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Theory of Viscosity in Nearly Ferromagnetic Fermi Liquids

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A recent theory of transport processes in nearly ferromagnetic Fermi liquids, based on the concept of particle scattering from persistent spin fluctuations, or paramagnons, is extended to calculate the viscosity coefficient η. The theoretical temperature dependence, in which η falls off increasingly less rapidly than T^{-2} as T is increased from absolute zero, fits well that of the observed viscosity of liquid He^3.

A THEORY of transport processes in nearly ferromagnetic Fermi liquids has been developed recently by the author[1] on the assumption that the predominant particle scattering at low temperatures occurs from persistent spin-density fluctuations, or "paramagnons," of the type discussed by Doniach *et al.*[2] and by Berk and Schrieffer.[3] A Boltzmann equation for this type of scattering, taken to occur between *bare-fermion states*, was set up and solved by the usual variational procedure to obtain expressions for the coefficients of thermal conductivity K and spin diffusion D. The latter were evaluated by using a simple model approximation for the paramagnon spectral density function $A_{\mathbf{q}}(\omega)$ based on the correct random-phase-approximation (RPA) result[2,4] at long wavelengths. In this paper we report an extension of the theory to calculate the viscosity coefficient η. We find that in the low-temperature limit, $T\rightarrow 0$, η varies as T^{-2} in accordance with the prediction of the Landau theory,[5] but varies increasingly less rapidly than T^{-2} as T is increased from zero. Thus the present result for the viscosity is in line with our previous work[1] in which similar departures from the Landau T^{-1} and T^{-2} laws for K and D, respectively, were found. The predicted temperature dependence of η fits well that of the observed low-temperature viscosity[6,7] of liquid He^3.

The theory of fermion transport presented in Ref. 1 (hereafter referred to as I) develops from a Boltzmann equation postulated to hold for the scattering of bare fermions from paramagnons:

$$\frac{\partial f_{\mathbf{k}s}}{\partial t}+\frac{\hbar\mathbf{k}}{m}\cdot\frac{\partial f_{\mathbf{k}s}}{\partial \mathbf{r}}=\left.\frac{\partial f_{\mathbf{k}s}}{\partial t}\right]_{\text{coll}}, \tag{1}$$

where

$$\left.\frac{\partial f_{\mathbf{k}\uparrow,\downarrow}}{\partial t}\right]=\frac{2\pi}{\hbar^2}I^2\sum_{\mathbf{q}}\int_{-\infty}^{\infty}\frac{d\omega}{(2\pi)}A_{\mathbf{q}}(\omega)\delta(\epsilon_{\mathbf{k}\pm\mathbf{q}\downarrow,\uparrow}-\epsilon_{\mathbf{k}\uparrow,\downarrow}\mp\hbar\omega)$$
$$\times\{(1-f_{\mathbf{k}\uparrow,\downarrow})f_{\mathbf{k}\pm\mathbf{q}\downarrow,\uparrow}[n(\omega)+\tfrac{1}{2}\pm\tfrac{1}{2}]$$
$$-f_{\mathbf{k}\uparrow,\downarrow}(1-f_{\mathbf{k}\pm\mathbf{q}\downarrow,\uparrow})[n(\omega)+\tfrac{1}{2}\mp\tfrac{1}{2}]\}. \tag{2}$$

Here $f_{\mathbf{k}s}$ denotes the single-fermion distribution function, defined for bare-fermion states of momenta $\hbar\mathbf{k}$, energy $\epsilon_{\mathbf{k}s}=\hbar^2k^2/2m$, and spin $s(\uparrow$ or $\downarrow)$. m denotes the bare-fermion mass, $A_{\mathbf{q}}(\omega)$ the paramagnon spectral density function, and $n(\omega)$ the Planck function $[\exp(\hbar\omega/k_BT)-1]^{-1}$ involving Boltzmann's constant k_B. As discussed in I the interaction between fermions is taken to be a δ function of strength I.

An expression for the viscosity coefficient can be derived from the above Boltzmann equation by following essentially the same procedure as used by Steinberg[8] in electron-phonon theory. We consider the fermi liquid to be in a steady state of shear motion with constant local velocity v_x in the x direction and a small uniform velocity gradient $a=\partial v_x/\partial y$ in the y direction. The deviation of $f_{\mathbf{k}s}$ from its local-equilibrium value $f_{\mathbf{k}s}{}^0$ is specified by the function $\phi_{\mathbf{k}s}$, linear in a, where $f_{\mathbf{k}s}-f_{\mathbf{k},s}{}^0=-\phi_{\mathbf{k}s}\partial f_{\mathbf{k}s}{}^0/\partial\epsilon_{\mathbf{k}s}$. The Boltzmann equation (1) may be linearized in a to give[8]

$$-\frac{\hbar^2}{m}ak_xk_y\frac{\partial f_{\mathbf{k}s}{}^0}{\partial\epsilon_{\mathbf{k}s}}=\frac{1}{k_BT}\int\frac{d^3k'}{(2\pi)^3}V(\mathbf{k}'s';\mathbf{k}s)\{\phi_{\mathbf{k}'s'}-\phi_{\mathbf{k}s}\}, \qquad (s\neq s') \tag{3}$$

where $V(\mathbf{k}'s';\mathbf{k}s)$ is given by the relations

$$V(\mathbf{k}'s';\mathbf{k}s)=V(\mathbf{k}s;\mathbf{k}'s'), \tag{4}$$

$$V(\mathbf{k}'\downarrow;\mathbf{k}\uparrow)=\frac{(2\pi)^3I^2}{\hbar}\sum_{\mathbf{q}}\int_{-\infty}^{\infty}d\omega\, n(\omega)f_{\mathbf{k}\uparrow}{}^0(1-f_{\mathbf{k}'\downarrow}{}^0)A_{\mathbf{q}}(\omega)$$
$$\times\delta(\epsilon_{\mathbf{k}'\downarrow}-\epsilon_{\mathbf{k}\uparrow}-\hbar\omega)\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q}). \tag{5}$$

The left-hand side of (3), when multiplied by $\phi_{\mathbf{k}s}$, integrated over $\mathbf{k}$, and summed over s, is just a times

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the mean nonequilibrium momentum flux T_{xy},

$$T_{xy}=-\sum_s \int \frac{d^3k}{(2\pi)^3}\hbar k_x\left(\frac{\hbar k_y}{m}\right)\phi_{\mathbf{k}s}\frac{\partial f_{\mathbf{k}s}{}^0}{\partial\epsilon_{\mathbf{k}s}}. \tag{6}$$

Bearing in mind the symmetry property (4) and that the shear viscosity coefficient is defined as $-T_{xy}/a$ it follows from (3) and (6) that the viscosity coefficient is given by

$$\frac{1}{\eta}=\frac{(1/k_BT)\int\int d^3kd^3k'V(\mathbf{k}'\downarrow;\mathbf{k}\uparrow)\{\phi_{\mathbf{k}'\uparrow}-\phi_{\mathbf{k}\uparrow}\}^2}{\{(2\hbar^2/m)\int d^3k\, k_xk_y\phi_{\mathbf{k}\uparrow}(\partial f_{\mathbf{k}\uparrow}{}^0/\partial\epsilon_{\mathbf{k}\uparrow})\}^2}, \tag{7}$$

where we have simplified a more general expression than (7) by using the fact that $\phi_{\mathbf{k}\uparrow}=\phi_{\mathbf{k}\downarrow}$. In order to calculate (7) we need to know $\phi_{\mathbf{k}s}$ and $A_{\mathbf{q}}(\omega)$. From (3), $\phi_{\mathbf{k}s}$ has the form $\phi_{\mathbf{k}s}=\psi(|\mathbf{k}|)k_xk_y$. As a first approximation to $\phi_{\mathbf{k}s}$ we shall take $\psi(|\mathbf{k}|)=\psi(k_f)=\text{constant}$, where k_f denotes the Fermi wave vector. For the spectral density function $A_{\mathbf{q}}(\omega)$, we employ the model proposed in I:

$$\begin{aligned} A_{\mathbf{q}}(\omega)&=\alpha_0\omega/q && (0\leq|\omega|\leq\omega_q)\\ &=0, && (|\omega|>\omega_q)\\ \hbar\omega_q&=\frac{4}{\pi}\frac{\kappa_0{}^2\epsilon_f}{\bar{I}}\frac{q}{k_f} && (0\leq q\leq Q)\\ &=0, && (q>Q)\\ \alpha_0&=\pi\mathfrak{N}(\epsilon_f)\hbar^2k_f/2\kappa_0{}^4\epsilon_f. \end{aligned} \tag{8}$$

In (8) ϵ_f denotes the Fermi energy, $\mathfrak{N}(\epsilon_f)$, the single-particle density of states $\sum_{\mathbf{k}}\delta(\epsilon_{\mathbf{k}s}-\epsilon)$ and $\bar{I}\equiv\mathfrak{N}(\epsilon_f)I$, and $1/\kappa_0{}^2\equiv1/(1-\bar{I})$, the RPA exchange enhancement factor[2,4] for the interacting Fermi liquid. Q is a cutoff vector of order k_f. As discussed in I we believe (8) represents the essential qualitative structure of $A_{\mathbf{q}}(\omega)$, i.e., the linear dependence on ω for $|\omega|<\omega_q$, and the strong peaking of $A_{\mathbf{q}}(\omega)$ about $|\omega|=\omega_q$ as $\kappa_0{}^2\to0$. The resulting integrals encountered in (7) are similar to those met in the calculation of D and K in I and may be evaluated by following the methods employed there. Neglecting terms of order $(T/T_f)^2$ relative to others, where $T_f\equiv\epsilon_f/k_B$ denotes the Fermi temperature of the noninteracting Fermi gas, we obtain the result

$$1/\eta T^2=\alpha w(T/\theta), \tag{9}$$

where α is a constant,

$$\alpha=\frac{15\pi^3}{8}\frac{\bar{I}^2\bar{Q}^3W(0)}{\hbar\kappa_0{}^4T_f{}^2k_f{}^3}, \tag{10}$$

and θ is a characteristic paramagnon temperature,

$$\theta=\frac{4}{\pi}\frac{\kappa_0{}^2\bar{Q}}{\bar{I}}T_f, \tag{11}$$

with $\bar{Q}\equiv Q/k_f$. The function $w(t)$ is defined by the relations

$$\begin{aligned} w(t)&=W(t)/W(0),\\ W(t)&=\left(1-\frac{3\bar{Q}^2}{20}\right)J_2\left(\frac{1}{t}\right)-t^3\left[J_5\left(\frac{1}{t}\right)-\frac{3\bar{Q}^2}{20}t^2J_7\left(\frac{1}{t}\right)\right],\\ J_\nu(y)&=\int_0^y\frac{dxx^\nu}{(e^x-1)(1-e^{-x})}, \qquad (\nu\geq2). \end{aligned} \tag{12}$$

The above results are similar in nature to those obtained in I for the thermal conductivity and spin diffusion. For small (T/θ) the right-hand side of Eq. (9) may be expanded in powers of (T/θ) to give a leading contribution at low temperatures of the form

$$\frac{1}{\eta T^2}=\alpha-\beta\frac{T^3}{\theta^3}+O\left[\frac{T^5}{\theta^5}\right], \qquad (T\ll\theta) \tag{13}$$

where, by (12),

$$\frac{\alpha}{\beta}=\left(1-\frac{3\bar{Q}^2}{20}\right)\frac{J_2(\infty)}{J_5(\infty)}.$$

Thus as $T\to0$, η varies as T^{-2} in agreement with Landau Fermi-liquid theory.[5] The qualitative point we wish to make here, however, is that as the Fermi liquid approaches the ferromagnetic instability, i.e., as $\kappa_0{}^2\to0$, so that $\theta\to0$, the range of applicability of Landau theory rapidly diminishes. We note from (13) that for a fixed value of θ (equivalently, a given pressure) the deviation from the T^{-2} viscosity law falls off as T^3 for small (T/θ). By contrast the low-temperature deviations from the Landau T^{-2} and T^{-1} laws for the spin diffusion D and thermal conductivity K, respectively, calculated in I, are linear in T. Therefore, as $T\to0$, we expect the viscosity to approach a "Landau region" much faster than do either K or D.

It may be seen from (12) that once the cutoff $\bar{Q}$ is fixed $w(t)$ is a function of the variable t only. We may

TABLE I. Computed values of the function w defined by Eq. (12) for $\bar{Q}=1.4$.

T/θ	$w(T/\theta)$
0.00	1.000
0.02	0.999
0.04	0.997
0.06	0.989
0.08	0.975
0.10	0.953
0.20	0.779
0.30	0.612
0.40	0.491
0.50	0.407
0.60	0.346
0.70	0.300
0.80	0.265
0.90	0.237
1.00	0.215

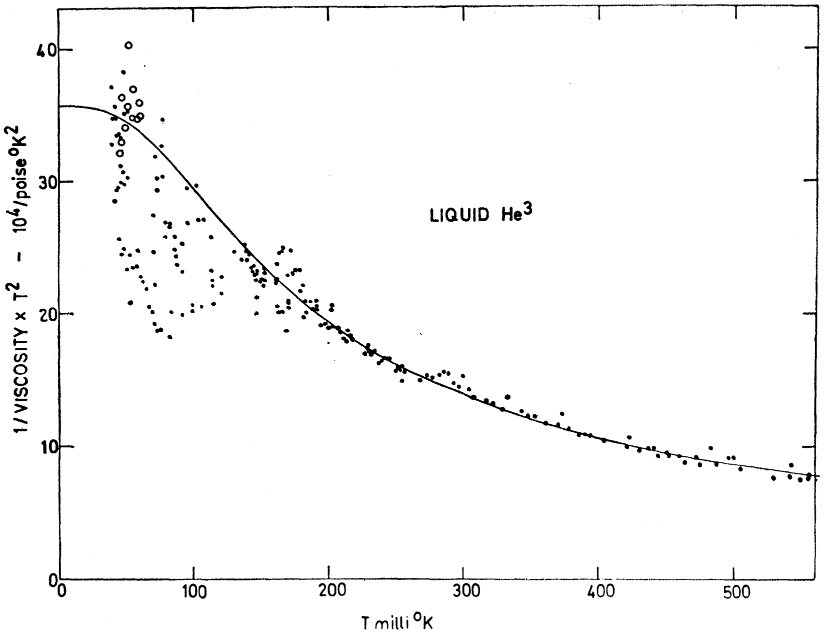

FIG. 1. $1/\eta T^2$ versus T for liquid He^3 at low pressure. The small full points are the combined saturated-vapor-pressure data of Betts, Osborne, Welber, and Wilks (Ref. 5) and Betts, Keen, and Wilks (Ref. 6). The open circles denote data of Abel, Anderson, and Wheatley (Ref. 10). The full line is the theoretical fit.

then regard Eq. (9) as a formula giving the viscosity in terms of the two parameters α and θ. In applying the theory to liquid He^3, $\bar{Q}$ and κ_0^2 may be fixed from the observed paramagnetic susceptibility and specific heat in the manner described in I. This procedure yields $\bar{Q}\simeq 1.4$ and $\kappa_0^2\simeq\frac{1}{9}$. In Table I we have computed the function $w(t)$ in the range $0\leq t\leq 1$ for this value of $\bar{Q}$. Using the table, we have attempted to fit Eq. (1) to the low-temperature data on the viscosity of liquid He^3, obtained by Betts, Osborne, Welber, and Wilks[6] and Betts, Keen, and Wilks[7] under the saturation vapor pressure. Because of the uncertainty in the measurements below about 120 m°K it was not possible to fix α by attempting to identify a low-temperature trend of the form $1/\eta T^2=\alpha$. However, excellent fit to the data above 120 m°K was obtained by taking $\alpha=0.36$ μP^{-1} $°K^{-2}$ and $\theta=0.56°K$. This is shown in Fig. 1. The latter choice of α is supported by independent low-temperature measurements (40–60-m°K range), also shown in Fig. 1, made by Abel, Anderson, and Wheatley.[9] The fitted values, $\theta=0.56°K$ and $\alpha=0.36$ μP^{-1} $°K^{-2}$, compare with $\theta\simeq 1.1°K$ and $\alpha\simeq 2$ μP^{-1} $°K^{-2}$ obtained by using the low-pressure estimates $\bar{Q}\simeq 1.4$ and $\kappa_0^2\simeq\frac{1}{9}$ in the theoretical formulas (11) and (10), respectively. Because of our use of the simple trial function, $\psi=$ constant, the latter value of α only represents an upper bound[10] for the theory.[11] The fitted value of $\theta=0.56°K$ is consistent with the value $\theta=0.50°K$ used in fitting the low-pressure diffusion data[12] in I, although somewhat lower than that, close to the theoretical estimate $\theta\simeq 1.1°K$, used in fitting the corresponding thermal-conductivity data.[13] The variation in these fitted values of θ, relative to the calculated value of θ, probably reflects the degree of inadequacy of our model (8) to treat a particular transport coefficient.

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