "first" sound, and plasma oscillations—he has discovered a class of boson-like collective modes which he calls "zero sound." Silin[2] has further investigated these oscillations, including some aspects of their behavior in magnetic fields. Gor'kov and Dzyaloshinskii[3] have considered the effects of anisotropic Fermi surfaces.

There are two types of zero sound. One is a propagating distortion of the shape of the Fermi surface; in anisotropic systems, it is chiefly a transverse current density wave. The second type is a spin-density wave. The term "spin-density wave," however, has had prior application to a rather different phenomenon in Fermi gases,[4] a long range, stationary magnetic ordering. To make specific reference to the second type of zero sound, we shall use the term "paramagnetic spin wave" (PSW).

Since Landau's discovery a decade ago, there has been no experimental detection of these modes in metals. It is the purpose of this paper to provide additional information about enough properties of zero sound in metals to make a successful experimental search reasonably probable.

Whether or not zero-sound excitations are possible for a given system depends upon the density, and the functional form of the particle interactions. The conditions are different for the two types of modes. Silin has shown that the existence of the first type of zero sound requires that the interaction between electrons in states close together on the Fermi surface be sufficiently large compared to the interaction of widely separated states. His calculations indicate that this criterion is unlikely to be met in simple metals. Similarly, Gor'kov and Dzyaloshinskii find that the interaction strength must exceed a certain rather high threshold, or the Fermi surface must be sufficiently distorted for zero sound to propagate. In the latter case, propagation can occur only along or near crystal symmetry directions.

The conditions for the existence of the spin type of modes are not so stringent. The above authors find solutions to their equations whenever the exchange interactions are repulsive, or attractive and larger than a certain threshold. We shall see that solutions of the latter case occur only in unusual situations or in interaction strength ranges where the gas is ferromagnetically unstable.

Thus an experimental search for zero sound would have the highest probability of success if directed towards paramagnetic spin waves in materials with repulsive exchange interactions. Repulsive exchange interactions imply attractive direct interactions. Therefore, likely materials for the detection of zero sound exhibit superconductivity, and such detection would provide independent justification of the hypothesis of attractive interactions among Fermi-surface electrons of superconductors in the normal state.

In the following, we are concerned exclusively with paramagnetic spin waves. We shall rederive a number of the earlier results and discuss some differences. Section II sets forth the general method of attack. Spin-density distributions, the frequency-wavelength relations, and the nature of the one-electron wave functions are derived for a simplified model. Section III is a calculation of the effects of external magnetic fields. In Secs. IV and V, we derive expressions for the damping rate due to several different processes. In Sec. VI, we derive a boundary condition, suggest a possible experiment, and determine a sensitivity criterion.

* Operated with support from the U. S. Air Force.

[1] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **32**, 59 (1957) [English transl.: Soviet Phys.—JETP **5**, 101 (1957)].

[2] V. P. Silin, Zh. Eksperim. i Teor. Fiz. **35**, 1243 (1958) [English transl.: Soviet Phys.—JETP **8**, 870 (1959)].

[3] L. P. Gor'kov and I. E. Dzyaloshinskii, Zh. Eksperim. i Teor. Fiz. **44**, 1650 (1963) [English transl.: Soviet Phys.—JETP **17**, 1111 (1963)].

[4] A. W. Overhauser, Phys. Rev. **128**, 1937 (1962).

We shall treat the electron gas in the approximation of time-dependent, self-consistent fields. We consider first an unexcited state in which there is no magnetization. If this system is driven by an externally imposed field, it will respond magnetically at the external frequency and wavelength. Each electron in the system "sees" the external field and, through its interactions with the remaining electrons, an additional, response field. The electron then contributes to the response field experienced by the others. The nature of the magnetic response thus involves the interactions and is very complicated. To reduce this complexity, we replace the true interaction operators with the Hartree-Fock operator, giving an "averaged" interaction. This allows us to treat one electron at a time, greatly simplifying the problem.

If we divide the response by the magnitude of the applied driving field, we obtain the magnetic susceptibility. Where a finite "response" is possible in the absence of any driving field, we have a normal mode of the system. In this case the susceptibility is infinite, and this circumstance, or its close approximation is our chief concern. A very large susceptibility implies a very small transverse electric field, and we shall, therefore, neglect the effects of transverse electric fields and currents.

In the approximation of the replacement of the true interactions with the Hartree-Fock operator, and the neglect of crystal imperfections, magnetic oscillations may be self-sustaining. When these approximations are given up by the introduction of scattering, the energy of excitation is bled away in the scattering events and the system relaxes to the unexcited state. This process proceeds according to the frequency and efficacy of the collisions.

The effectiveness of a collision in dissipating energy of excitation depends upon the degree to which the colliding electron participates in the collective motion, and this, in turn, depends on the velocity of the electron relative to the phase velocity of the collective wave. The participation is greatest, and with it the dissipation, when these velocities match.

The presence of electrons with velocities to match the wave is the necessary condition for "Landau damping." In fact, the dissipation is so much stronger under these conditions that when the electron interactions with the collective wave are treated in perturbation theory, the damping becomes singular and the "wave" ceases to exist.

It is possible, however to go beyond first-order perturbation theory in the treatment of the electron-collective wave interaction. In fact in the time-dependent Hartree-Fock approximation used here, the interaction may be treated to all orders. If we then study the dissipative effects of arbitrarily weak scattering, as is done in Secs. V and VI, we find that the decay lifetime for collective spin wave is independent of the fact of Landau damping. Now the regime of arbitrarily weak scattering is the same as the regime of arbitrarily *strong* external driving field, in so far as the damping problem is concerned; this is discussed in Appendix B.

The treatment which we present here then suggests the interesting possibility that in materials of sufficiently high quality and long scattering lifetime, or with sufficiently high driving amplitudes, resonances may be excited and narrow lines observed against the background of the continuum of single particle excitations. Of course, the material quality must always be sufficiently high that the required illuminating power levels do not destroy samples or produce other obscuring nonlinearities. We must emphasize that our results do not predict the vanishing of Landau damping under ordinary conditions of low driving power. Rather, the energy absorption of Landau damping appears as the initial stage of the establishment of the excited, polarized, oscillating condition from which our calculation begins, and unless energy can be supplied faster than the collisions can drain it away, no resonances will be seen.

The calculations which follow use single particle wave functions in which the self-consistent fields are incorporated *ab initio*. This leads naturally to a single treatment of damping for both weakly damped and Landau-damped frequency-wavelength conditions. It also provides a natural structure of basis functions for the study of the dynamics of the collision process carried out in Appendix B. The procedure is entirely equivalent to working with a density matrix which is diagonal in the presence of the excitation.

II. THE DISPERSION RELATION

The basic Hamiltonian is well known.

$$H=\sum_i \mathbf{p}_i^2+\tfrac{1}{2}\sum_{i,j} V(\mathbf{r}_i-\mathbf{r}_j)-\sum_i \boldsymbol{\sigma}_i\cdot\mathfrak{IC}_{ex}\mu. \qquad (1)$$

Here μ is the electron magnetic moment and $\mathfrak{IC}_{ex}$ is the external magnetic field. The sums range over the electrons of the system. We assume a solution of the time-dependent Schrödinger equation for the whole system in the form of a determinant of one-electron functions, solutions of the equations[5]

$$h_i\phi_i=i\partial\phi_i/\partial t, \qquad (2)$$

where

$$h_i=\mathbf{p}_i^2+V_{sc}-\sigma_{iz}\mu\mathfrak{IC}_{ex}. \qquad (3)$$

We have taken $\hbar$ and $\hbar^2/2m$ equal to 1. V_{sc} is formed from the Hartree-Fock operator.

We assume a form for V_{sc} which describes a periodic, moving field acting on the electron spin, and a constant, homogeneous field which we lump together with the externally applied field. It is convenient to give the periodic field the character of a spiraling polarization lying in the plane normal to the external field. Thus

[5] See, for instance, A. D. McLachlan and M. A. Ball, Rev. Mod. Phys. 36, 844 (1964).

we try

$$V_{sc}-\sigma_{zi}\mu\mathcal{H}_{ex}=e(\mathbf{k})\sigma_z+d(\mathbf{k})+c(\mathbf{k})\{\sigma_i^+e^{-2i(\mathbf{q}\cdot\mathbf{r}-\omega t)}+\sigma_i^-e^{2i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\}, \quad (4)$$

where $e(\mathbf{k})$, $d(\mathbf{k})$, and $c(\mathbf{k})$ are to be determined. In general, the various terms in (4) depend on the state of the particle, which we shall label with the vector $\mathbf{k}$. If we temporarily ignore the orbital effects of the external field, the solutions of (2) are

$$\phi_i=\Omega^{-1/2}\{A_1(\mathbf{k})e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}_i+i\omega t}\alpha+A_2(\mathbf{k})e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_i-i\omega t}\beta\}e^{-iE_{\mathbf{k}}t}, \quad (5)$$

where

$$A_1(\mathbf{k})=\frac{E_{\mathbf{k}}-(\mathbf{k}+\mathbf{q})^2+e+\omega}{\{c^2+[E_{\mathbf{k}}-(\mathbf{k}+\mathbf{q})^2+e+\omega]^2\}^{1/2}},$$

$$A_2(\mathbf{k})=\frac{c}{\{c^2+[E_{\mathbf{k}}-(\mathbf{k}+\mathbf{q})^2+e+\omega]^2\}^{1/2}},$$

$$E_{\mathbf{k}}=\mathbf{k}^2+\mathbf{q}^2\pm[(2\mathbf{k}\cdot\mathbf{q}-e-\omega)^2+c^2]^{1/2}+d(\mathbf{k}). \quad (6)$$

Here Ω is the volume of normalization. The upper and lower signs in $E_{\mathbf{k}}$ distinguish what we shall call upper and lower branch states, respectively. The functions have the property that

$$A_1^+(\mathbf{k})=A_2^-(\mathbf{k}); \quad A_2^+(\mathbf{k})=-A_1^-(\mathbf{k}).$$

There is a different one-electron wave function for each value of $\mathbf{k}$ consistent with periodic boundary conditions. In the interest of symmetry between the spin-up and -down parts of the wave functions and to simplify some expressions, we have made an unconventional choice of origin in the space of $\mathbf{k}$; the states of lowest one-electron energy will occur near $\mathbf{k}=\pm\mathbf{q}$.

Let N different states be occupied. We form the self-consistent field acting on any one of them from the Hartree-Fock operator

$$V_{sc}\phi(\mathbf{k},\mathbf{r})=\sum_{\mathbf{k}_{occ}'}\int\phi^*(\mathbf{k}',\mathbf{r}')\phi(\mathbf{k}',\mathbf{r}')V(\mathbf{r}-\mathbf{r}')d\mathbf{r}'\phi(\mathbf{k},\mathbf{r})$$

$$-\sum_{\mathbf{k}_{occ}'}\int\phi^*(\mathbf{k}',\mathbf{r}')\phi(\mathbf{k},\mathbf{r}')V(\mathbf{r}-\mathbf{r}')d\mathbf{r}'\phi(\mathbf{k}',\mathbf{r}). \quad (7)$$

The direct part of the interaction is a constant which we may ignore. The exchange part has the form of a potential like that of (4). That (7) and (4) shall be identical requires

$$c(\mathbf{k})=-\sum_{\mathbf{k}_{occ}'}\frac{V(\mathbf{k}-\mathbf{k}')}{\Omega}A_1(\mathbf{k}')A_2(\mathbf{k}'),$$

$$d(\mathbf{k})=-\sum_{\mathbf{k}_{occ}'}\frac{V(\mathbf{k}-\mathbf{k}')}{2\Omega},$$

$$e(\mathbf{k})=-\left\{\sum_{\mathbf{k}_{occ}'}\frac{V(\mathbf{k}-\mathbf{k}')}{2\Omega}(A_1^2(\mathbf{k}')-A_2^2(\mathbf{k}'))\right\}-\mu\mathcal{H}_{ex}. \quad (8)$$

Here $V(\mathbf{k},\mathbf{k}')$ is the $\mathbf{k}-\mathbf{k}'$ Fourier component of the interaction potential, $V(\mathbf{r}-\mathbf{r}')$. These are the equations of self-consistency. They are a slight generalization of the self-consistent equations of Ref. 4, in which $\omega=0$.

Our one-electron states have no definite spins. There is a component of spin in the z direction proportional to $A_1^2-A_2^2$. This is very nearly ±1 for most states. We shall refer to spin "up" or "down" on the basis of the sign of this quantity. There is also a component of spin in the x-y plane which follows the spiraling self-consistent field. This component is equal to $-A_1A_2$. We shall refer to this quantity as the polarization of the state.

The spiraling parts of all the lower band states are in phase. The upper band state polarizations are similarly in phase with themselves, but shifted by π with respect to the lower band. More general wave functions can be formed; these are discussed in Appendix B.

Associated with the periodic disturbance of wave vector $2\mathbf{q}$ and frequency 2ω is an energy gap at those states for which $\mathbf{k}_\uparrow^2-\mathbf{k}_\downarrow^2=2\omega$. States at or near this gap are in resonance with the oscillation and are completely polarized by it; their z-spin component disappears. States away from the gap are more or less polarized according to their proximity.

The location of the gap depends on the relation of ω to $\mathbf{q}$. When the plane of the gap intersects the Fermi surface and resonating states are occupied, we shall call the mode "Landau damped."

We can calculate a one-particle energy function by taking the expectation of the total Hamiltonian and finding the change in this expectation upon annihilating a single particle. This function may be verified to be $\langle h_i\rangle$ or $\langle(i\partial/\partial t)_i\rangle$, where $(i\partial/\partial t)_i$ acts only on the ith state in the determinant. This function is

$$\mathcal{E}(\mathbf{k})=E_{\mathbf{k}}-\omega[A_1^2(\mathbf{k})-A_2^2(\mathbf{k})]. \quad (9)$$

Figure 1 is a sketch of a surface over which the one-particle energy is constant, all states within the surface being occupied, and all states outside being empty. The

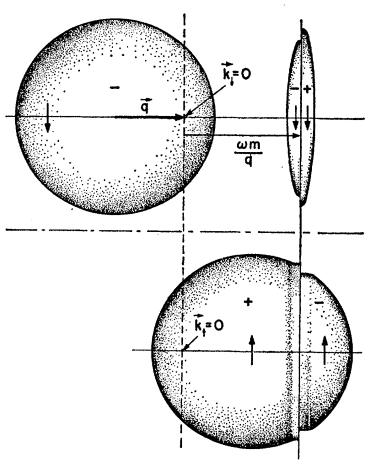

FIG. 1. Spin-down and spin-up Fermi surfaces for a Landau-damped PSW of large wave vector and exaggerated amplitude. The plus and minus signs refer to the choice of sign in Eq. (10).

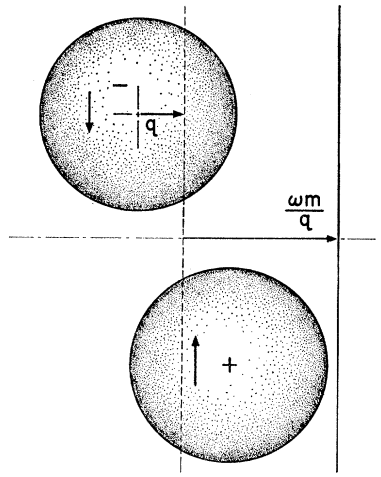

FIG. 2. Fermi surfaces for the undamped PSW.

dotted section contains the origin in k space. The solid section is the energy-gap plane. The pieces of surface enclosing like spin states have been grouped together. The arrows indicate the up- or down-spin pieces; the plus and minus signs refer to the sign choice in (6), the lower and upper bands, respectively. The different pieces of Fig. 1 of like arrows enclose states of like z components of spin; the pieces labeled with like signs enclose states of like spiraling components. The amplitude of the disturbance is grossly exaggerated in the figure. The true gap size would be of order $1/N$. Figure 2 depicts a situation in which the gap plane does not intersect the Fermi surface.

To determine the relationship between $\mathbf{q}$ and ω we need to solve the system of coupled integral equations in (8). To this end, it is convenient to introduce a new variable, ϕ.

$$\sin\phi=-2A_1A_2=c/[(2\mathbf{k}\cdot\mathbf{q}-\omega')^2+c^2]^{1/2};\qquad \omega'=\omega+e. \quad (10)$$

It then follows that

$$\cos\phi=A_1^2-A_2^2=(2\mathbf{k}\cdot\mathbf{q}-\omega')/[(2\mathbf{k}\cdot\mathbf{q}-\omega')^2+c^2]^{1/2}.$$

The sum of $\sin\phi$ over the occupied states is twice the polarization of the spin-wave mode, and since $\sin\phi$ is roughly proportional to $c(\mathbf{k})$, the sum defines an average of $c(\mathbf{k})$ which we shall call c and refer to as the "amplitude." We shall later show that the energy difference between the excited PSW state and the ground state is proportional to c^2 and to the volume of the system. Therefore, for a fixed energy of excitation, c goes to zero as $1/N^{1/2}$, and it is appropriate to study the system of Eqs. (8), in the limit of small c.

For modes which are not Landau damped, $A_1^2-A_2^2=\pm1$ plus terms of order c^2. Adding $d(\mathbf{k})$ and $e(\mathbf{k})\sigma_z$ then gives just the spin-interaction energy with the external field plus the ordinary exchange energy, a part of which represents a small correction to the external field. In particular, we have

$$d(\mathbf{k})+e(\mathbf{k})\sigma_z=-\mu\sigma_z\mathfrak{H}_{ex}-\sum_{\mathbf{k}'}{}'\frac{V(\mathbf{k}-\mathbf{k}')}{\Omega}. \quad (11)$$

The prime on the summation signifies summation only over states of like spin. Now $e(\mathbf{k})$ may be calculated directly, and we have only a single integral equation in $c(k)$ to solve. For Landau damped modes, $|A_1^2-A_2^2|$ is different from 1 only in a layer of states with width proportional to c. These add a correction term to $e(\mathbf{k})$ of order c, while the c-dependent term in e introduces a c^2 correction term in the equation for $c(\mathbf{k})$ (because A_1A_2 is proportional to c), and thus for both damped and undamped situations the integral equations decouple in the limit of small c.

Let us consider the strong screening or "delta-function" limit in which we take $V(\mathbf{k},\mathbf{k})=V_0$, a constant. We have only constants for $c(\mathbf{k})$, $d(\mathbf{k})$, and $e(\mathbf{k})$; thus, $c(\mathbf{k})=c$, $e(\mathbf{k})=e=\mu\mathfrak{H}_{eff}$, and $d(\mathbf{k})=d$ is a vacuous constant.

$$c=-\frac{V_0}{\Omega}\sum_{k_{occ}'}A_1(\mathbf{k}')A_2(\mathbf{k}')$$

$$=\frac{V_0}{2\Omega}\sum_{k_{occ}'}\frac{\mp c}{[(2\mathbf{k}'\cdot q-\omega')^2+c^2]^{1/2}}. \quad (12)$$

The choice of sign depends on the branch of states over which the sum is taken; it is positive for the lower branch and negative for the upper. The integrations of (12) are easily performed. In the limit of small c, we have

$$\frac{-16\pi^2}{V_0}=\frac{1}{(2q)^3}$$

$$\times\left\{[(2qk_{F\downarrow})^2-(2q^2+\omega')^2]\ln\left|\frac{2qk_{F\downarrow}-2q^2-\omega'}{-2qk_{F\downarrow}-2q^2-\omega'}\right|\right.$$

$$+[(2qk_{F\uparrow})^2-(2q^2-\omega')^2]\ln\left|\frac{2qk_{F\uparrow}-2q^2+\omega'}{-2qk_{F\uparrow}-2q^2+\omega'}\right|$$

$$\left.-4qk_{F\downarrow}(2q^2+\omega')-4qk_{F\uparrow}(2q^2-\omega')\right\}. \quad (13)$$

The arrows labeling k_F refer to the spin-up and spin-down Fermi spheres.

It is of interest to study (13) under the hypothesis that the difference in Fermi wave vectors may be neglected. We expand for small q and find

$$\Gamma_0\equiv\frac{4\pi^2}{V_0k_F}=1-\frac{\eta}{2}\ln\left|\frac{1+\eta}{1-\eta}\right|-\frac{q^2}{3k_F^2(1-\eta^2)^2};$$

$$\eta=\frac{\omega+e}{2qk_F}. \quad (14)$$

This equation, without the term in q^2 and with $e=0$, was presented by Landau.[1] Unless the disturbance is Landau-damped ($\eta<1$), there are no solutions for positive V_0, which means that the electron interactions must be attractive. For the damped solutions, $V_0>4\pi^2/k_F$. In this region of interaction strength, however, the exchange energy "reward" for populating one band at the expense of the other exceeds the kinetic energy "cost." Under these circumstances, the gas is ferromagnetically unstable.

Equation (14) has the approximate solution

$$\eta=1\pm 2e^{-2\beta}, \tag{15}$$

where

$$\beta=1-\frac{4\pi^2}{V_0k_F}-\frac{q^2}{k_F^2}\frac{1}{3(1-\eta^2)^2}.$$

Since our discussion is concerned with superconducting materials, the quantities in (15) may be easily determined. The expression $-V_0k_F/4\pi^2$, written in this way to make explicit the dependence on electron density through k_F, is equal to the expression $N(0)V$ appearing in the BCS[6] theory of superconductors. (The two V's differ by a factor $-\Omega$; ours is independent of system size.) We use the BCS expression for the critical temperature

$$\kappa T_c=1.14\hbar\omega_{\text{Debye}}\exp[-1/N(0)V]. \tag{15a}$$

Since the Debye temperature and the critical temperature are directly measured, tabulated properties, the equation may be easily inverted to give a value for $N(0)V$.

Lead is a material with strong electron-phonon coupling,[7] high transition temperature and low Debye temperature. We find $(4\pi^2/V_0k_F)_{\text{Pb}}=-2.7$. Aluminum is a material with weaker electron-phonon coupling, lower transition temperature, and high Debye temperature. $(4\pi^2/V_0k_F)_{\text{Al}}=-5.9$. These two numbers indicate the range we may expect for our parameter.

These numbers can be used to evaluate $\delta=|1-\eta|$ at $q=0$. We have $\delta=1.3\times10^{-3}$ for Pb, and $\delta=1.9\times10^{-6}$ for Al. We can use (13) to find the region where the departure from linear behavior of ω' in q becomes significant. We approximate the maximum linear q value by setting the q^2 correction term in (14) equal to 1 and substituting the $q=0$ values of η. The approximation indicates a linear spectrum in lead up to $q=4.5\times10^{-3}\,k_F$ and in aluminum up to $q=6.6\times10^{-6}\,k_F$.

In Fig. 3 we exhibit a complete spectrum, determined numerically from Eq. (13). For larger values of V_0, the curve extends to higher q and the separation between the branches increases. As V_0 goes to 0 the curve shrinks down into the origin. An interesting feature is the maximum q beyond which there are no PSW's. Figure 3 is plotted for an exaggerated value of V_0, but we have also studied curves for values appropriate to lead and aluminum. We have found maximum q's of $1.8\times10^{-3}\,k_F$ for lead and $4.5\times10^{-6}\,k_F$ for aluminum. These q's are consistent with those we estimated for the onset of the "nonlinear" region. The term in q^2 of (14) never attains 1. The departure of the curve from linear behavior and the reversal occur very suddenly over a range of only a few tenths of a percent of the total extent of the curve.

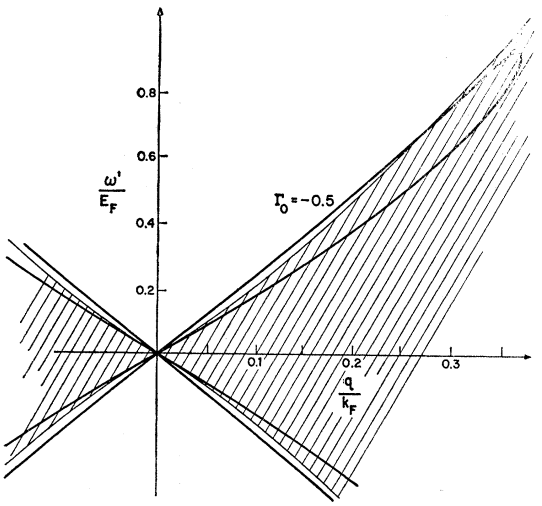

FIG. 3. PSW spectrum for the case of attractive interactions. ω' is one half the separation of the PSW frequency and the Larmor frequency, $\Gamma=4\pi^2/Vk_F$. Within the shaded areas is the region of Landau damping.

Corresponding to the maximum in q, there is a maximum in $\tilde{\omega}$. For lead this comes at $3.6\times10^{-3}\,E_F$, of the order of 50 meV. We expect a maximum in this cutoff frequency for the case of lead as compared to other elemental superconductors. The corresponding cutoff comes at about 200 μV in aluminum. The terminating frequency for lead falls in the far infrared; that for aluminum is a high microwave frequency.

We shall see later that the maximum q is a sensitive function of the paramagnetic polarization of the gas. This offers the possibility of extending the frequency range somewhat. The extension is more pronounced for systems with weaker interactions. A field of 10 kG will increase the maximum for aluminum by about 50%.

Figure 4 shows two spectra with V_0 positive. As V is decreased, the curve shrinks down and disappears into the origin at $V_0=4\pi^2/k_F$. This figure represents an artificial situation, for, as we remarked, the paramagnetic state is unstable for these interaction strengths.

III. MAGNETIC EFFECTS

We have already seen one effect of applying an external magnetic field. The frequency of the modes, $\tilde{\omega}=2\omega$, is shifted by $2e$, the Larmor precession fre-

[6] J. Bardeen, L. Cooper, and R. Schriefer, Phys. Rev. **108**, 1175 (1957).

[7] Therefore, Eq. (15a), a consequence of a weak-coupling approximation, can only offer crude approximation for lead—this is sufficient.

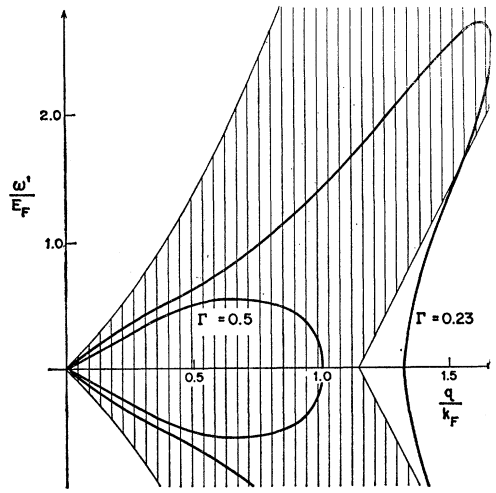

FIG. 4. PSW spectrum for the case of repulsive interactions.

quency. This, however, is the result of an earlier specification. In Eq. (4), a form was assumed for the effective one-electron potential in which the spiraling polarization of the PSW was normal to the z direction as defined by the external field. Two such PSW's are possible, a left-handed and a right-handed spiral. A third possibility, a mode with the polarization in the z direction, does not lead to an exactly soluble self-consistent problem as did the first two. It must be treated in perturbation theory. The result is structurally the same for the parallel as for the perpendicular disturbance except that the former is not subject to the Larmor precession; the frequency is unshifted by the field.

A second effect of the application of an external field is a Pauli paramagnetic polarization of the gas, a preferential occupation of one spin-band relative to the other. The modifications of the PSW spectrum introduced by this effect are the same for both perpendicular and parallel modes.

The fractional polarization of the gas will be of the order of the spin-up–spin-down energy differences compared to the Fermi energy, a number in the neighborhood of 10^{-6} for readily available fields. Thus we may expect that the gross features of the dispersion curves are unaffected by the polarization and that the external field may be treated as a perturbation. However, the modes with energies comparable to or less than the spin splitting may be profoundly altered, and in the long-wavelength limit, the dispersion curves may be quite different from those we have discussed up to now.

We treat (12) in the long-wavelength limit by expanding in powers of q. In so doing, it is necessary to assume a form for the expansion of $\omega(q)$. We assume

$$\omega'=\sum_{j=0}^{\infty} a_j q^{j+m}$$

and try to determine the a_j. This procedure succeeds in two cases, $m=0$ and $m=2$, for which we find

$$\omega_1'=(\Gamma_0-1)4(E_F/e)q^2,$$

$$\omega_2'=\frac{e}{\Gamma_0}+(\Gamma_0-1)\frac{4}{3}\frac{E_F}{e}q^2;\quad \Gamma_0=\frac{4\pi^2}{V_0k_F}. \tag{16}$$

At $q=0$, the frequency of the first branch is just that of a single spin precessing in the effective field, e. We remarked in setting up the self-consistent field problem that this effective field includes both the external field and a correction due to the excess of electrons of one spin. From (11) this correction is clearly equal to

$$\frac{V_0\Delta N}{2\Omega}=\frac{V_0}{2\Omega}\frac{\Omega}{8\pi^3}4\pi k_F^2\Delta k_F=\frac{V_0k_F^2}{4\pi^2}\Delta k_F=\frac{e}{\Gamma_0}. \tag{17}$$

Hence the second branch at $q=0$ describes the precession of a magnetic moment in the uncorrected external field. The first branch thus describes an excitation of a single spin turned against the preferred direction of the field, where the second branch gives the frequency of the collective precession of the entire net spin of the gas in the field. The first branch is Landau-damped and the second is not.

In the absence of external fields, we found four branches at $q=0$, two for both positive and negative frequencies. Now the symmetry between plus and minus frequencies is gone, and the number of branches is reduced to 2.[8] In Fig. 5 we show the dispersion curves for q small enough that polarization effects are important, but with $\Delta k_F\ll k_F$.

Let us now consider the case of small external field and q not necessarily small. We define $\lambda=(k_{F\uparrow}-k_{F\downarrow})/k_F$ and expand ω' in powers of λ for fixed q. Equation (12) is of the form $F(\lambda,q,\omega')=0$, giving

$$\frac{d\omega'}{d\lambda}=-\frac{\partial F}{\partial\lambda}\Big/\frac{\partial F}{\partial\omega'}.$$

We differentiate.

$$\frac{d\omega'}{d\lambda}=\frac{-\ln|q_1|+\ln|q_2|}{(2q^2-\omega')\ln|q_1|-(2q^2+\omega')\ln|q_2|}(2qk_F)^2,$$

where (18)

$$q_1=\frac{2qk_F-2q^2+\omega'}{-2qk_F-2q^2+\omega'}\quad\text{and}\quad q_2=\frac{2qk_F-2q^2-\omega'}{-2qk_F-2q^2-\omega'}.$$

[8] These results are in accord with some earlier calculations, although differing in detail because of the different physical situations. D. C. Mattis [Phys. Rev. 132, 2521 (1963)] has found spin-wave excitations in a magnetized electron gas using a two-band model. A. D. Brailsford (unpublished) has considered the fully polarized single-band case. Both these authors find a spin-wave branch, corresponding to ω_2', and an optical branch corresponding to ω_1'. Similarly they find that the two branches are the opposite ends of a continuous curve. We are not in agreement with Silin (Ref. 2) who finds two branches corresponding to $\pm$ our ω_2'. Unfortunately, Silin's discussion is too scanty to allow us to do more than note the discrepancy.

Since we are considering λ small and expanding about $\lambda=0$,

$$\Delta\omega'=\frac{d\omega'}{d\lambda}\Delta\lambda=\frac{d\omega'}{d\lambda}\lambda=\frac{d\omega'}{d\lambda}\frac{\Delta k_F}{k_F}=-\frac{d\omega'}{d\lambda}\frac{e}{E_F}. \quad (19)$$

Equation (18) lacks transparency. We may examine it in the region $q\ll k_F$.

$$\frac{d\omega'}{d\lambda}=k_F{}^2 2\ln\left|\frac{(\omega'-2qk_F)}{(\omega'+2qk_F)}\right| \Big/ \left(\frac{4k_Fq\omega'}{(4q^2k_F{}^2-\omega'^2)}-\ln\left|\frac{(\omega'-2qk_F)}{(\omega'+2qk_F)}\right|\right). \quad (20)$$

The ln terms may be removed by use of (13) to obtain

$$\frac{d\omega'}{d\lambda}=-2k_F{}^2\frac{\Gamma_0-1}{\Gamma_0-1-\eta/(1-\eta^2)}$$

$$\cong-2(\Gamma_0-1)(\eta^2-1)k_F{}^2, \quad (21a)$$

and by use of (14) we obtain

$$\frac{d\omega'}{d\lambda}\cong\pm 8(\Gamma_0-1)e^{2(\Gamma_0-1)}E_F. \quad (21b)$$

Thus the lowest-order effect of the electron polarization on the frequency of a given mode is a q-independent shift proportional to the product of the Larmor frequency and the term $4(\Gamma_0-1)e^{2(\Gamma_0-1)}$. We have argued that reasonable values for Γ_0 are of the order -5. The region where this perturbation approach fails, where $\Delta\omega'$ becomes comparable to ω', thus occurs at low radio-frequencies for reasonable magnetic fields. In a field of 10 kG, the frequency shift is about 2 Mc/sec. This shift becomes of the same order of magnitude as the frequency itself for wavelengths of the order of a meter.

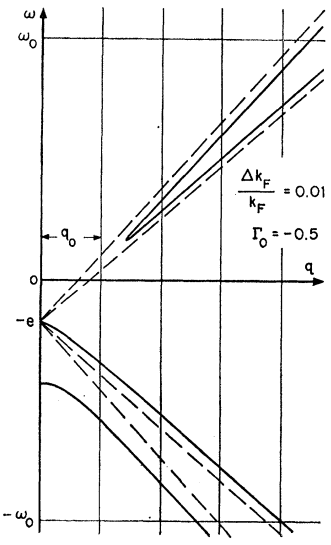

FIG. 5. Details of the influence of magnetic fields on the PSW spectrum. The solid lines are the spectrum of the polarized gas. The dashed lines are the spectrum of the unpolarized gas. The interaction strength and the polarization are exaggerated; the Larmor shift is understated.

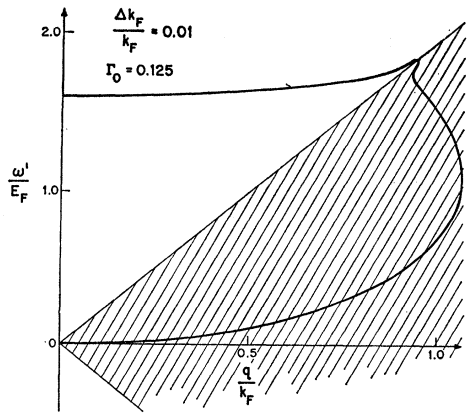

FIG. 6. PSW spectrum of the partially polarized gas with repulsive interactions. The interaction strength and the polarization are exaggerated.

Therefore, for most purposes the frequency shift is negligibly small, and the modifications in the details of the dispersion curves at small q may be ignored.

There is a subtler effect of the polarization which we must consider, however. In Fig. 3 we show two branches of the spectrum, and have indicated that there are reflected branches in other quadrants. The first effect of the magnetic field is to shift the spectrum by the Larmor frequency; the second is the change of the structure at low q as in Fig. 5. The dashed lines show the spectrum shifted by the Larmor frequency, but with the electron polarization neglected. The solid lines show the effect of a small dc polarization of the gas. If we look at (21) we see that the effect of the polarization is to shift the Landau-damped branches, $\eta^2<1$, one way, and the undamped branches the other. The two branches with $\omega'>0$ move together and the other two move apart. An estimate of value of q below which there are no modes $\omega'>0$ may be obtained from the magnitude of the shift and the difference in slope of the two branches. The shift, as we have just seen, is very small, but so is the slope difference. There are no modes for $0<\omega'<4|\Gamma_0-1|e$. Unfortunately, the modes wiped out by this effect were potentially the most interesting experimentally. As we shall discuss, observation of PSW's calls for unconventionally thick films, and the nature of the remaining modes requires still thicker films. Further, the number of observable modes is limited whereas without this "wiping out" effect, many more could have been studied.

As the polarization at fixed field is imagined increased, the intersection of the two upper branches of Fig. 5 moves out, while at the same time the far limit of the curve moves in and eventually there are no modes at all on those branches. The separation of the lower branches increases; the far limit correspondingly moves to higher q and $|\omega|$.

In Fig. 6 we show the gas spectrum for $V_0>0$. There is a point of intersection of $q=0$ at $\omega'=0$, $\omega=e$, for both

signs of V. If the magnetic field is entirely the exchange field of the electrons, then the second of the points for $V>0$ is at $\omega=0$, and that branch is the much studied spin-wave spectrum of the ferromagnetic electron gas.[9]

The magnetic field also affects the orbital motion of the electrons. In the event that propagation of the PSW is along the direction of the applied field, quantization of the orbits may be carried out and the problem of self-consistency solved exactly. For other directions of propagation the two problems become badly tangled. A simple approximation is possible, however.

To the kinetic-energy terms of the Hamiltonian $-\nabla_i^2$ we add the terms linear in the applied field. The effect of these additional terms in Eq. (2) may be included in the wave functions of (5) by taking time-dependent $\mathbf{q}$ and $\mathbf{k}$

$$\dot{q}_x=\frac{e\mathfrak{H}\mathcal{C}}{mc}q_y \quad \dot{q}_y=-\frac{e\mathfrak{H}\mathcal{C}}{mc}q_x$$

$$\dot{k}_x=\frac{e\mathfrak{H}\mathcal{C}}{mc}k_y; \quad \dot{k}_y=-\frac{e\mathfrak{H}\mathcal{C}}{mc}k_x$$

$$\dot{A}(\mathbf{k})=0$$

$$\dot{E}(\mathbf{k})=0. \tag{22}$$

Thus the lowest orbital effect of the field is a rotation of the entire electron gas about the field direction, PSW and all, at the cyclotron frequency. The wave fronts of the PSW rotate and in the laboratory frame have a velocity equal to the sum of the ordinary phase velocity and the tangential velocity of rotation about some fixed point. By examining the relative sizes of the orbital term included and the term in $3\mathcal{C}_{ex}^2(x^2+y^2)$ dropped, it is easy to see that the condition for the validity of the approximation is that the latter velocity shall be very much smaller than the former. In this approximation, the PSW propagating along the field is unaffected. The PSW propagating across the field travels along a circular arc.

IV. DAMPING

The self-consistent field approximation to the electron interaction has permitted a simple treatment of the dynamics of the spin density of an electron gas. Were this not an approximation, a PSW excitation would persist in time with unchanged amplitude. The neglected parts of the interactions cause collisions and rearrangements of the $\mathbf{k}$-state occupation distribution. The spin-wave amplitude changes in response to these rearrangements. In general, we expect to find a PSW amplitude decaying in time at a rate dependent on the circumstances of the system.

The PSW has little effect on the majority of the collisions. The electrons scatter, conserving wave vector, energy, and spin, as if there were no spin wave present. In an occasional, anomalous collision, however, the total $\mathbf{k}$ changes by $2\mathbf{q}$, the total single-particle energy changes by 2ω, and a spin is flipped. This behavior suggests absorption or emission of a magnon.

We have not quantized the spin-density coordinate. For the purpose of this discussion, however, we will assume that it can be quantized, and that it is meaningful so to regard it. We will use the anomalous collision rate as a measure of the magnon scattering rate and so calculate the damping as the net excess of absorption over emission scatterings per unit time. This procedure is more fully justified in Appendix B.

A number of different processes contribute to the damping. Landau has calculated the damping from two-particle collision processes using arguments about the relationship between the damping and the classical viscosity coefficient. We shall reconsider the question in order to show that our use of the anomalous scattering rate leads to the same results and to show in more detail how the scattering events lead to damping. We will also treat the question of the damping due to single electron scattering from magnetized and nonmagnetic scattering centers.

We begin with the simplest case. At the point $\mathbf{R}$ we assume a scattering center with which the electrons interact according to the potential $U(\mathbf{r}-\mathbf{R})$. Golden rule perturbation theory gives a rate for scattering from $\mathbf{k}$ to $\mathbf{k}'$ (assuming initially that $\mathbf{k}$ is occupied and $\mathbf{k}'$ is not) proportional to

$$\frac{1}{\Omega^2}|U(\mathbf{k}-\mathbf{k}')(A_1(\mathbf{k})A_1(\mathbf{k}')+A_2(\mathbf{k})A_2(\mathbf{k}'))|^2\delta(E_{\mathbf{k}}-E_{\mathbf{k}'}). \tag{23}$$

We note that $E(\mathbf{k})$ is conserved in this process. If the two states are both of the same approximate spin species, then this condition is equivalent to conservation of single-particle energy. But if the states are in different bands, then $\mathcal{E}_{\mathbf{k}\downarrow}$ exceeds $\mathcal{E}_{\mathbf{k}'\uparrow}$ by $\tilde{\omega}=2\omega$, and single-particle energy is not conserved. This, then, describes an anomalous event.

We have now two scattering rates: an ordinary rate proportional to

$$\frac{2\pi}{\hbar}\frac{U^2(\mathbf{k}-\mathbf{k}')}{\Omega^2}\delta(\mathcal{E}_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}'}), \tag{24}$$

and an anomalous rate proportional to

$$\frac{2\pi}{\hbar}\frac{U^2(\mathbf{k}-\mathbf{k}')}{\Omega^2}\left(\frac{\sin\phi}{2}+\frac{\sin\phi'}{2}\right)^2\delta(\mathcal{E}_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}'}-2\omega) \tag{25}$$

which we get by writing $(A_1A_1'+A_2A_2')^2$ as $\frac{1}{2}(1+\cos\phi\cos\phi'+\sin\phi\sin\phi')$ and $\cos\phi$ as

$$\pm(1-\sin^2\phi)^{1/2}\cong\pm\left(1-\frac{\sin^2\phi}{2}\right).$$

[9] For a recent instance, see A. K. Rajagopal, Phys. Rev. **142**, 152 (1966).

An electron may scatter from $\mathbf{k}$ to $\mathbf{k}'$ and absorb a magnon in the process. The reverse also occurs, and the relative rates depend upon the availability of appropriate final states. An electron on a sharply defined Fermi surface can not emit a magnon in collision, all the states of energy 2ω below the Fermi surface being filled. At low temperatures only the absorption process can occur. If the Fermi surface is sufficiently blurred by temperature, the counting of anomalous events must include both processes. A microwave frequency of 10^{10} cps corresponds to a temperature of 0.4°K. We indicated that the spectrum in lead cuts off at 50 meV or at about 10^{13} cps, so it is clear that both ranges are experimentally accessible.

We first carry out the damping calculation for modes which are not Landau damped and consider Landau damping separately later.

We multiply the single event probability of (25) by the appropriate occupation probabilities, subtract the emissions from the absorptions, and sum over all possible events. For simplicity, we take $U(\mathbf{k},\mathbf{k}')$ to be constant and obtain

$$\frac{dn}{dt}=2\pi\frac{U^2}{4\Omega^2}\left(\frac{\Omega}{8\pi^3}\right)^2$$
$$\times\int_0^\infty\{f(\mathcal{E}_{\mathbf{k}\downarrow})[1-f(\mathcal{E}_{\mathbf{k}'\uparrow})]-f(\mathcal{E}_{\mathbf{k}'\uparrow})[1-f(\mathcal{E}_{\mathbf{k}\downarrow})]\}$$
$$\times k^2k'^2\delta(\mathcal{E}_{\mathbf{k}\downarrow}-\mathcal{E}_{\mathbf{k}'\uparrow}-2\omega)dkdk'$$
$$\times\left\{\int 2\sin^2\phi dSdS'-2\int\sin\phi\sin\phi' dSdS'\right\}. \quad (26)$$

While the necessary integrations can be performed without approximation, much more transparent expressions result if we consider the small q limit. The integrations are easily performed, giving

$$\frac{dn}{dt}=-\frac{U^2}{8\pi^3}\frac{2\omega}{8}\frac{c^2}{4q^2}\left\{\frac{4}{\eta^2-1}-\ln^2\frac{\eta-1}{\eta+1}\right\}. \quad (27)$$

To find the damping rate, we need to compare this energy dissipation rate with the total energy of excitation. This energy may be found by evaluating the expectation of the Hamiltonian as a function of the amplitude, c.

$$\langle H\rangle=\sum_{\mathbf{k}_{occ}}z^2+\rho^2+q^2-2qz\cos\phi-\tfrac{1}{2}\sum_{\mathbf{k}_{occ}}\sum_{\mathbf{k}_{occ}'}\frac{V(\mathbf{k}-\mathbf{k}')}{2\Omega}$$
$$\times(1+\cos\phi\cos\phi'+\sin\phi\sin\phi'), \quad (28)$$

where $\mathbf{k}^2=z^2+\rho^2$. The limits of the summation are determined by the Fermi surface. ρ is a radial cylindrical coordinate where the polar axis is determined by the direction of $\mathbf{q}$. It is interesting that this expectation is time-independent.[10]

[10] P. A. M. Dirac, Proc. Camb. Phil. Soc. **26**, 376 (1930).

Equation (28) is interpretable as a single sum of contributions from each occupied state, where

$$\mathfrak{E}_{\mathbf{k}}=z^2+\rho^2+q^2-2qz\cos\phi$$
$$-\tfrac{1}{2}\sum_{\mathbf{k}_{occ}'}\frac{V(\mathbf{k},\mathbf{k}')}{2\Omega}[1+\cos\phi\cos\phi'+\sin\phi\sin\phi']$$
$$=E_{\mathbf{k}}+\frac{c(\mathbf{k})}{2}\sin\phi-\frac{e(\mathbf{k})}{2}\cos\phi-\omega\cos\phi \quad (29)$$

$\mathfrak{E}_{\mathbf{k}}$ summed over all particles gives the total system energy.

$\langle H\rangle$ may change with the amplitude through the change in the energy per particle and through the change in the region of occupation. For our derivative we have

$$\frac{d}{dc}\int\mathfrak{E}_{\mathbf{k}}d\mathbf{k}=\mathfrak{E}_0\int\pi\frac{d\rho^2}{dc}dz+\int\frac{d\mathfrak{E}_{\mathbf{k}}}{dc}d\mathbf{k}. \quad (30)$$

$\mathfrak{E}_0$ is the value of $\mathfrak{E}_{\mathbf{k}}$ at zero amplitude at the Fermi surface. The first integral in (30) is the derivative of the total number of electrons with respect to c and is 0. Thus we need only calculate the change in system energy with fixed occupation.

The differentiation of the functions in (29) gives

$$\int\frac{d\mathfrak{E}_{\mathbf{k}}}{dc}d\mathbf{k}=\int\left(-\omega-\sum_{\mathbf{k}'}\frac{V(\mathbf{k},\mathbf{k}')}{2\omega}\cos\phi'\right)\frac{d\cos\phi}{dc}$$
$$-\int\frac{\sin^3\phi}{2}d\mathbf{k} \quad (31)$$

and

$$\frac{d}{dc}\cos\phi=-\frac{\sin^2\phi\cos\phi}{c}. \quad (32)$$

When the excitation is not Landau damped, all the occupied states are far removed from the gap, and $\sin\phi$ is proportional to c. The first term of (31) is therefore proportional to c and dominates the second term, proportional to c^3. The total energy of excitation is thus proportional to Nc^2 which makes c proportional to $N^{-1/2}$, as we mentioned earlier.

When the mode is Landau damped, the states near the gap have $\sin\phi$ of order 1, and, as we shall see in Sec. V, the energy becomes proportional to Nc, making c proportional to N^{-1}. The first term of (31) still dominates the second by order c^{-1}. We can rewrite the first integrand in (31) as

$$\left(-\omega-\tfrac{1}{2}\sum_{\mathbf{k}\uparrow'}[V(\mathbf{k},\mathbf{k}')-V(\mathbf{k},\mathbf{k}'+2\mathbf{q})]\right)\frac{d\cos\phi(\mathbf{k}_\uparrow)}{dc}$$
$$+\left(-\omega-\tfrac{1}{2}\sum_{\mathbf{k}\downarrow'}[V(\mathbf{k},\mathbf{k}')-V(\mathbf{k},\mathbf{k}'-2\mathbf{q})]\right)$$
$$\times\frac{d\cos\phi(\mathbf{k}_\downarrow)}{dc}. \quad (33)$$

We have confined ourselves to the case of constant $V(\mathbf{k},\mathbf{k}')$, leaving only $\omega(d\cos\phi_k/dc)$ to be summed. Thus we have

$$\Delta\mathcal{E}=-\frac{c}{2}\frac{d\mathcal{E}}{dc}=(2\omega)(n+1/2)=\frac{\Omega c}{16\pi^3}\int\frac{d}{dc}\omega\cos\phi d\mathbf{k}$$

$$=\frac{\Omega c^2\omega}{16\pi^2 q}\left\{\frac{2\eta}{\eta^2-1}+\ln\frac{\eta-1}{\eta+1}\right\}. \quad (34)$$

The values of η which we have found to characterize PSW's allow us to neglect the logarithm terms in (27) and (34). We find, then, a fractional linewidth

$$\frac{d(n+\frac{1}{2})}{dt}\frac{1}{(\eta+1/2)(2\omega)}=-\frac{1}{(2\omega)\tau}=-\frac{U^2}{2\pi q\Omega}$$

$$=-\frac{U^2k_F}{2\pi\Omega h(2\omega)}\frac{2m}{\hbar^3}. \quad (35)$$

We have included finally the terms of unit magnitude to permit conversion to standard units. The factor of (system volume)$^{-1}$ occurs because we have as yet only considered a single scattering site. To find the total damping rate, we multiply (35) by the total number of impurity sites, which replaces Ω^{-1} by the impurity concentration.

The ordinary scattering produces a resistivity which is also proportional to the U^2 of Eq. (35). It is a straightforward matter to find this resistivity.[11] If, for instance, we calculate the time rate of loss of momentum of an electron distribution perturbed by an electric field and equate this to the time rate of momentum gain from the field, we are led to a conductivity, and thereby a conduction relaxation time.

$$\sigma=\frac{Ne^2}{m}\frac{\Omega 2\pi}{U^2k_F}=\frac{Ne^2\tau_{\text{cond}}}{m}. \quad (36)$$

Thus (35) may be written

$$\frac{1}{\omega\tau_{\text{PSW}}}=\frac{1}{\omega\tau_{\text{cond}}}. \quad (37)$$

That the proper relaxation time to use in studying PSW's is that of electron transport is an interesting result. While we would not be surprised at such a result for spin-independent zero sound, which we remarked earlier is a current density wave, we might have expected to see a characteristic spin relaxation time here. It suggests that basically the spin density of the spin wave results from an influx of (say) spin-up electrons in a region and an outgo of spin-down electrons. We shall see this suggested again in the discussion of the boundary conditions in the final section.

Let us now suppose that the impurity sites can flip spins; the electrons interact according to the potential $J(\mathbf{r}-\mathbf{R})\boldsymbol{\sigma}\cdot\mathbf{S}_R$. $\mathbf{S}_R$ is the angular momentum operator of the impurity on site $\mathbf{R}$. We shall assume that the different spin levels of the impurities are degenerate.

Again we calculate a matrix element, and a corresponding scattering rate.

$$\frac{2\pi J^2}{\Omega^2}\{|A_1(\mathbf{k})A_2(\mathbf{k}')|^2(\langle M+1|S^+|M\rangle)^2$$

$$+|A_2(\mathbf{k})A_1(\mathbf{k}')|^2\langle M-1|S^-|M\rangle^2\}$$

$$\times\delta(E_k-E_{k'}-2\omega). \quad (38)$$

In this case, the majority of scattering events flip electron spins and do not change the magnon number. The anomalous events are those in which $\mathbf{k}$ and $\mathbf{k}'$ refer to the same Fermi sphere. For these spin type interactions, $E_k-E_{k'}-2\omega$ is zero, in contrast to the result for nonmagnetic potential scattering above where $E_k-E_{k'}$ had to vanish.

We average the transition rate of (38) over impurity spin configurations and obtain a net electron scattering rate.

$$2\pi(J^2/\Omega^2)(S^2+S-\langle S_z^2\rangle_{\text{av}})\{|A_1(\mathbf{k})A_2(\mathbf{k}')|^2$$

$$+|A_2(\mathbf{k})A_1(\mathbf{k}')|^2\}\delta(E_k-E_{k'}-2\omega). \quad (39)$$

The anomalous parts give a magnon transition rate

$$2\pi\frac{J^2}{\Omega^2}(S^2+S-\langle S^2\rangle_{\text{av}})\left\{\frac{\sin^2\phi}{4}+\frac{\sin^2\phi'}{4}\right\}$$

$$\times\delta(\mathcal{E}_k-\mathcal{E}_{k'}-2\omega) \quad (40)$$

obtained in the same way as (25).

Now we saw in evaluating the integrals in the nonmagnetic case of (25) that the terms in $\sin\phi\sin\phi'$ leading to the logarithmic term in (27) could be neglected. Therefore, we find that the ratio of the anomalous to the ordinary scattering matrix elements is the same for both the magnetic and nonmagnetic types of scattering event. Equation (37) thus remains valid when the possibility of magnetic scattering is taken into account, although more complicated corrections would be necessary if the spin levels of the impurity sites were significantly split. In this case, the anomalous scattering from the second term of (39) would be diminished as the relative probabilities of the impurity states changed, and ultimately the ratio of anomalous to ordinary scattering would be reduced by one-half for the magnetic as compared to the nonmagnetic cases.

Let us consider now the collisions of the electrons among themselves. We evaluate the matrix element for scattering two electrons from states $\mathbf{k}_1$ and $\mathbf{k}_2$ to $\mathbf{k}_3$

[11] N. F. Mott and H. Jones, *The Theory of the Properties of Metals and Alloys* (Clarendon Press, Oxford, England, 1936), pp. 247 ff.

and $\mathbf{k}_4$.

$$\langle \mathbf{k}_3,\mathbf{k}_4|\, V(\mathbf{r}-\mathbf{r}')\,|\mathbf{k}_1,\mathbf{k}_2\rangle=\frac{1}{\Omega^2}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)V$$
$$\times[(A_1^{(1)}A_1^{(4)}+A_2^{(1)}A_2^{(4)})(A_1^{(2)}A_1^{(3)}+A_2^{(2)}A_2^{(3)})$$
$$-(A_1^{(1)}A_1^{(3)}+A_2^{(1)}A_2^{(3)})(A_1^{(2)}A_1^{(4)}+A_2^{(2)}A_2^{(4)})]. \quad (41)$$

As we found in impurity scattering there are ordinary and anomalous events, the latter characterized by spin and energy nonconservation. Both the direct and the exchange scattering, the first and second terms in (41), respectively, contribute to the magnon damping. If we begin with $\mathbf{k}_1$ and $\mathbf{k}_2$ on the same Fermi sphere, say both "up," the final state for an ordinary event will have $\mathbf{k}_3$ and $\mathbf{k}_4$ again both on the same sphere. The anomalous event is recognizable by having one final-state electron on the "down" Fermi sphere. Similarly if we begin with $\mathbf{k}_1$ and $\mathbf{k}_2$ on different spheres, the anomalous event will end with both on the same sphere. The total $\mathbf{k}$ is conserved in (41) but because the spheres are shifted in origin by $2\mathbf{q}$ with respect to one another, an anomalous event, ending with one electron on the "wrong" sphere, will fail by $2\mathbf{q}$ to conserve momentum.

We carry out the multiplications of (41) and after cancellation find

$$V\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\{A_1^{(1)}A_2^{(2)}A_2^{(3)}A_1^{(4)}$$
$$+A_1^{(2)}A_2^{(1)}A_2^{(4)}A_1^{(3)}-A_1^{(1)}A_2^{(2)}A_2^{(4)}A_1^{(3)}$$
$$-A_1^{(2)}A_2^{(1)}A_2^{(3)}A_1^{(4)}\}(1/\Omega^2). \quad (42)$$

With $\mathbf{k}_1$ and $\mathbf{k}_2$ "up" for the bracketed term we have

$$(A_2^{(2)}-A_2^{(1)})=\frac{\sin\phi_2}{2}-\frac{\sin\phi_1}{2} \quad (43)$$

if $\mathbf{k}_3$ is on the "down" sphere, and

$$(A_2^{(1)}-A_2^{(2)})=\frac{\sin\phi_1}{2}-\frac{\sin\phi_2}{2} \quad (44)$$

if $\mathbf{k}_4$ is "down." If $\mathbf{k}_1$ is "up" and $\mathbf{k}_2$ is "down," we have

$$(A_2^{(3)}-A_2^{(4)})=\frac{\sin\phi_3}{2}-\frac{\sin\phi_4}{2} \quad (45)$$

for $\mathbf{k}_3$ and $\mathbf{k}_4$ "up," and

$$(A_1^{(4)}-A_1^{(3)})=\frac{\sin\phi_4}{2}-\frac{\sin\phi_3}{2} \quad (46)$$

for $\mathbf{k}_3$ and $\mathbf{k}_4$ "down." There are similar expressions for $\mathbf{k}_1$ and $\mathbf{k}_2$ both "down" and for $\mathbf{k}_1$ "down" and $\mathbf{k}_2$ "up." A doubling of the final damping rate will include the effect of these collisions. The bracketed terms are plus or minus 1 for both initial and final pairs of states on opposite spheres.

In the case of nonmagnetic impurity scattering, for each ordinary scattering matrix element from $\mathbf{k}$ to $\mathbf{k}'$ ignoring spin, there was one anomalous matrix element with relative weight $(\sin\phi-\sin\phi')$. Here, by contrast, we find for each ordinary event from $\mathbf{k}_1$, $\mathbf{k}_2$ to $\mathbf{k}_3$, $\mathbf{k}_4$ ignoring spin, there are two anomalous events, weighted by $(\sin\phi_1-\sin\phi_2)$ or $(\sin\phi_3-\sin\phi_4)$. In the range $\hbar\omega\ll\kappa T$ if we could sum independently over states, we could immediately write down the PSW damping rate as the average scattering rate of electrons within κT of $\mathcal{E}_F$ just by comparison with the nonmagnetic impurity result.

In the range $\hbar\omega\gtrsim\kappa T$, the difference between the elastic ordinary events and the inelastic anomalous events becomes important. The structural difference between one and two particle scattering also complicates the relation between the two damping processes, and the sums over states are not independent but are correlated by the energy and momentum conservation relations. The result of this is to reduce the magnon damping, making τ_{PSW} much greater than the electron lifetime.

Quoting from the Appendix, we find

$$\frac{1}{\omega\tau_{\mathrm{PSW}}}=\frac{1}{\omega\tau_{\kappa T}}(\eta^2-1)\frac{2.248}{3\pi^2}\left(\pi^2+\frac{\mu^2}{4}\right),$$

where $\tau_{\kappa T}$ is the lifetime of an electron removed κT in energy from $\mathcal{E}_F$ and $\mu=\hbar(2\omega)/\kappa T$. At helium temperatures, $\tau_{\kappa T}$ is of order 10^{-8} sec, giving τ_{PSW} about 10^{-6} sec. Since τ_{cond} can be made to exceed 10^{-9} sec or so only by very careful sample preparation, it is clear that the only significant damping process will be single electron scattering from crystal imperfections.

V. LANDAU-DAMPED MODES

The problem of the loss of energy from an oscillation with wavelength and frequency appropriate for exciting single particles presents an interesting contrast with the case where such excitation is not possible. In the following discussion, as in the foregoing, we take the point of view that the properties of the gas may first be calculated without reference to collisions and that the effects of the collisions may then be studied perturbatively. That is, we study the lifetime of free oscillations without considering how they were originally excited. Experimental conditions, however, involve simultaneous excitation and damping, forced oscillations. This difference is significant for the Landau damped modes. It concerns the functional relationship between the amplitude of the driven waves and their energy content and energy loss rate, which relationship we shall presently discuss.

The necessary condition for the circumstance we study is that the external driving field be of sufficient amplitude to establish the "no collisions" steady state in spite of collision-induced dissipation. The limit of weak collisions is, therefore, to be understood in refer-

ence to the driving field amplitude, and it is necessary to emphasize that we are not here considering the usual forced oscillations, zero amplitude limit.

In our discussion of the non-Landau-damped situation in the preceding section, we found that the scattering of electrons could be divided into two distinct classes: the ordinary collisions, and the anomalous collisions in which the single electron energies were not conserved. This division still holds for the Landau damped case for those events in which neither the initial nor the final state lies near the gap. In this latter case, however, the division fails. The quantity conserved in collision, $E_{\mathbf{k}}$, agrees with the single-particle energy $\mathcal{E}_{\mathbf{k}}$ for the gap states, and differs from it by $\pm\omega$ for the states away from the gap. This means that every event in which an electron scatters from the gap region or into it is anomalous, and since the gap states themselves are a small minority of the totality of occupied states, virtually every scattering of a gap state electron or hole is anomalous. This is not surprising, for it is the gap states which are tightly coupled to the PSW, interacting strongly with it.

To find the damping rate, we proceed as in the non-Landau-damped case by calculating the energy loss rate in anomalous collisions and dividing by the energy of the excitation. The principal difference is that in the non-Landau-damped case all the anomalous collisions had the same energy nonconservation factor, while here the factor, $E_{\mathbf{k}}-\mathcal{E}_{\mathbf{k}}-(E_{\mathbf{k}'}-\mathcal{E}_{\mathbf{k}'})=\omega(\cos\phi_{\mathbf{k}}-\cos\phi_{\mathbf{k}'})$, is a strong function of nearness to the gap.

The matrix element for scattering from $\mathbf{k}$ to $\mathbf{k}'$ is as before

$$(U/\Omega)(A_1(\mathbf{k})A_1(\mathbf{k}')+A_2(\mathbf{k})A_2(\mathbf{k}')). \tag{47}$$

Taking $\mathbf{k}'$ as a state far from the gap, A_1' is 1 or 0 according as the state is on the "up" or "down" spin sphere. We can neglect the small terms of order c which were crucial in the previous case. Thus for scattering from a gap state to a spin-up state we have a rate

$$\frac{U^2}{\Omega^2}\pi A_1^2=\frac{U^2}{\Omega^2}\pi\frac{1}{2}(1+\cos\phi), \tag{48}$$

and to a spin-down state

$$\frac{U^2}{\Omega^2}\pi A_2^2=\frac{U^2}{\Omega^2}\pi\frac{1}{2}(1-\cos\phi). \tag{49}$$

The energy nonconservation is $\omega(1-\cos\phi)$ for the first case and $-\omega(1+\cos\phi)$ for the second. Thus the total energy loss rate for scattering out of gap states is

$$\left(\frac{\Omega}{8\pi^3}\right)^2\int d\mathbf{k}d\mathbf{k}'\{\tfrac{1}{2}\omega(1-\cos^2\phi)(\pi U^2/\Omega^2)f(\mathcal{E})\times[1-f(\mathcal{E}')]\delta(\mathcal{E}'-\mathcal{E}-\omega[1-\cos\phi])-\tfrac{1}{2}\omega(1-\cos^2\phi)(\pi U^2/\Omega^2)f(\mathcal{E})\times[1-f(\mathcal{E}')]\delta(\mathcal{E}'-\mathcal{E}+\omega[1+\cos\phi])\}. \tag{50}$$

The integrations in (50) may be carried out by writing $\int d\mathbf{k}$ as

$$\int dS(\mathcal{E})\frac{d\mathcal{E}}{|\nabla_{\mathbf{k}}\mathcal{E}|}=\int\pi dz d\mathcal{E}_{\mathbf{k}}, \tag{51}$$

where $dS(\mathcal{E})$ is an element of surface area on the surface of constant $\mathcal{E}$. Equation (51) may be easily verified for any $\mathcal{E}_{\mathbf{k}}$ which can be written in the form $\mathcal{E}_{\mathbf{k}}=\mathbf{k}^2+g(\mathbf{k}\cdot\mathbf{q})$. The energy integration gives a factor

$$\Delta\mathcal{E}/(1-e^{\Delta\mathcal{E}/\kappa T}).$$

If we combine the energy loss rate due to scattering out of gap states with the energy loss due to scattering in, we are left with

$$\left(\frac{1}{8\pi^3}\right)^2 2\pi^3 k_F\frac{U^2\omega}{2}\int\Delta\mathcal{E}(1-\cos^2\phi)dz=\frac{1}{64\pi^3}U^2\omega k_F\int(1-\cos\phi)\sin^2\phi dz \tag{52}$$

for scattering from the gap to spin up and

$$\frac{1}{64\pi^3}U^2\omega k_F\int(1+\cos\phi)\sin^2\phi dz$$

for scattering to spin down. The integrations over z are easily performed. We find a total collective mode energy loss rate of

$$d\mathcal{E}/dt=-U^2\omega c k_F^2/32\pi^2. \tag{53}$$

The energy of the excitation is again calculated from (31).

$$\mathcal{E}=\frac{c}{2}\frac{\Omega}{8\pi^3}\int\frac{d\mathfrak{E}_{\mathbf{k}}}{dc}d\mathbf{k}=\frac{\Omega}{8\pi^3}\int[\tfrac{1}{2}\omega\sin^2\phi\cos\phi+\tfrac{1}{2}c\sin^3\phi]\pi\rho^2 dz. \tag{54}$$

Because $\cos\phi$ and $\sin\phi$ have the opposite sign in the up and down bands, we can write (54) as

$$\frac{\Omega}{8\pi^3}\int(\tfrac{1}{2}\omega\sin^2\phi\cos\phi+\tfrac{1}{2}c\sin^3\phi)\pi(\rho_\downarrow^2-\rho_\uparrow^2)dz. \tag{55}$$

Now, $\mathcal{E}_F=z^2+\rho^2+q^2-c\csc\phi-\omega\cos\phi$ from Eq. (10). Thus we have

$$\rho_\downarrow^2-\rho_\uparrow^2=2(c\csc\phi+\omega\cos\phi). \tag{56}$$

The integrations are all elementary.

$$\mathcal{E}=(\Omega/8\pi^2 q)\times\{\tfrac{1}{4}\omega^2 c\pi-\omega c^2(\ln c-\ln 2qz_m)+\tfrac{1}{4}\omega c^2+\tfrac{1}{4}c^3\pi\}. \tag{57}$$

Thus, finally

$$1/\tau_{\mathrm{PSW}}=-(d\mathcal{E}/dt)/\mathcal{E}=U^2k_F/2\pi\Omega$$

which is exactly the same result as for the non-Landau-damped case.

Landau damping occurs when there are electrons with particle velocities which match the collective mode phase velocity. These electrons, in resonance with the collective mode, can absorb energy from it. By the same token, they can also relinquish energy to the mode. Assuming no external influences, the mode and the resonating electron continually exchange energy. This is the point of view we have taken throughout this paper, and it is reflected in the form of the single-electron wave functions. When external influences, such as impurity centers or phonons or lattice defects scatter the resonating electron, the coherence between the electron and the collective mode is disrupted. Since such scattering is more likely with a more energetic electron, this process results in a loss of energy from the collective mode. If the electron-scattering time is short, the mode can hardly be said to exist at all.

In fact, the polarized condition which is the starting point for the above calculation cannot be established. If the system is driven by an external field, the absorbed power must be proportional to the strength of the external field and the strength of the response, which is again proportional to the driving field; the system absorbs energy proportional to c^2. But we have just seen above that in the limit of infinitely weak collisions, it will dissipate energy at a rate proportional to c. These two conditions are clearly incompatible in the limit of small c. There must, therefore, exist, some minimum value of c dependent on the strength of the scattering below which the results of Eq. (53) are inappropriate.

Equation (53) was obtained by considering the expectation of the system Hamiltonian before and after a scattering event, then using the difference in this expectation as a measure of the energy dissipated. This procedure is more extensively discussed in Appendix B. There we find that in addition to this fixed amount of energy dissipated per particle scattered, there are a number of transients associated with the scattering process reflecting the manner in which a disturbed particle exchanges energy with the field. These transients oscillate and die away at a rate which depends on the strength of the field. The characteristic distinction between the Landau damped and the non-Landau damped modes is the appearance of transients with long periods associated with the states on the gaps. The weak scattering limit for which we have calculated the damping in (53) allows these transients time to decay completely.

While the approach used in Appendix B could be carried out for short scattering lifetimes carrying along the effects of the transients, it is much simpler to perform a more conventional calculation, obtain the usual Landau damping and observe that the "resonances" associated with these modes are completely damped out.

What we have demonstrated above is the possibility that with sufficiently long scattering lifetimes or sufficiently high driving fields, it may be possible to observe resonances in the single particle continuum. The necessary condition is that the product $2c\tau>1$. For a lifetime of 10^{-8} sec, this requires a total (external plus response) rf magnetic field strength in excess of 50 G.

VI. DETECTION

Ordinary spin waves in magnetized materials with localized moments may be "seen" by exciting geometrical resonances in thin films. The existence of thin-film resonances depends on the nature of the boundary conditions to which the modes are subject.

The interactions of the electrons conserve angular momentum. The Hamiltonian we have been using, Eq. (1), conserves spin. This allows us to write an equation of continuity.

$$\dot{M}=\mu\nabla\cdot(\mathbf{j}\uparrow-\mathbf{j}\downarrow), \tag{58}$$

where M is the spin density in the (say) x direction, and $\mathbf{j}\uparrow$ is the particle-current density of electrons polarized "up" with respect to x. In second quantized form

$$\begin{aligned} M&=(\psi^+(\mathbf{r},\uparrow)\psi(\mathbf{r},\uparrow)-\psi^+(\mathbf{r},\downarrow)\psi(\mathbf{r},\downarrow))\mu,\\ j&=\psi^+(\mathbf{r},\uparrow)\nabla\psi(\mathbf{r},\uparrow)-[\nabla\psi^+(\mathbf{r},\downarrow)]\psi(\mathbf{r},\downarrow)\\ &\quad+\psi^+(\mathbf{r},\downarrow)\nabla\psi(\mathbf{r},\downarrow)-[\nabla\psi^+(\mathbf{r},\uparrow)]\psi(\mathbf{r},\uparrow), \end{aligned} \tag{59}$$

from which Eq. (58) may be proved by writing the Hamiltonian in second quantized form and evaluating the commutator $[M,H]$.

We assume that the reflecting of the electrons by the surface occurs without spin flip, which immediately implies that the normal component of $\mathbf{j}$ at the surface must vanish. This is our boundary condition. We use (58) to express it as a condition on the magnetization. In Fourier-analyzed form, with excitations propagating normally to the surface, we have

$$\sum_q\frac{\omega_q M_q}{q}=\mu\sum_q j_q=0. \tag{60}$$

For a single Fourier component, Eq. (58) states that the spin current $\mathbf{j}$ and the spin density M are in quadrature in space. Since the surface must be a node of $\mathbf{j}$, it is an antinode of M.

The polarization due to a single, spiral PSW is

$$\mathbf{P}=P_0[\hat{x}\cos(2\mathbf{q}\cdot\mathbf{r}-2\omega t)+\hat{y}\sin(2\mathbf{q}\cdot\mathbf{r}-2\omega t)]. \tag{61}$$

If we combine this with the spiral of opposite handedness, the polarization becomes

$$\mathbf{P}=2P_0[\hat{x}\cos2\omega t+\hat{y}\sin2\omega t]\cos2\mathbf{q}\cdot\mathbf{r}. \tag{62}$$

When $\mathbf{q}$ is in the z direction, the boundary condition is satisfied at $z=n\pi/q$.

$$q=+\left\{\frac{\omega+e}{\eta\hbar k_F}m\right\};\quad \eta\cong1.$$

We notice also that the boundary condition is satisfied if $q=0$.

An obvious experiment is one modeled after a conventional thin-film–spin-wave resonance measurement. Radiation of fixed frequency is incident normally on a film with a magnetic field applied perpendicular to the film plane. As the field is changed, different standing-wave modes are brought into and out of resonance with the applied radiation. Viewed by reflection, the film becomes absorptive at resonance.

In Fig. 5 we have overdrawn lines to indicate the circumstances of this experiment. A fixed frequency ω_0 is defined by the apparatus; a lowest q, q_0, that for which one half wave is contained in the film, and is defined by the thickness. As the field is increased, the various branches of the spectrum move in the manner discussed in Sec. III. As the branches cross points of intersection of ω_0 and nq_0, there are resonances.

Because the point of disappearance of the upper states in Fig. 5 moves away so rapidly with field, the frequency must be within a few percent of resonance at zero field in order that any of these modes may be observed. As many lower state resonances can be observed as there are points of intersection contained within the triangle $q=0$, $\omega=\omega_0$, and $\omega=qv_F\eta$.

The Fermi velocity in Pb has been measured directly and found to be 7.5×10^7 cm/sec. At a frequency of 10^{10} Hz, this means there will be no resonances for samples less than 38-μ thick, one resonance for 38- to 75-μ samples, two resonances for 75- to 113-μ samples, etc.

At this frequency, the penetration depth is a few percent of a wavelength. Hence by transmission, the samples should become anomalously transparent at resonance.

We must finally check to see that the effects of the spin waves on the properties of a specimen will be large enough to observe. Let us consider driving the electron system with an external field. A new term appears in the one-electron Hamiltonian Eq. (3).

$$\mu\mathcal{H}_{\rm rf}[\hat{x}\cos(2\mathbf{q}\cdot\mathbf{r}-2\omega t)+\hat{y}\sin(2\mathbf{q}\cdot\mathbf{r}-2\omega t)]\cdot\boldsymbol{\sigma} = c_{\rm ext}[\sigma^+e^{-2i(\mathbf{q}\cdot\mathbf{r}-\omega t)}+\text{c.c.}]. \quad (63)$$

We can combine the term in (63) with the self-consistent field to obtain

$$c_{\rm tot}=c_{\rm res}+c_{\rm ext}=c_{\rm res}+\mu\mathcal{H}_{\rm rf}. \quad (64)$$

Now in the integral equation for $c(\mathbf{k})$, our dispersion equation (12), $c(\mathbf{k}')$ appearing under the summation comes from the response of the one-electron wave functions to the total spiraling field and is, therefore, $c_{\rm tot}$ of (64). However, $c(\mathbf{k})$ on the left-hand side is the field seen by one electron because of the polarization of all the others and does not include the external field. Hence we have

$$c_{\rm res}=\sum_{k\circlearrowright}\frac{V}{2\Omega}\frac{c_{\rm tot}}{(2\mathbf{k}\cdot\mathbf{q}-\omega')} =c_{\rm ext}\left[\left(\frac{V}{16\pi^3}\int\frac{1}{2\mathbf{q}\cdot\mathbf{k}-\omega'}d\mathbf{k}\right)^{-1}-1\right]^{-1}. \quad (65)$$

We can find the magnetization density of the gas by summing the fractional polarization over the occupied states, multiplying by the electron magnetic moment, and dividing by the system volume. After some rearrangement of terms, we find

$$\mathfrak{M}=\mu c_{\rm ext}\frac{1}{16\pi^3}\int\frac{d\mathbf{k}}{2\mathbf{k}\cdot\mathbf{q}-\omega'}\Big/\left(1-\frac{V}{16\pi^3}\int\frac{d\mathbf{k}}{2\mathbf{k}\cdot\mathbf{q}-\omega'}\right) =\mu c_{\rm ext}\frac{(S_1/V)}{2\Gamma_0-S_1}, \quad (66)$$

where as before

$$\Gamma_0=4\pi^2/Vk_F;\quad S_1=2+\eta\ln(\eta-1)/(\eta+1);\quad \eta=\omega'/2qk_F;$$

and the condition for a collective mode is $2\Gamma_0=S_1$.

We are interested primarily in the behavior of $\mathfrak{M}$ near resonance. We can therefore expand (66) by a two-term Taylor series expansion of S_1.

$$\mathfrak{M}=\mu^2\mathcal{H}_{\rm rf}\frac{2\Gamma_0}{V}\frac{\omega_0(\eta_0^2-1)}{\eta_0(\omega-\omega_0)}, \quad (67)$$

where ω_0 is the center-resonance frequency, which is complex because of the damping.

The energy of excitation supplied by the external field is the volume integral $\int\mathfrak{M}\mathcal{H}_{\rm rf}d\mathbf{r}$. This energy is dissipated at the damping rate, hence the power which must be supplied is $\int(\mathfrak{M}/\tau)\mathcal{H}_{\rm rf}d\mathbf{r}$, and since $1/\tau$ is the frequency width of the resonance, we can find the power as the integral over external frequency of $\int\mathfrak{M}\mathcal{H}_{\rm rf}d\mathbf{r}$. Hence,

$$\mathcal{P}=\frac{2\mu^2\Gamma_0}{V}\frac{\omega_0(\eta_0^2-1)}{\eta_0}\int\frac{1}{\omega-\omega_0}d\omega\int\mathcal{H}_{\rm rf}^2d\mathbf{r} =\frac{\mu^2\Gamma_0k_F}{2\pi^2}\frac{\omega_0(\eta_0^2-1)}{\eta_0}\pi\frac{\mathcal{A}\delta}{2}\mathcal{H}_{\rm rf}^2(0), \quad (68)$$

where $\mathcal{H}_{\rm rf}(0)$ is the value of $\mathcal{H}_{\rm rf}$ at the surface of the sample, $\mathcal{A}$ is the sample area, and δ is the penetration depth of the radiation, the rf skin depth. We have assumed that δ is much less than the PSW wavelength.

It is interesting to make a comparison between the power in (68) and the power dissipation in an ordinary electron spin resonance in the same system. We have a well-known equation of motion for the spin density;

$$d\mathbf{S}/dt=\gamma\mathbf{S}\times\mathcal{H}=\gamma\mathbf{S}\times[\mathcal{H}_{\rm ex}\hat{z}+\mathcal{H}_{\rm rf}(\hat{x}\cos2\omega t+\hat{y}\sin2\omega t)],$$

which we linearize to give the oscillating component in terms of the constant component.

$$S_{rf}=-\gamma S_0 \mathcal{H}_{rf}(\omega-\gamma\mathcal{H}_{ex})^{-1}=-\gamma S_0\mathcal{H}_{rf}(\omega-\omega_0)^{-1}.$$

The constant component is the spin polarization of the gas by the constant, homogeneous, external field. After substitution and simplification, we find a spin magnetization density

$$\mathfrak{M}=\frac{3\mu^2\mathfrak{N}\mathcal{H}_{ex}}{\mathcal{E}_F}\frac{\mathcal{H}_{rf}}{\omega-\omega_0},$$

where $\mathfrak{N}$ is the electron density. Comparing this with (67), and cancelling terms, we see that the ratio of powers in the PSW to ESR experiments is

$$\Gamma_0^2(\eta_0^2-1)/\eta^2\cong\Gamma_0^2(\eta^2-1).$$

For lead, this number is 1.8×10^{-2}. For aluminum, the ratio will be of order 10^{-5}.

We have seen earlier that the PSW spectrum extends to its highest in lead, permitting greatest freedom in the choice of experimental frequency ranges. Since the damping is quite strong, one would like to be able to use as high a frequency as possible to obtain the sharpest, highest lines. We have just seen that minimum required sensitivity of the detection apparatus will be lowest for lead, and that this minimum sensitivity, while high, is not unattainably so.

ACKNOWLEDGMENTS

I gratefully acknowledge the assistance and cooperation of P. H. Trent, who programmed the SDS 930 for the numerical computations of this work. I should also like to acknowledge valuable conversations with A. L. McWhorter and M. M. Litvak.

APPENDIX A

Figure 7 shows initial- and final-state vectors for a possible anomalous scattering event conserving energy and momentum. The origins are shown as the center of the Fermi sphere. Momentum conservation requires the coincidence of the midpoints of the spans from $\mathbf{k}_1$ to $\mathbf{k}_2$ and $\mathbf{k}_3$ to $\mathbf{k}_4$, as indicated. Energy conservation is indicated for this highly specialized example.

Practically all of the scattering takes place among states within $\tilde{\omega}$ or $2\kappa T$ of the Fermi energy, being so restricted by energy conservation and the occupation probability factors. This is crudely indicated in Fig. 8, a detail of the region of the Fermi surface about $\mathbf{k}_1$. In order for scattering to occur, $\mathbf{k}_1$ must lie within the outermost surface, for beyond, there are no occupied states to scatter. At the same time, $\mathbf{k}_2$ must lie within the outer sphere. Assuming fixed $\mathbf{\Lambda}$, the plus sign on line segment ℓ indicates the relative locations of $\mathbf{k}_2$ and the Fermi surface. The plus is located at the reflection of the head of $\mathbf{k}_1$ in the line OB. Thus, roughly in order for a scattering event to occur, $\mathbf{k}_1$ must fall within the

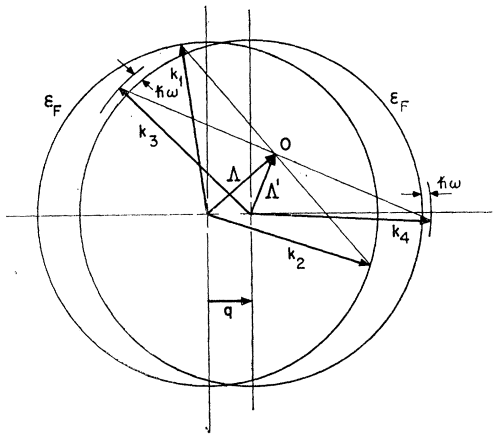

FIG. 7. A possible configuration of initial wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and final-wave vectors, $\mathbf{k}_3$, $\mathbf{k}_4$, measured from the center of the Fermi sphere. The coincidence of the heads of $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ is demanded by conservation of momentum. Conservation of energy is indicated crudely.

shaded region of Fig. 8. The energy of the state $\mathbf{k}_1$ is a linear function of the displacement of $\mathbf{k}_1$ from the Fermi surface, plus a correction term of relative order q/k_F which we shall ignore. The energy of the state $\mathbf{k}_2$ is a similar function of the displacement of the + from

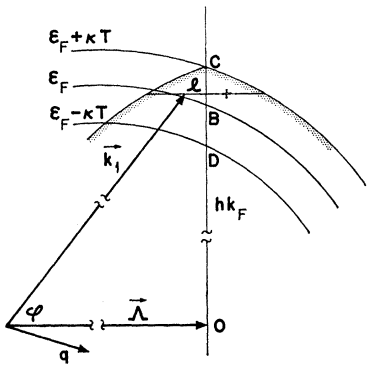

FIG. 8. Details of the Fermi surface in the regions near the initial state. The pluses are the images of the other member of each pair reflected in O.

the Fermi surface. As $\mathbf{k}_1$ is imagined moved along ℓ, the sum of the two energies is a constant. A similar construction shown in Fig. 9 applied to the final-state wave vectors.

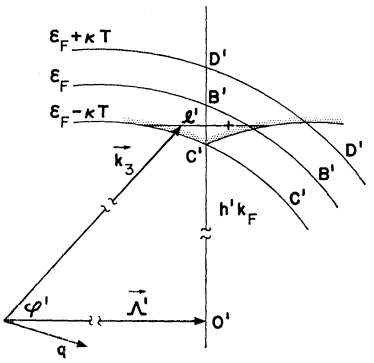

FIG. 9. Fermi-surface details near final states.

With each ℓ, there is an initial-state energy associated, and with each ℓ' a final-state energy. If these differ by $\tilde{\omega}$, the absorbed magnon energy, then any $\mathbf{k}_1$ on the surface of revolution of ℓ about $\mathbf{\Lambda}$ and any $\mathbf{k}_3$ on the surface of revolution of ℓ' about $\mathbf{\Lambda}'$ describe energy and momentum-conserving anomalous events.

Let s measure the displacement of ℓ from some fixed point along the line OB, say the point B, and let l measure the displacement of $\mathbf{k}_1$ along ℓ from OB. Then the energy of the state $\mathbf{k}_1$ is $\mathcal{E}_F+2k_F(sh+xl)$ and that of $\mathbf{k}_2$ is $\mathcal{E}_F+2k_F(sh-xl)$, where xk_F is the magnitude of $\mathbf{\Lambda}$.

There are analogous expressions for $\mathcal{E}_{k3}$ and $\mathcal{E}_{k4}$. Now O and O' are relatively displaced by $\mathbf{q}$ from the centers of their respective spheres. For this reason, h differs from h' and x differs from x' by relative amounts of the order q/k_F. We shall ignore these corrections.

To find the damping, we need to evaluate the sum [assuming constant $V(\mathbf{k})$],

$$\frac{V^2}{2\pi\Omega^2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\left(\frac{\sin\phi_1-\sin\phi_2}{2}\right)^2 \times\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-2\mathbf{q})\delta(\mathcal{E}_1+\mathcal{E}_2+2\omega-\mathcal{E}_3-\mathcal{E}_4) \times f(\mathcal{E}_1)f(\mathcal{E}_2)[1-f(\mathcal{E}_3)][1-f(\mathcal{E}_4)]. \quad \text{(A1)}$$

We sum first over $\mathbf{k}_4$ and convert the remaining sums to integrals.

$$\frac{\Omega V^2}{2\pi(8\pi^3)^3}\int\left[\frac{\sin\phi_1-\sin\phi_2}{2}\right]^2 \times\delta(\mathcal{E}_1+\mathcal{E}_2+2\omega-[\mathcal{E}_3+\mathcal{E}_4])f(\mathcal{E}_1)f(\mathcal{E}_2) \times[1-f(\mathcal{E}_3)][1-f(\mathcal{E}_4)]d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3. \quad \text{(A2)}$$

The energy delta function may be written as a delta function on s'.

$$\delta(\mathcal{E}_3+\mathcal{E}_4-\mathcal{E}_0)=\delta(2k_F{}^2[x^2+(h+s')^2]-\mathcal{E}_0) =\frac{\delta(s'-s_0)}{4hk_F}, \quad \text{(A3)}$$

where s_0 is the value of s' for which energy conservation is satisfied.

Substituting (A3) into (A2) gives

$$\frac{2\pi\Omega V^2}{(8\pi^3)^3}\int\frac{\delta(s'-s_0)}{4hk_F}f(\mathcal{E}_1[s,l])f(\mathcal{E}_2[s,l]) \times[1-f(\mathcal{E}_3[s',l'])][1-f(\mathcal{E}_4[s',l'])] \times F_{\text{av}}dsdl\,hd\psi ds'dl'hd\psi'8\pi x^2dx\,k_F{}^3. \quad \text{(A4)}$$

The angles ψ and ψ' are angles of rotation of $\mathbf{k}_1$ about $\mathbf{\Lambda}$ and $\mathbf{k}_3$ about $\mathbf{\Lambda}'$, respectively. $d\mathbf{k}_2$ has been replaced by $2d\mathbf{\Lambda}=2[(xk_F)^2\sin\theta d\theta d\rho d(xk_F)]$. The function on ϕ_1 and ϕ_2 has been replaced by its mean value

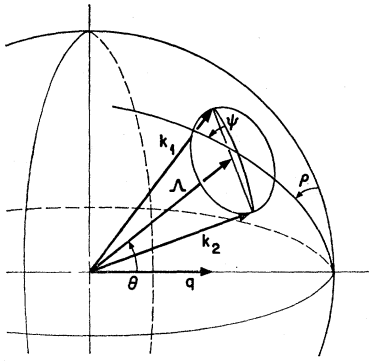

FIG. 10. The Fermi sphere. ρ is the angle between a plane containing $\mathbf{q}$ and a fixed vector and the plane containing $\mathbf{q}$ and $\mathbf{\Lambda}$. ψ is the angle between the plane containing $\mathbf{q}$ and $\mathbf{\Lambda}$ and the plane containing $\mathbf{k}_1$ and $\mathbf{k}_2$. θ is the angle between $\mathbf{q}$ and $\mathbf{\Lambda}$.

$$F_{\text{av}}=\int\left[\frac{\sin\phi_1-\sin\phi_2}{2}\right]^2\frac{d\psi}{2\pi}\frac{\sin\theta d\theta}{2}\frac{d\psi'}{2\pi} \quad \text{(A5)}$$

which we will discuss in turn. Figure 10 shows the variables of integration.

The integrals over ℓ, ℓ', and s' in (A4) give elementary functions.

$$\frac{2\pi\Omega V^2}{(8\pi^3)^3}\int\frac{(\kappa T)^3}{(2k_F)^3x^2h}\frac{(2\pi hk_F)^2}{4hk_F}F_{\text{av}}\frac{1}{2}\frac{z(z+\mu)}{(e^z-1)(e^{-z-\mu}-1)}dz \times2(xk_F)^2d(xk_F)\sin\theta d\theta d\rho, \quad \text{(A6)}$$

where $\mu=\tilde{\omega}/\kappa T$; $z=4k_Fhs/\kappa T$. We integrate over z and obtain

$$\frac{2\pi\Omega V^2}{(8\pi^3)^3}\frac{(\kappa T)^3 2k_F{}^4}{8(2k_F)^3} \times\left\{\int F_{\text{av}}\frac{(2\pi h)^2}{x^2h^2}x^2dx\sin\theta d\theta d\rho\right\}\frac{\mu^3+4\pi^2\mu}{6(1-e^{-\mu})}. \quad \text{(A7)}$$

The net energy loss rate is found by subtracting the emitting collision rate from the absorbing rate, giving finally

$$\frac{\Omega V^2k_F(\kappa T)^3(\mu^3+4\pi^2\mu)}{192(8\pi^3)^2}\int F_{\text{av}}dx\sin\theta d\theta d\rho. \quad \text{(A8)}$$

Now let us consider F_{av}. We have

$$\sin\phi_1=\frac{c}{2\mathbf{q}\cdot\mathbf{k}_1-\omega'}=\frac{c}{2qk_F}\left[\frac{1}{x(\cos\theta)+\delta-h\sin\theta\cos\psi}\right]$$

and

$$\sin\phi_2=\frac{c}{2qk_F}\left[\frac{1}{x(\cos\theta)+\delta+h\sin\theta\cos\psi}\right]$$

by the construction of Fig. 10. Therefore,

$$\tfrac{1}{4}(\sin\phi_1-\sin\phi_2) =\left(\frac{c}{2qk_F}\right)^2\frac{h^2\sin^2\theta\cos^2\psi}{[(x\cos\theta+\delta)^2-h^2\sin^2\theta\cos^2\psi]^2}. \quad \text{(A9)}$$

The integration over ψ may be performed, but it results in a complicated function, precluding the integrations over θ and x. The integrand is a positive function with broad maxima at $\psi=0$ and π. We overestimate the integral by a factor of order unity by replacing the integrand with its maximum value. The resulting integration over θ is elementary, but tedious. In the limit of small δ,

$$F_{\rm av}=\frac{c^2\eta^2}{\omega^2}\left\{\frac{1+h^2}{2h}\ln\frac{1+h}{1-h}-1\right\}. \tag{A10}$$

Thus the sum in (A1) has become

$$\frac{\Omega V^2 k_F(\kappa T)^3}{192(8\pi^3)^2}(\mu^3+4\pi^2\mu)\frac{c^2\eta^2}{\omega^2}4\pi \times\int_0^1-\left\{\frac{1+h^2}{2h}\ln\frac{1-h}{1+h}+1\right\}dx. \tag{A11}$$

This is the net magnon absorption rate for a particular type of collision, say $\mathbf{k}_1$ and $\mathbf{k}_2$ "up" to $\mathbf{k}_3$ "up" and $\mathbf{k}_4$ "down." As we discussed, there are four types of anomalous collision in which an up spin is turned down, indicating magnon absorption. Likewise, there are four types of process in which a down spin is turned up, indicating emission. Thus for the net magnon absorption rate, we have

$$\frac{\Omega V^2 k_F(\kappa T)^3(\mu^3+4\pi^2\mu)}{(8\pi^3)^2 12}\frac{c^2\eta^2\pi}{\omega^2} \times\int_0^1-\left\{\frac{1+h^2}{2h}\ln\frac{1-h}{1+h}+1\right\}dx. \tag{A12}$$

The value of the integral is 2.248. The magnon lifetime is found as before by dividing by the magnon number.

$$\frac{d(n+\frac{1}{2})}{dt}\frac{1}{n+\frac{1}{2}}=\frac{[\Omega V^2 k_F(\kappa T)^3(\mu^3+4\pi^2\mu)(\pi/12(8\pi^3)^2)](c^2\eta^2/\omega^2)2.248}{(\Omega c^2\omega/16\pi^2 q)[2\eta/(\eta^2-1)-\ln[(\eta-1)/(\eta+1)]]}\cong\frac{V^2(\kappa T)^3(\mu^3+4\pi^2\mu)(\eta^2-1)2.248}{\omega^2\pi^3 192}. \tag{A13}$$

It is well known[12] that the rate of electron or hole scattering by collisions with other carriers is proportional to the square of the energy separation of the state in question from the Fermi energy. The total number of collisions in the system per unit time can thus be approximated by

$$n_{\rm coll}=\frac{\Omega}{8\pi^3}2\int_0^{\kappa T}4\pi k_F\alpha(\Delta E)^2 d(\Delta E)=\frac{\Omega k_F}{\pi^2}\frac{1}{\tau_{\kappa T}}\frac{\kappa T}{3}, \tag{A14}$$

where $\tau_{\kappa T}$ is the lifetime of an electronic state κT above the Fermi energy and α is a constant.

We can also find an expression for the total collision rate by using (A7), setting $F_{\rm av}=1$, $\mu=0$, and multiplying by 4 to include the different types of ordinary scattering event (such as k_1, k_3 "up," k_2, k_4 "down," etc.)

$$V^2=\frac{16\pi}{\tau_{\kappa T}(\kappa T)^2}. \tag{A15}$$

We equate the rates in (A15) and (A14) to eliminate V^2 in favor of $\tau_{\kappa T}$ and use this in (A12) to express the magnon damping time in terms of $\tau_{\kappa T}$.

$$\frac{\Delta\omega}{\omega}=\frac{(\kappa T)(\mu^3+4\pi^2\mu)(\eta^2-1)2.248}{\tau_{\kappa T}\omega^2\pi^2 12} =\frac{(4\pi^2+\mu^2)(\eta^2-1)2.248}{\omega\tau_{\kappa T}12\pi^2}=\frac{1}{\omega\tau_{\rm PSW}}. \tag{A16}$$

We can evaluate $\tau_{\kappa T}$ from the expression for electron lifetime as given by Quinn and Ferrell,[12] using parameters for the Fermi surface electrons in lead from Anderson and Gold.[13] We find $\tau_{\kappa T}=4.1\times10^{-7}T^{-2}$ sec. At liquid-helium temperatures, this is of order 10^{-8} sec. The factor η^2-1 in (A16) is of order 10^{-3} in lead, giving a magnon lifetime around 10^{-5} sec or longer due to interelectron scattering.

APPENDIX B

Our discussion of the properties of PSW's has been carried out in toto using the wave functions of Eqs. (5) and (6). These wave functions describe electron states with time-dependent, spiralling spin polarization. This polarization is constant and homogeneous in magnitude. It is either in perfect phase or out of phase by π with the driving field, the term proportional to c in the one-particle Hamiltonian of Eq. (2). As we saw in Sec. VII, this driving field may be composed of an external applied field and an average response field found self-consistently. Our approximate many body wave functions formed out of our one electron states give rise to an average response field in phase with the external field or out of phase by π.

This description neglects the effects of damping. We expect in general the response of a dissipative system to lag somewhat behind the driving field. The resulting component of the response in quadrature with the field provides the mechanism for power transfer from field to system.

We can relate the power absorption to the polarization phase through the use of the equation of con-

[12] J. J. Quinn and R. A. Ferrell, Phys. Rev. **112**, 812 (1958).

[13] J. R. Anderson and A. V. Gold, Phys. Rev. **139**, A1459 (1965).

tinuity of Sec. VI.

$$\nabla\cdot(\mathbf{j}_\uparrow-\mathbf{j}_\downarrow)=\frac{\dot{M}}{\mu}=\frac{d}{dz}(j_\uparrow-j_\downarrow)_z, \tag{58}$$

where z is in the direction of q. A particle in the field experiences a force along the field gradient and absorbs energy in proportion to its velocity. Hence for the power absorption density, we have

$$\mathcal{P}(r)=-[\nabla\mathcal{U}(r)]\cdot(\mathbf{j}_\uparrow-\mathbf{j}_\downarrow). \tag{B1}$$

All our quantities vary sinusoidally in space, hence $\nabla\mathcal{U}(z)=q\mathcal{U}(z+\pi/2q)$ and likewise for j and M, from which we find

$$\mathcal{P}(r)=-\mathcal{U}(r)\frac{d}{dt}M(r). \tag{B2}$$

When we integrate this power density over the volume of the system, the only contribution comes from the components of $\dot{M}$ in phase with the field. Our discussion is specific to the case of spin density variations along a single direction, but the result, Eq. (B2), is more general than this. We shall use (B2) for the spiral configurations we have been considering and observe that the conclusions agree with the simple results of Sec. IV.

The wave functions with which we have dealt so far exhibit no polarization component in quadrature with the field and able to exchange energy with it. To understand the absorption and the out of phase components of the response, we must generalize the class of functions from which we form determinants.

The one particle Schrödinger equation, Eq. (2), has two orthogonal solutions for each $\mathbf{k}$. We have so far considered no fractional occupations of the corresponding states. Now any linear combination of these two wave functions again satisfies Eq. (2) and represents therefore a possible one particle wave function for use in the many-body determinant. We so generalize one state.

$$\phi_\mathbf{k}=a_l\phi_l(\mathbf{k},\mathbf{r})+a_u\phi_u(\mathbf{k},\mathbf{r}). \tag{B3}$$

The u and l refer to the upper and lower branches as discussed after Eqs. (5) and (6). If we now calculate the spin polarization as before, the contribution of the occupation of this state is of two parts. The first is in phase with the field

$$[A_1A_2(|a_l|^2-|a_u|^2)+(A_2^2-A_1^2)\{(a_la_u^*+a_l^*a_u)\times\cos(E_u-E_l)t+i(a_la_u^*-a_l^*a_u)\sin(E_u-E_l)t\}]\times[\hat{x}\cos(\tilde{\mathbf{q}}\cdot\mathbf{r}-\tilde{\omega}t)+\hat{y}\sin(\tilde{\mathbf{q}}\cdot\mathbf{r}-\tilde{\omega}t)] \tag{B4}$$

and no longer constant in magnitude. The second part is in quadrature with the field

$$\{i(a_la_u^*-a_l^*a_u)\cos(E_u-E_l)t-(a_la_u^*+a_l^*a_u)\sin(E_u-E_l)t\}\times[\hat{x}\sin(\tilde{\mathbf{q}}\cdot\mathbf{r}-\tilde{\omega}t)-\hat{y}\cos(\tilde{\mathbf{q}}\cdot\mathbf{r}-\tilde{\omega}t)]. \tag{B5}$$

It is the interference between ϕ_u and ϕ_l that produces the out of phase components and is hence crucial in the understanding of the energy absorption. Let us, therefore, consider a scattering event from the state $\mathbf{k}'$ into the state $\mathbf{k}$, considering all four orthogonal states involved and follow the polarization as the scattering proceeds. We shall then use that polarization to calculate the energy absorbed per particle scattered.

The bulk of the following discussion is an elaborate exercise in trigonometry, the major part of which is not displayed. We begin with the assumption that at $t=0$, the state $\mathbf{k}_l'$ is occupied and the states $\mathbf{k}_u'$, $\mathbf{k}_l$, and $\mathbf{k}_u$ are all empty. Scattering occurs off impurity centers as in Sec. IV. We calculate the various amplitudes, a_i as in (B3), and form the polarization as a function of time. The polarization in the states $\mathbf{k}_u$, $\mathbf{k}_l$ is proportional to products of a_u with a_l. These products are of second order in the scattering potential U, since the a's are of first order. We confine this discussion to scattering not involving spin flip, in which case there are no direct matrix elements connecting $\mathbf{k}_l'$ with $\mathbf{k}_u'$. The amplitude a_u' is therefore of second order in the scattering potential, but since $a_l'=1$ is of zero order, products of the two are of second order; the polarization in the states $\mathbf{k}'$ must be considered as well. Taking the a_i as calculated by time-dependent perturbation theory (first or second order as is needed) and substituting into (B4) and (B5) gives the polarization which we may then differentiate with respect to time. Using this derivative in (B2) and multiplying by the "driving" potential

$$\mathcal{U}(\mathbf{r})=c[\hat{x}\cos(\tilde{\mathbf{q}}\cdot\mathbf{r}-\tilde{\omega}t)+\hat{y}\sin(\mathbf{q}\cdot\mathbf{r}-\tilde{\omega}t)] \tag{B6}$$

and integrating over space gives the power absorption. We form a packet of states about the state $\mathbf{k}$ to give meaning to the term "scattering into $\mathbf{k}$" and integrate with respect to time to find the energy absorbed. We find that there are two types of contribution to this final energy; terms which are proportional to the time elapsed, and terms which oscillate in time between fixed limits. We consider the case of one of the "anomalous" scattering events of Sec. IV, and considered the occupation probability of the "almost spin flipped state," $\mathbf{k}_u$. This probability grows linearly with time. Thus when we calculate the energy absorbed per particle scattered, we find a constant plus some oscillating transients which die away like $1/t$. The constant proves to be $+$ or $-\tilde{\omega}$, as we used in Sec. IV, for the case of non-Landau-damped modes. The cases of other starting occupations of $\mathbf{k}_u'$ and $\mathbf{k}_l$ may be treated analogously; we shall not display them here.

From time-dependent perturbation theory, we have

$$a_l = -U(A_1A_1'+A_2A_2')\frac{\sin[(E_l'-E_l)/2]t}{E_l'-E_l}2ie^{-i[(E_l'-E_l)/2]t},$$

$$a_u = -U(A_2A_1'-A_1A_2')\frac{\sin[(E_l'-E_u)/2]t}{E_l'-E_u}2ie^{-i[(E_l'-E_u)/2]t},$$

$$a_u' = \frac{U^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')(-2i)}{E_l'-E_l}\left\{\frac{\sin[(E_l'-E_u')/2]t}{E_l'-E_u'}e^{-i[(E_l'-E_u')/2]t}\right.$$
$$\left. - \frac{\sin[(E_l-E_u')/2]t}{E_l-E_u}e^{-i[(E_l-E_u')/2]t}\right\} + \frac{U^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')(-2i)}{E_l'-E_u}$$
$$\times\left\{\frac{\sin[(E_l'-E_u')/2]t}{E_l'-E_u'}e^{-i[(E_l'-E_u')/2]t} - \frac{\sin[(E_u-E_u')/2]t}{E_u-E_u'}e^{-i[(E_u-E_u')/2]t}\right\}. \quad \text{(B7)}$$

At this point the development becomes unwieldy. Following the prescription outlined above yields an expression for the power absorption containing 20 terms. Of these, 4 come from the first term in (B4) and are of higher order in c. Of the remainder, 14 are oscillatory between fixed limits, giving an oscillating, decaying contribution to the energy per particle scattered. All of the remaining terms coming from (B4) are transients of this form. Two terms contribute to the energy per particle, and we shall show the details of the development only to the extent necessary to obtain them.

The first-order quantities, a_u and a_l, taken from (B7) and substituted into (B5), yield a polarization

$$2U^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')\frac{\sin[(E_l'-E_l)/2]t}{[(E_l'-E_l)/2]}\frac{\sin[(E_l'-E_u)/2]t}{[(E_l'-E_u)/2]}\sin[(E_u-E_l)/2]t$$
$$\times(\hat{x}\sin 2(\mathbf{q}\cdot\mathbf{r}-\omega t)-\hat{y}\cos 2(\mathbf{q}\cdot\mathbf{r}-\omega t)) \quad \text{(B8)}$$

after some trigonometric combinations. Differentiating with respect to time, multiplying by $\mathcal{U}(\mathbf{r})$ from (B6) and integrating over $d\mathbf{r}$ gives

$$-2U^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')2c\omega\frac{\sin[(E_l'-E_l)/2]t}{[(E_l'-E_l)/2]}\frac{\sin[(E_l'-E_u)/2]t}{[(E_l'-E_u)/2]}\sin[(E_u-E_l)/2]t. \quad \text{(B9)}$$

We now integrate over the states of a wave packet centered on $\mathbf{k}_l$. We take the slowly varying factors out of the integration and obtain

$$-\rho(E_l)U^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')4\pi c\omega\frac{1-\cos(E_u-E_l)t}{[(E_u-E_l)/2]} \quad \text{(B10)}$$

for the power absorption, where $\rho(E_l)$ is the density of states about E_l.

The second-order amplitude, a_u', also gives a contribution as we mentioned. For the power absorption we have, in analogy to (B9)

$$2\omega cU^2(A_1A_1'+A_2A_2')(A_2A_1'-A_1A_2')\left\{\frac{-\sin(E_l'-E_l)t-\sin(E_u'-E_l')t}{(E_l'-E_l)[(E_l-E_u')/2]}+\frac{-\sin(E_l'-E_u)t-\sin(E_u'-E_l')t}{(E_l'-E_u)[(E_u-E_u')/2]}\right.$$
$$\left.+\frac{\sin(E_u'-E_l')t}{(E_l'-E_l)[(E_l'-E_u')/2]}+\frac{\sin(E_u'-E_l')t}{(E_l'-E_u)[(E_l'-E_u')/2]}\right\}. \quad \text{(B11)}$$

Again we integrate over the packet of final states around $\mathbf{k}_l$, and obtain for the power absorption a constant plus oscillatory terms.

$$\rho(E_l)\pi 4\omega c U^2(A_1A_1'+A_2A_2)(A_2A_1'-A_1A_2')\times\left\{\frac{1}{[(E_l'-E_u')/2]}\right\}+\cdots. \quad \text{(B12)}$$

Integrating over time from 0 to t gives the energy absorbed from the field by the scattering as a function of time.

From (B7) we may find a_l as a function of time, the total scattered probability. Dividing by the integral over the packet we obtain the energy absorbed per particle scattered.

$$2\omega c\frac{A_2A_1'-A_1A_2'}{A_1A_1'+A_2A_2'}\left\{\frac{1}{E_u-E_l}-\frac{1}{E_u'-E_l'}\right\}=\omega\frac{A_2A_1'-A_1A_2'}{A_1A_1'+A_2A_2'}(\sin\phi+\sin\phi'). \quad \text{(B13)}$$

After expressing $\sin\phi$ and $\sin\phi'$ in terms of the A_i, this just becomes $\omega(\cos\phi-\cos\phi')$ the result we used in Sec. V.

The transient terms which we have neglected oscillate with frequencies E_u-E_l, $E_u'-E_l'$, E_u-E_u', E_u-E_l', and $E_u'-E_l$. They occur in the form of functions like $\sin\Delta t/\Delta$, which only become small relative to the terms increasing as t for times such that $\Delta t\gtrsim 1$. In the Landau damped situation, say where the state $\mathbf{k}'$ is on the gap, $E_u'-E_l'$ becomes $2c$. If the scattering lifetime of the states is not long enough to allow the transients to decay, their effects must be included. Thus for high scattering rates, high compared to $\frac{1}{2}c$, the field "pumping rate," the customary theory of Landau damping is applicable and the Landau damped "modes" are broadened to unobservability. In the case of long scattering lifetimes or high pumping rates corresponding to strong excitation, the lines sharpen up, as we discussed in Sec. VI and observable resonances may occur.

We have seen that the inphase components of the polarization contribute only transients to the energy absorption. We may, therefore, relate the damping directly to the component of net magnetization in quadrature with the field, that is, to the imaginary component of the susceptibility.

$$\chi_{\text{imag}}=\frac{c_{\text{tot}}}{c_{\text{ext}}}\frac{dn}{dt},$$

$$\text{since}\quad -\frac{dE}{dt}=c_{\text{tot}}\tilde{\omega}M_1=-\tilde{\omega}\frac{dn}{dt}, \quad \text{(B14)}$$

where dn/dt is the net magnon absorption rate, the excess rate of absorbing over emitting anomalous collisions.

We cannot clearly associate an average out of phase magnetization to each state of the gas, for as we have just seen, both the initial and final states in a collision contribute a portion. Let us, however, arbitrarily assign the entire amount to the final state. The average out of phase magnetization of state $\mathbf{k}$ (however we make the division) depends on the rate at which electrons scatter in anomalously.

$$\sum_{\mathbf{k}'}\frac{2\pi U^2}{\Omega}\frac{(\sin\phi+\sin\phi')^2}{c}\times(1-f(E_{\mathbf{k}\downarrow}))f(E_{\mathbf{k}\uparrow}')\delta(E_{\mathbf{k}\downarrow}-E_{\mathbf{k}\uparrow}'+\tilde{\omega}). \quad \text{(B15)}$$

Only states within $\tilde{\omega}$ of the Fermi surface contribute to the out of phase magnetization, and, on a surface of constant energy, the average out of phase part varies as $(\sin\phi+\text{const})$.[2] It is interesting to contrast this result with the device of using a complex frequency to obtain the imaginary part of the susceptibility. Assuming the mode is not Landau damped, we have for the magnetization of state k

$$\frac{c}{2\mathbf{k}\cdot\mathbf{q}-\omega-i\eta}. \quad \text{(B16)}$$

In the limit of small damping, the imaginary part (the out-of-phase part) is thus, for small damping,

$$i\eta\frac{c}{(2\mathbf{k}\cdot\mathbf{q}-\omega)^2}=i\eta\frac{\sin^2\phi}{c}. \quad \text{(B17)}$$

Comparing (B17) with (B15), we see that in order to reproduce the results of the detailed, microscopic damping theory we have presented here, it is necessary to use a state-dependent complex frequency; the detailed theory does not permit the use of a unique imaginary part of the frequency.