

Multiple Scattering of High-Energy Dirac Particles

S. MUKHERJEE

Department of Physics, Indian Institute of Technology, Kharagpur, West Bengal, India

(Received 1 May 1967)

The distribution function for multiple scattering of high-energy Dirac particles has been calculated, starting from an approximate solution of the Dirac equation. With the Thomas-Fermi potential itself, the calculated results are found to be in good agreement with the experimental results. This is in contradiction to the calculations of Nigam, Sundaresan, and Wu, who found it necessary to introduce an arbitrary screening parameter. The multiple scattering of polarized particles is also discussed.

I. INTRODUCTION

THE theory of multiple scattering of fast Dirac particles is of much help in interpreting many experimental observations. In a comprehensive review article Scott¹ has given an account of the development of the theory as well as the various improvements suggested by different workers. To apply this theory to any specific scattering problem one requires first a knowledge of the scattering distribution for a single collision. Molière² derived his single-scattering formula by the BWK method. Nigam, Sundaresan, and Wu³ have criticized some of the mathematical steps of Molière; they have used instead the second-Born-approximation results of Dalitz⁴ for scattering from an exponentially screened Coulomb field. They introduced a screening parameter which was adjusted to make the theoretical results agree with the experiments of Hanson *et al.*⁵ Reed and Nodvick⁶ calculated the same with the Molière representation of the Thomas-Fermi potential but retained the screening parameter. The introduction of the arbitrary screening parameter is, however, not a happy feature and shows the inadequacy of the theory. Also the suggestion of Nigam *et al.* that the Thomas-Fermi potential is unrealistic for small-angle scattering cannot be accepted. The major part of this paper will be devoted to this problem. In Sec. III we shall consider the multiple scattering of polarized particles. Mühschlegel and Koppe⁷ have studied the polarization effects in the first Born approximation. An improvement of their calculations with the help of the Dalitz formula does not appear feasible, considering the complicated expressions this will lead to. A simple and fairly accurate single-scattering formula may, therefore, be of much help. In the following calculations we shall utilize the scattering results obtained by following the method discussed in a previous paper.⁸ The method essentially consists in solving the iterated

Dirac equation by a distorted-wave approximation. This leads to a scattering matrix T given by⁹

$$T(\mathbf{P}, \mathbf{P}') = [2\epsilon + \boldsymbol{\alpha} \cdot (\mathbf{P}' - \mathbf{P})] \Phi(\theta), \quad (1.1)$$

where

$$\Phi(\theta) = -\frac{N}{4\pi} \int \exp(i\mathbf{q} \cdot \mathbf{r}) V(r) F(ia\epsilon/\rho, 1, i\rho r - i\mathbf{P} \cdot \mathbf{r}) d\mathbf{x}, \quad (1.2)$$

where $\mathbf{q} = \mathbf{P} - \mathbf{P}'$ is the momentum transfer. The units chosen are $\hbar = m = c = 1$. ϵ is the energy and the $\boldsymbol{\alpha}$'s are the Dirac matrices. The constant $a = Ze^2/\hbar c$. N is a normalization constant which can be determined by considering the case of a point charge where T is known. This gives

$$N = e^{(1/2)\pi a\epsilon/\rho} [\Gamma(1 - ia\epsilon/\rho) / \Gamma(1 + ia\epsilon/\rho)]. \quad (1.3)$$

For the scattering potential we shall consider the Molière representation of the Thomas-Fermi potential, viz.,

$$V(r) = \frac{a}{r} \sum_{i=1}^3 a_i \exp(-b_i r/a_0), \quad (1.4)$$

where a_0 is the Thomas-Fermi radius.¹⁰ The form (1.4) of $V(r)$ makes it possible to evaluate the integral (1.2) analytically and thus a closed expression for T may be obtained.

The scattering of polarized particles is given by the scattering amplitude A defined by

$$\sigma(\mathbf{P}, \mathbf{P}', \boldsymbol{\mathfrak{P}}) \rho(\boldsymbol{\mathfrak{P}}') = A(\mathbf{P}, \mathbf{P}') \rho(\boldsymbol{\mathfrak{P}}) A^\dagger(\mathbf{P}, \mathbf{P}'), \quad (1.5)$$

where σ is the scattering cross section, ρ the density matrix for spin, and $\boldsymbol{\mathfrak{P}}$ and $\boldsymbol{\mathfrak{P}}'$ are the polarization vectors for the incident and the scattered beam. The 2×2 matrix A may be written in terms of T as⁷

$$A = \frac{2\epsilon}{\epsilon + 1} \frac{1}{2}(1 + \beta) \Lambda_+(\mathbf{P}') T(\mathbf{P}, \mathbf{P}') \Lambda_+(\mathbf{P}) \frac{1}{2}(1 + \beta), \quad (1.6)$$

where Λ_+ is the positive-energy-projection operator. A much simpler relation between A and T may be determined by noting that the operator A converts the large

⁹ See Eqs. (11) and (16) of Ref. 8. The normalization constant N given in that paper is incorrect and should be replaced by that in (13) of the present paper.

¹⁰ R. D. Reed [Phys. Rev. **138**, A1000 (1965)] has shown that the Rozental representation of the Thomas-Fermi potential gives almost the same scattering results.

¹ W. T. Scott, Rev. Mod. Phys. **35**, 231 (1963).

² G. Molière, Z. Naturforsch. **2a**, 133 (1947); **3a**, 78 (1948).

³ B. P. Nigam, M. K. Sundaresan, and Ta-You Wu, Phys. Rev. **115**, 491 (1959).

⁴ R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

⁵ A. O. Hanson, L. H. Lanzl, E. M. Lyman, and M. B. Scott, Phys. Rev. **84**, 634 (1951).

⁶ R. D. Reed and J. S. Nodvick, Phys. Rev. **135**, B199 (1964).

⁷ B. Mühschlegel and H. Koppe, Z. Physik **150**, 474 (1958).

⁸ S. Mukherjee and S. D. Majumdar, Ann. Physik **16**, 360 (1965).

components of the incident wave into the large components of the scattered wave. This establishes the following relation between the matrix elements (with \mathbf{P} along the z axis):

$$A_{ij} = T_{ij} + (-1)^{j+1} (p/(\epsilon+1)) T_{i,j+2} \quad (i, j=1, 2). \quad (1.7)$$

A can also be written as

$$A = f(\theta)I + ig(\theta) \sin\theta \boldsymbol{\sigma} \cdot \mathbf{n}, \quad (1.8)$$

where $\mathbf{n} = (\mathbf{P} \times \mathbf{P}') / |\mathbf{P} \times \mathbf{P}'|$. We have from (1.7) and (1.8)

$$f(\theta) = \frac{1}{2} \text{Trace} A = \frac{1}{2} (T_{11} + T_{22}) + (p/2(\epsilon+1)) (T_{13} - T_{24}) \quad (1.9)$$

and

$$g(\theta) \sin\theta = e^{i\varphi} A_{12} = e^{i\varphi} [T_{12} - (p/(\epsilon+1)) T_{14}] \\ = -e^{-i\varphi} A_{21} = -e^{-i\varphi} [T_{21} + (p/(\epsilon+1)) T_{23}], \quad (1.10)$$

where (θ, φ) gives the direction of \mathbf{P}' . The two functions f and g determine the scattering completely. It is, however, convenient to discuss the polarization effects in terms of four functions $J, D, E,$ and F given by⁷

$$J = f^*f + \sin^2\theta g^*g,$$

$$D = i(f^*g - fg^*),$$

$$E = -(f^*g + fg^*),$$

and

$$F = 2g^*g. \quad (1.11)$$

For the scattering matrix (1.1), both f and g are real and there is no Mott polarization ($D=0$). This approximation results in considerable simplification, since Waldmann's transport equations which describe the multiple scattering are completely decoupled in the absence of Mott polarization. The Dalitz formula gives a nonzero value for D , viz.,

$$D(\theta) = \frac{1}{2\pi^2} \frac{\epsilon^2 - 1}{\epsilon^2} \frac{(4p^2 \sin^2 \frac{1}{2}\theta)^2}{(4p^2 \sin^2 \frac{1}{2}\theta + \lambda^2)} \sin\theta \text{Im}(I - J), \quad (1.12)$$

where

$$\text{Im}(I - J) = \frac{\pi^2}{2p^3 \cos^2 \frac{1}{2}\theta} \left\{ \frac{\lambda^2 + 2p^2 \sin^2 \frac{1}{2}\theta}{2(\sin^2 \frac{1}{2}\theta) \Lambda} \ln \frac{\Lambda + 2p^2 \sin^2 \frac{1}{2}\theta}{\Lambda - 2p^2 \sin^2 \frac{1}{2}\theta} + \ln \frac{\lambda}{(4p^2 + \lambda^2)^{1/2}} \right\},$$

and $\Lambda = [\lambda^4 + 4p^2(\lambda^2 + p^2 \sin^2 \frac{1}{2}\theta)]^{1/2}$ for a potential $V(r) = (1/r) e^{-\lambda r}$, but this expression is not suitable for application of the small-angle approximation because of the convergence difficulties in calculating its integral transform.¹¹

II. MULTIPLE SCATTERING OF AN UNPOLARIZED BEAM

In the small-angle approximation, the multiple-scattering-distribution function $f(\chi)$ for an unpolarized incident beam is given by

$$f(\chi) = \frac{1}{2\pi} \int \xi d\xi \exp[\tilde{j}(\xi) - \tilde{j}_0] J_0(\xi\chi), \quad (2.1)$$

with

$$\tilde{j}(\xi) - \tilde{j}_0 = 2\pi n t \int \theta d\theta J(\theta) [J_0(\xi\theta) - 1], \quad (2.2)$$

where n is the number of scattering centers per unit volume and t is the path traversed by the particle, which will be assumed to be equal to the thickness of the foil. For the potential (1.4) the scattering matrix is given by

$$T(\mathbf{P}, \mathbf{P}') = e^{(1/2)\pi a\epsilon/p} \frac{\Gamma(1 - ia\epsilon/p)}{\Gamma(1 + ia\epsilon/p)} [2\epsilon + \boldsymbol{\alpha} \cdot (\mathbf{P}' - \mathbf{P})] a \sum_{i=1}^3 \frac{a_i}{(4p^2 \sin^2 \frac{1}{2}\theta + b_i^2/r_0^2)} \left(\frac{4p^2 \sin^2 \frac{1}{2}\theta + b_i^2/r_0^2}{b_i^2/r_0^2 - 2ipb_i/r_0} \right)^{ia\epsilon/p}. \quad (2.3)$$

¹¹ In the small-angle approximation (1.12) gives

$$D(\theta) \propto \frac{\theta^4}{(\theta^2 + \theta_1^2)} \left\{ \frac{2\theta_1^2 + \theta^2}{2(\theta_1^4 + 4\theta_1^2 + \theta^2)^{1/2}} \right. \\ \left. \times \ln \frac{2(\theta_1^4 + 4\theta_1^2 + \theta^2)^{1/2} + \theta^2}{2(\theta_1^4 + 4\theta_1^2 + \theta^2)^{1/2} - \theta^2} + \theta \ln \frac{\theta_1}{(4 + \theta_1^2)^{1/2}} \right\},$$

where $\theta_1 = \lambda/p$, so that $D(\theta)$ is negligibly small for small-angle scattering. But its transform^{1,7}

$$\tilde{d}(\xi) = \int \theta^2 d\theta D(\theta) J_1(\xi\theta),$$

which enters into the solution of Waldmann's equation if D is taken nonzero, does not exist.

Utilizing relations (1.9), (1.10), and (1.11), we have in the small-angle approximation

$$\begin{aligned}
 J(\theta) &= f^2 + \theta^2 g^2 \\
 &= e^{\pi a \epsilon / p} \frac{4 \epsilon^2 a^2}{p^4} \left[1 + \theta^2 \frac{(\epsilon - 1)^2}{4 \epsilon^2} \right] \\
 &\quad \times \left\{ \sum_{i=1}^3 \frac{a_i^2}{(\theta^2 + \theta_i^2)^2} \exp \left(-2a \frac{\epsilon}{p} \tan^{-1} \frac{2}{\theta_i} \right) + 2 \operatorname{Re} \sum_{i < j} \frac{a_i a_j}{(\theta^2 + \theta_i^2)(\theta^2 + \theta_j^2)} \left(\frac{\theta^2 + \theta_j^2}{\theta^2 + \theta_i^2} \right)^{i \epsilon / p} \left(\frac{\theta_i^2 + 2i \theta_i}{\theta_j^2 - 2i \theta_j} \right)^{i \epsilon / p} \right\}, \quad (2.4)
 \end{aligned}$$

where $\theta_i = b_i \theta_0$ and $\theta_0 = 1 / p r_0$. We also define the characteristic angle θ_c by

$$\theta_c^2 = 4 \pi N_0 a^2 \frac{Z + 1}{Z} \frac{\rho_0 t \epsilon^2}{A p^4}, \quad (2.5)$$

where ρ_0 is the density and A the atomic number of the scatterer. The factor $(Z + 1) / Z$ has been introduced to account for the electron-electron scattering. To start with we shall neglect the term $\theta^2 g^2$ in $J(\theta)$. The integral (2.2) then consists of the following terms:

$$\begin{aligned}
 2 \theta_c^2 e^{\pi a \epsilon / p} \sum a_i^2 \exp \left[-2a(\epsilon / p) \tan^{-1} 2 / \theta_i \right] \int \frac{\theta d \theta}{(\theta^2 + \theta_i^2)^2} [J_0(\xi \theta) - 1] \\
 = 2 \theta_c^2 e^{\pi a \epsilon / p} \sum a_i^2 \exp \left[-2a(\epsilon / p) \tan^{-1} 2 / \theta_i \right] [(\xi / 2 \theta_i) k_1(\xi \theta_i) - (2 \theta_i^2)^{-1}], \quad (2.6)
 \end{aligned}$$

where k_1 is a modified Bessel function, and

$$2 \theta_c^2 e^{\pi a \epsilon / p} 2 \operatorname{Re} \sum_{i < j} a_i a_j \left(\frac{\theta_i^2 + 2i \theta_i}{\theta_j^2 - 2i \theta_j} \right)^{i \epsilon / p} \int \frac{\theta d \theta}{(\theta^2 + \theta_i^2)(\theta^2 + \theta_j^2)} \left(\frac{\theta^2 + \theta_j^2}{\theta^2 + \theta_i^2} \right)^{i \epsilon / p} [J_0(\xi \theta) - 1]. \quad (2.7)$$

In evaluating the integral in (2.7), terms of order $a^2 \epsilon^2 / p^2$ and higher will be neglected. Then the part of the integral independent of a gives

$$I_1 = (\theta_i^2 - \theta_j^2)^{-1} [k_0(\xi \theta_j) - k_0(\xi \theta_i) - \ln(\theta_i / \theta_j)],$$

while the other term after a partial integration reduces to

$$I_2 = \frac{i a \epsilon \xi}{4 p (\theta_i^2 - \theta_j^2)} \int \ln^2 \left(\frac{\theta^2 + \theta_j^2}{\theta^2 + \theta_i^2} \right) J_1(\xi \theta) d \theta. \quad (2.8)$$

Now the important range of values of ξ in (2.1) is $\xi \simeq 1 / \theta_c$. Hence with $\theta_i \ll \theta_c < \theta_j$, we write

$$\begin{aligned}
 I_2 &= \frac{i a \epsilon \xi^2}{8 p (\theta_i^2 - \theta_j^2)} \int_0^{\theta_s} \ln^2 \left(\frac{\theta^2 + \theta_j^2}{\theta^2 + \theta_i^2} \right) \theta d \theta \\
 &= \frac{i a \epsilon \xi^2}{16 p (\theta_i^2 - \theta_j^2)} \left[2(\theta_i^2 - \theta_j^2) \sum_{k=1}^{\infty} \frac{1}{K^2} \left(\frac{\theta_i^2 - \theta_j^2}{\theta_i^2} \right)^k - \theta_j^2 \ln^2 \frac{\theta_i^2}{\theta_j^2} \right]. \quad (2.9)
 \end{aligned}$$

Collecting the terms and expanding the Bessel functions, we have to the order of ξ^2

$$\begin{aligned}
 \bar{j}(\xi) - \bar{j}_0 &= 2 \theta_c^2 e^{\pi a \epsilon / p} \left[\frac{1}{8} \xi^2 \sum_{i=1}^3 a_i^2 \exp \left[-2a(\epsilon / p) \tan^{-1} 2 / \theta_i \right] \ln \frac{\gamma^2 \xi^2 \theta_i^2}{4 e} + 2 \operatorname{Re} \sum_{i < j} a_i a_j \left(\frac{\theta_i^2 + 2i \theta_i}{\theta_j^2 - 2i \theta_j} \right)^{i \epsilon / p} \right. \\
 &\quad \times \left. \left[\frac{1}{8} \frac{\xi^2}{\theta_i^2 - \theta_j^2} \left(\theta_i^2 \ln^2 \frac{1}{4} (\gamma^2 \xi^2 \theta_i^2) - \theta_j^2 \ln^2 \frac{1}{4} (\gamma^2 \xi^2 \theta_j^2) \right) - \frac{1}{4} \xi^2 + \frac{i a \epsilon \xi^2}{8 p} \sum_{k=1}^{\infty} \frac{1}{K^2} X_{ij}^k - \frac{i a \epsilon \xi^2}{16 p} \frac{\theta_j^2}{\theta_i^2 - \theta_j^2} \ln^2 \frac{\theta_j^2}{\theta_i^2} \right] \right], \quad (2.10)
 \end{aligned}$$

where $\ln \gamma$ is Euler's constant and $X_{ij} = (\theta_i^2 - \theta_j^2) / \theta_i^2$.¹² The expression in (2.10) can be written in a compact form in terms of a screening angle θ_α defined by

$$\lambda \ln \theta_\alpha = -\frac{1}{2} \lambda - \lim_{\theta_s \rightarrow \infty} \left[\int_0^{\theta_s} q(\theta) d \theta / \theta - \lambda \ln \theta_s \right], \quad (2.11)$$

¹² The values of the sum of the series $M_{ij} = \sum (1/k^2) X_{ij}^k = \sum (1/K^2) [(b_i^2 - b_j^2) / b_i^2]^k$ are given by (for the Molière representation)

$$M_{12} = 1.473104, \quad M_{23} = 1.402476, \quad M_{13} = 1.627226.$$

where

$$\lambda = e^{\pi a \epsilon / p} \operatorname{Re} \sum_i \sum_j a_i a_j \left(\frac{\theta_i^2 + 2i \theta_i}{\theta_j^2 - 2i \theta_j} \right)^{i \epsilon / p}, \quad (2.12)$$

and $q(\theta)$ is given in terms of the scattering cross section $\sigma(\theta)$ as

$$n t \sigma(\theta) = 2 \theta_c^2 q(\theta) / 4(1 - \cos \theta)^2. \quad (2.13)$$

The definition (2.11) of θ_α is very convenient because

we can write (2.10) as

$$\tilde{f}(\xi) - \tilde{f}_0 = \frac{1}{4}\lambda\theta_c^2\xi^2 \ln(\gamma^2\xi^2\theta_c^2/4e), \quad (2.14)$$

which, apart from the factor λ , is of the same form as Molière's expression. The factor λ , however, presents no difficulty, as we can define the new variable η by

$$\eta = \xi\theta_c(\lambda B)^{1/2}, \quad (2.15)$$

where the parameter B satisfies the equation

$$B - \ln B = \ln(e\lambda\theta_c^2/\gamma^2\theta_c^2). \quad (2.16)$$

This renders (2.14) exactly into Molière's form, viz.,

$$\tilde{f}(\eta) - \tilde{f}_0 = \Omega(\eta) - \Omega_0 = -\eta^2/4 + (\eta^2/4B)x \ln(\eta^2/4). \quad (2.17)$$

The reduced scattering distribution is then given by

$$f(\varphi) = \frac{1}{2\pi} \int \eta d\eta J_0(\varphi\eta) \exp[-\eta^2/4 + (\eta^2/4B) \ln(\eta^2/4)], \quad (2.18)$$

where the reduced angular variable is

$$\varphi = \chi/\theta_c B^{1/2} \lambda^{1/2}. \quad (2.19)$$

The exponential factor in (2.18) becomes large positive for large η , thus making the integral diverge if taken to infinity. The exponent is equal to zero for $\eta=0$ and $\eta=2e^{B/2}$ and these will be considered as the limits for the integral. The major contribution, however, comes from the range $\eta=0$ to $\eta \simeq B$. Molière² and Bethe¹³ expanded the exponential factor in inverse powers of B , and integrated the first three terms. The approximation is not very accurate, the value for $\varphi=0$ and $B=7.33$ being 1.2% less than the exact value obtained by numerical integration. In general the exact curve is steeper and smoother than that obtained with the Molière-Bethe expansion (\mathcal{G}_1 in Table II gives the reduced distribution function for $B=8.648$).

In Table I we have compared the results obtained by the present method with the results of other workers. It is seen that with the Molière representation of the Thomas-Fermi potential for gold, the present method gives a value of $\chi_{1/e}$ close to the experimental value. This is of great significance and removes the doubts about the validity of the Thomas-Fermi potential for small-angle scattering. No additional screening is found necessary, while the method of Nigam *et al.* needs an additional screening with the parameter $\mu=1.80$. That the statistical Thomas-Fermi method gives the general feature of the atomic potential is a well-established fact and the present calculations reaffirm this. The agreement is poor in the case of Be and it is worthwhile to compare the results obtained with the Hartree-Fock potential, taking account the modifications due to the crystalline structure of metallic Be. This will be considered in our next paper.

In evaluating $J(\theta)$, we neglected the $\theta^2 g^2$ term. If this term is retained, we shall have the integral \mathcal{G}_3

¹³ H. A. Bethe, Phys. Rev. **89**, 1256 (1953).

TABLE I. Multiple scattering of electrons from gold and beryllium.

Material	Au	Be
Experiment		
ρ_{at} (mg/cm ²)	37.28	491.3
E_{kin} , MeV	15.67	15.24
θ_c , deg	1.539°	1.506°
$\chi_{1/e}$, deg	3.76°	4.25°
B	8.648	10.263
B (Nigam <i>et al.</i> , ^a $\mu=1.00$)	8.131	10.267
B (Nigam <i>et al.</i> , ^a $\mu=1.80$)	6.766	8.954
$\chi_{1/e}$	3.626°	4.507°
$\chi_{1/e}$ (Nigam <i>et al.</i> , ^a $\mu=1.80$)	3.659°	4.179°

^a These are quoted from R. D. Reed [Phys. Rev. **138**, A1000 (1965)], where results for other values of μ are also given.

[Eq. (3.5)] instead of \mathcal{G}_1 [Eq. (3.3)] as the reduced distribution function. As Table II shows, the two integrals are not much different, and the value of $\chi_{1/e}$ calculated from \mathcal{G}_3 is practically the same. The accuracy of the present method is, however, restricted by (a) the limitation of the small-angle approximation and (b) the approximation made in neglecting terms of order $a^2 e^2/p^2$. While the error due to the latter approximation is difficult to estimate consistently, the accuracy of the small-angle approximation may be discussed qualitatively by comparing it with the more exact method of Goudsmit and Saunderson.¹⁴ Lewis¹⁵ and Bethe¹³ have shown the relation between the two methods. In fact, the method of Goudsmit and Saunderson introduces an additional factor in the integrand of (2.18) in our calculations which may safely be neglected. Nigam *et al.*, who followed the method of Goudsmit and Saunderson, also neglected this factor in their final calculations. Scott has also pointed out that the error made by Nigam *et al.* in using the asymptotic formula for the logarithmic derivative of the γ function $\psi(l) \simeq \ln(l + \frac{1}{2})$, down to $l=0$ in evaluating the distribution function is of the same order as the factor the exact method introduces. It is, therefore, sufficient to consider Molière's small-angle approximation which, because of its simplicity, makes it possible to study also the multiple scattering of polarized particles. This will be undertaken in the next section.

III. MULTIPLE SCATTERING OF A POLARIZED ELECTRON BEAM

The scattering of polarized particles is described by the product of the distribution function and the density matrix for the spin. Thus the scattered beam is given by

$$F(\chi, \psi) \rho(\mathfrak{P}') = \frac{1}{2} [F(\chi, \psi) + \pi(\chi, \psi) \cdot \mathfrak{d}], \quad (3.1)$$

where

$$\pi(\chi, \psi) = F(\chi, \psi) \mathfrak{P}', \quad (3.2)$$

¹⁴ S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, A24 (1940); **58**, B36 (1940).

¹⁵ H. W. Lewis, Phys. Rev. **78**, 526, (1950).

TABLE II. Values of the integrals. $B=8.648$.

φ	\mathcal{I}_1	\mathcal{I}_2	\mathcal{I}_3	\mathcal{I}_4	\mathcal{I}_5	\mathcal{I}_6	\mathcal{I}_7
0.0	2.14470	0.00000	2.14664	0.00000	2.14664	2.14886	0.00000
0.4	1.76115	0.38921	1.76279	0.04697	1.76217	1.76404	0.38991
0.8	0.99643	0.59633	0.99746	0.05313	0.99605	0.99721	0.59718
1.2	0.41514	0.61716	0.41566	0.03319	0.41434	0.41491	0.61777
1.6	0.14274	0.55084	0.14299	0.01521	0.14218	0.14244	0.55119
2.0	0.04710	0.47101	0.04722	0.00627	0.04680	0.04693	0.47121
3.0	0.00510	0.32756	0.00513	0.00102	0.00503	0.00507	0.32763

\mathfrak{P}' being the polarization vector of the scattered beam. If $D=0$, a single collision does not change the degree of polarization, i.e., $\mathfrak{P}^2=\mathfrak{P}'^2$. The polarization vector simply rotates around \mathbf{n} , the normal to the scattering plane. One has still to investigate the possibility of a depolarization in multiple scattering due to the fact that particles coming out in a given direction have actually traversed different paths and hence may have their polarization vectors pointing in different directions. The angle of rotation of the polarization vector presents another feature of interest in the theory of multiple scattering.

Mühlschlegel and Koppe have solved Waldmann's equations for $F(\chi, \psi)$ and $\pi(\chi, \psi)$; the solutions are given in an elaborate form by Scott.¹ We shall utilize these solutions but, whereas the earlier workers obtain their results by the Molière-Bethe expansion, we shall express our results in terms of the following seven integrals:

$$\mathcal{I}_1 = \int \eta d\eta \exp[\Omega - \Omega_0] J_0(\varphi\eta), \tag{3.3}$$

$$\mathcal{I}_2 = \int d\eta \exp[\Omega - \Omega_0] J_1(\varphi\eta), \tag{3.4}$$

$$\mathcal{I}_3 = \int \eta d\eta \exp[\Omega - \Omega_0 - \frac{1}{4}s(\Omega'' + \eta^{-1}\Omega')] J_0(\varphi\eta), \tag{3.5}$$

$$\mathcal{I}_4 = \int \eta d\eta \exp[\Omega - \Omega_0 + \frac{1}{4}s\Omega''] \times \{(-s^{1/2}/\omega) \sinh\omega\Omega'\} J_1(\varphi\eta), \tag{3.6}$$

$$\mathcal{I}_5 = \int \eta d\eta \exp[\Omega - \Omega_0 + \frac{1}{4}s\Omega''] \times \{\cosh\omega\Omega' + (s/4\omega\eta) \sinh\omega\Omega'\} J_0(\varphi\eta), \tag{3.7}$$

$$\mathcal{I}_6 = \int \eta d\eta \exp[\Omega - \Omega_0 + \frac{1}{4}s\Omega''] \times \{\cosh\omega\Omega' - (s/4\omega\eta) \sinh\omega\Omega'\} J_0(\varphi\eta), \tag{3.8}$$

$$\mathcal{I}_7 = \int d\eta \exp[\Omega - \Omega_0 + \frac{1}{4}s\Omega''] \times \{\cosh\omega\Omega' - (s/4\omega\eta) \sinh\omega\Omega'\} J_1(\varphi\eta), \tag{3.9}$$

where $\Omega - \Omega_0 = -\eta^2/4 + (\eta^2/4B) \ln(\eta^2/4)$; the primes in-

dicate differentiation by η . The quantity ω is given by

$$\omega = (s + s^2/16\eta^2)^{1/2}, \tag{3.10}$$

where the parameter s is defined in our case by

$$s = [(\epsilon - 1)^2/\epsilon^2] \theta_c^2 \lambda B. \tag{3.11}$$

The values of these integrals, relevant for the case of scattering of 15.67-MeV electrons from gold, cited in Table I, have been evaluated numerically (by an IBM 1620 digital computer) and are given in Table II.

Let us consider a longitudinally polarized electron beam, i.e., \mathfrak{P} along the z axis. Then we shall have

$$F(\varphi, \psi) = (2\pi)^{-1} \mathcal{I}_3, \\ \pi_x \pm i\pi_y = \exp(\pm i\psi) (|\mathfrak{P}|/2\pi) \mathcal{I}_4, \\ \pi_z = (|\mathfrak{P}|/2\pi) \mathcal{I}_5. \tag{3.12}$$

The degree of polarization τ_L is given by $\tau_L^2 = \pi^2/F^2$. In their calculation, Mühlschlegel and Koppe found some depolarization which, as pointed out by Scott, resulted from a mathematical error. Scott proved that $\tau_L^2 = |\mathfrak{P}|^2$ to the order of s^2 ; s was defined there as

$$s = [(\epsilon - 1)^2/\epsilon^2] \theta_c^2 B.$$

Our calculations indicate a very small depolarization and this increases with φ (see Table III).

The angle of rotation of the polarization vector is given by $\theta_L = \sin^{-1}(|\pi_x + i\pi_y|/F)$, which Mühlschlegel and Koppe expressed as

$$\theta_L = s^{1/2} \varphi [1 - sa(\varphi)], \tag{3.13}$$

with

$$a(\varphi) = \frac{1}{1^2} \left[\frac{1 - \varphi \mathcal{I}_2}{\mathcal{I}_1} + \frac{\mathcal{I}_2}{B\varphi \mathcal{I}_1} \right]. \tag{3.14}$$

The results of Mühlschlegel and Koppe are not very

TABLE III. Polarization in multiple scattering. $|\mathfrak{P}|=1$.

φ	τ_L^2	$\tau_T^2 (\psi=0)$	θ_L (rad)	$a(\varphi)$	$\bar{a}(\varphi)$
0.0	1.000000	1.000000	0.00000	0.04367	
0.4	1.000000	1.000000	0.02665	0.04528	0.04505
0.8	1.000000	0.999999	0.05329	0.05094	0.05181
1.2	1.000000	0.999998	0.07993	0.06401	0.06533
1.6	0.999999	0.999996	0.10656	0.09251	0.09686
2.0	0.999998	0.999994	0.13316	0.15077	0.15543
3.0	0.999957	0.999951	0.19932	0.48973	0.63524

accurate. However, if we express our results in the form (3.13) by changing $a(\varphi)$ we get the column for $\bar{a}(\varphi)$ in Table III. The values of $a(\varphi)$ given by (3.14) and evaluated with correct values of g_1 and g_2 are also tabulated there. It may be pointed out that only the transverse component of polarization can be detected experimentally, and $a(\varphi)$, in fact, serves as a measure of this transverse component.

If the incident beam is polarized transversely, with \mathfrak{P} along the x axis, and the scattered beam observed along $(\varphi, 0)$, we have

$$\begin{aligned}\pi_x(\varphi, 0) &= (|\mathfrak{P}|/2\pi)[e^{-s/4B}\varphi^{-1}g_2 + g_6 - \varphi^{-1}g_7], \\ \pi_y &= 0, \\ \pi_z(\varphi, 0) &= (|\mathfrak{P}|/2\pi)g_4.\end{aligned}\quad (3.15)$$

It is easily seen that the angle θ_T , by which the polariza-

tion vector is rotated, is equal to θ_L , discussed above. But for the azimuth $\psi = \pi/2$, we have

$$\begin{aligned}\pi_x(\varphi, \pi/2) &= (|\mathfrak{P}|/2\pi)[e^{-s/4B}g_1 - \varphi^{-1}e^{-s/4B}g_2 + \varphi^{-1}g_7], \\ \pi_y = \pi_z &= 0,\end{aligned}\quad (3.16)$$

there being no rotation of the polarization vector. Thus we see an azimuthal asymmetry in polarization, the angle θ_T decreasing from a maximum of $\theta_T = \theta_L$ at $\psi = 0$ to $\theta_T = 0$ at $\psi = \pi/2$. The degree of polarization τ_T for $\psi = 0$ is also tabulated, and also shows very little depolarization.

ACKNOWLEDGMENT

The author is greatly indebted to Professor B. C. Basu for his help in the computational work.

Mössbauer Spectra of FeMo Alloys*

H. L. MARCUS

Texas Instruments, Attleboro, Massachusetts

AND

L. H. SCHWARTZ

Northwestern University, Evanston, Illinois

(Received 4 May 1967)

Mössbauer spectra of FeMo alloys have been obtained for a series of closely spaced compositions from 2.1 to 6.0 at. % Mo. The hyperfine field and peak widths show a composition dependence suggestive of overlap of the disturbed surroundings of impurity atoms, in accord with results of neutron-diffraction studies of dilute impurities in iron. Analysis of the data in conjunction with the recent unpublished magnetization results of Aldred led to the deduction of a fractional increase in average iron magnetic moment, $(1/\mu_{Fe}) (d\mu_{Fe}/dC)$, of 0.25 per at. % and a magnetic moment per Mo atom of $0.4 \mu_B$ directed antiparallel to the iron moment, the latter value in good agreement with the neutron-diffraction results.

THE Mössbauer spectrum of iron alloys is complex and has been analyzed as the superposition of six-line spectra due to iron atoms in different local environments.¹⁻³ Approximate analyses have been made by assuming random distributions of the impurity atoms which reduce the hyperfine field (hf) of 1st, 2nd, ..., near neighbors by h_1, h_2, \dots , kOe. Wertheim *et al.*¹ represented the results of studies of several alloys by the equation

$$H(n, m) = H_{Fe}(1 + an + bm)(1 + kC), \quad (1)$$

giving the hf, $H(n, m)$, for an Fe atom with n impurity near neighbors and m impurity next-near neighbors in terms of $aH_{Fe} = h_1$, $bH_{Fe} = h_2$, and the constant k for

atomic concentration C . Stearns⁴ has extended this analysis to include the effects of 3rd- and 4th-neighbor impurity effects. Cranshaw *et al.*⁵ have emphasized the shortcomings of this simplistic approach and in their analysis of the spectra of FeSi alloys demonstrated the necessity for corrections due to local order, quadrupole splitting, and higher than first-order dependence on the number of impurity neighbors in a given shell.

In related experiments, Collins and Low⁶ have measured the magnetic-moment distributions around impurities in iron as evidenced in diffuse elastic neutron scattering. For elements to the left of iron in the Periodic Table (V, Mo, etc.), these workers found a small decrease on the moments of first and second neighbors of the impurity with a small increase in moment on the iron atoms at greater distance, the range

* Supported by Advanced Research Projects Agency through the Northwestern University Materials Research Center Contract No. SD-67 and by National Aeronautics and Space Administration through a traineeship to H. L. Marcus.

¹ G. K. Wertheim, V. Jaccarino, J. H. Wernick, and D. N. E. Buchanan, *Phys. Rev. Letters* **12**, 24 (1964).

² M. B. Stearns, *Phys. Rev.* **129**, 1136 (1963).

³ T. E. Cranshaw, *Rev. Mod. Phys.* **36**, 395 (1964).

⁴ M. B. Stearns, *Phys. Rev.* **147**, 439 (1966).

⁵ T. E. Cranshaw, C. E. Johnson, M. S. Ridout, and G. A. Murray, *Phys. Letters* **21**, 481 (1966).

⁶ M. F. Collins and G. G. Low, *Proc. Phys. Soc. (London)* **86**, 535 (1965).