Note added in proof. Recent experimental results[14] indicate that the rate for $\eta \rightarrow \pi^0\gamma\gamma$ computed in VMD model is too small by an order of magnitude. This may be due to the fact that the VMD amplitude involves higher powers of momenta than are required by gauge invariance.[15] Nevertheless, since the amplitude given by pure S-wave coupling ($F_{\mu\nu}F_{\mu\nu}\phi_\eta\phi_\pi$) for $\eta \rightarrow \pi^0 e^+e^-$ is proportional to electron mass, the vector-meson intermediate states will still be dominant for this decay process. If we assume pure S-wave coupling for $\eta \rightarrow \pi^0\gamma\gamma$ and VMD for $\eta \rightarrow \pi^0 e^+e^-$, the branching ratio will then be lowered accordingly by an order of magnitude to $\approx 10^{-6}$.

[14] See, for example, M. Feldman, W. Frati, R. Gleeson, J. Halpern, M. Nussbaum, and S. Richert, Phys. Rev. Letters 18, 868 (1967).

[15] This was pointed out to me by Professor R. H. Dalitz in a private conversation.

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High-Energy Sum Rule for Kp Scattering*

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A sum rule of the superconvergent type is obtained for forward, elastic Kp scattering. It connects Regge parameters with an integral over total cross sections. A numerical evaluation is carried out and the Regge parameters thus determined are compared to those obtained from a fit to the high-energy data. Impressive agreement is found.

It was recently pointed out[1] that if an amplitude $f(\nu)$ decreases faster than ν^{-1} at high energies, it satisfies a superconvergent relation

$$\int_{-\infty}^{\infty} d\nu \, \mathrm{Im} f(\nu) = 0. \tag{1}$$

Subsequently, Logunov *et al.*[2] and Igi and Matsuda[3] have shown that it is possible to write down an analog of relation (1) for an amplitude which is not convergent but whose high-energy behavior is given. Briefly, the procedure consists in writing the given amplitude as a sum of two pieces, one of which is nonconvergent and the other is convergent enough to satisfy the condition of superconvergence. One then writes a superconvergence relation for this latter piece which, of course, is the difference of the given amplitude and its (known) nonconvergent piece. These authors applied the above procedure to the case of forward pion nucleon scattering and obtained a striking agreement with experiment. For a further examination of the underlying assumptions and usefulness of this procedure, we have analyzed the case of forward Kp scattering. We present here an account of this calculation. We find that the results obtained from our sum rule agree very well with the experimental data available at present.

We consider the forward $K^{\pm}p$ elastic scattering amplitude $f^{\pm}(\nu)$ defined as [4]

$$f^{(\pm)}(\nu) = \frac{1}{4\pi}[A(K^{\pm}p) + \nu B(K^{\pm}p)]. \tag{2}$$

Since we need a crossing even absorptive part of an amplitude for a superconvergent relation, we introduce the combination

$$F_{-}(\nu) = \tfrac{1}{2}[f^{(-)}(\nu) - f^{(+)}(\nu)]. \tag{3}$$

The absorptive part of this amplitude satisfies the crossing property

$$\mathrm{Im} F_{-}(-\nu) = +\mathrm{Im} F_{-}(\nu).$$

Following the method described above, we decompose $F_{-}(\nu)$ as a sum of a nonconvergent part and a convergent one. For simplicity, we assume that the nonconvergent part can be represented by a sum of Regge-pole terms. We have

$$F_{-}(\nu) \equiv \sum_i F^{(i)}(\nu) + \epsilon_{-}(\nu), \tag{4}$$

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[3] K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967).

[4] The amplitudes A and B are defined in G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957). We follow here the notations of this article.

where we define

$$F^{(i)}(\nu)\equiv-\beta_i\frac{P_{\alpha_i}(-\nu/m_K)-P_{\alpha_i}(+\nu/m_K)}{2\sin\pi\alpha_i}. \quad (5)$$

The index i runs over ρ, ω, and ϕ.

We assume that the function $\epsilon_-(\nu)$ satisfies

$$\epsilon_-(\nu)<\nu^{-1} \quad \text{as} \quad \nu\to\infty. \quad (6)$$

We have the dispersion relation

$$\epsilon_-(\nu)=\sum_j R_j\left(\frac{1}{\nu_j-\nu}-\frac{1}{\nu_j+\nu}\right)+\frac{1}{\pi}\int_{m_K}^{\infty}d\nu'\,\mathrm{Im}\epsilon_-(\nu')\left(\frac{1}{\nu'-\nu}-\frac{1}{\nu'+\nu}\right), \quad (7)$$

where j runs over Σ^0, Λ, $Y_1^*(1385)$, and $Y_0^*(1405)$ and

$$\nu_j=(M_j^2-M^2-m_K^2)/2M.$$

Here M is the proton mass.

Multiplying (7) with ν, taking the limit $\nu\to\infty$, and making use of Eqs. (4) and (5), we obtain the sum rule

$$8\pi^2\sum_j R_j+\int_{m_K}^{\infty}d\nu\,\{(\nu^2-m_K^2)^{1/2}[\sigma^{(-)}(\nu)-\sigma^{(+)}(\nu)]-4\pi\sum_i\beta_iP_{\alpha_i}(\nu/m_K)\}=0, \quad (8)$$

where $\sigma^{(\pm)}$ refer to the total cross sections for $K^\pm p$ scattering. The residues R_j are given below[5]:

$$R_\Sigma=(g_{\Sigma Kp}^2/4\pi)X_-(\Sigma), \quad R_\Lambda=(g_{\Lambda Kp}^2/4\pi)X_-(\Lambda),$$
$$R_{Y_0^*}=(h_0^2/4\pi)X_+(Y_0^*),$$
$$R_{Y_1^*}=(h_1^2/4\pi)32/3(M/M_{Y_1^*})^2X_-(Y_1^*)X_+^2(Y_1^*),$$

where

$$X_\pm(j)=(1/8M^2)[(M_j\pm M)^2-m_K^2].$$

To check the validity of the sum rule (8), we proceed as follows. We determine the value of β_i from the experimental data for $K^\pm p$ total cross sections[6] at 6 BeV/c and higher assuming that the Regge asymptotic behavior for the amplitude $F_-(\nu)$ is already established at 6 BeV/c. We then compare this value of β_i to the ones obtained from the sum rule (8).

Let us estimate the various terms in (8).

To determine $g_{\Sigma Kp}$ and $g_{\Lambda Kp}$, we use $SU(3)$ symmetry which gives

$$g_{\Sigma Kp}=(1-2f)g_{NN\pi} \quad \text{and} \quad g_{\Lambda Kp}=-(1/\sqrt{3})(1+2f)g_{NN\pi}.$$

Here $g_{NN\pi}^2/4\pi=14.5$. The value of f is not precisely known but various estimates put it in the neighborhood of 0.3. The coupling constants h_0 and h_1, according to the estimates of Frye and Warnock,[7] are $h_0^2/4\pi=0.32$, $h_1^2/4\pi=1.9$.

So much for the Born terms. Turning to the evaluation of the integral in (8), we remark that since we assume that the Regge behavior for the amplitude $F_-(\nu)$ is already established by 6 BeV/c, we take this energy as the upper limit of integration. To carry out the integration over cross sections, we used the data in Refs. 8 and 9 for K^-p and K^+p, respectively. Integration over the second term is carried out analytically.[10]

To keep the number of parameters to a minimum, we first consider the case of $i=\rho$ only. From the analysis of πN scattering, the value of α_ρ is determined[2,3,11] to be roughly 0.54~0.56. Using a typical value of $\alpha_\rho=0.54$ we find[10] from the sum rule (8) that $4\pi\beta_\rho=12.9(1/m_K)$. To ascertain the cutoff independence of this result, we took the upper limit of integration as 18 BeV/c and evaluated the integral over cross sections using the data of Ref. 6. In this case, we find that $4\pi\beta_\rho=12.8(1/m_K)$. The values of β_ρ quoted above are obtained with the choice of $f=0.3$. However, since the Born contribution is rather small compared to the integral over cross sections, the value of β_ρ obtained from the sum rule (8) changes only slightly[12] when we change the value of f say to 0.2 or 0.4. Therefore, from now on we shall consider $f=0.3$ only.

The value of β_ρ obtained from the sum rule is to be compared with the value calculated by a Regge fit to the total-cross-section data above 6 BeV/c. For this fit, we employ the formula

$$4\pi\beta_\rho=m_K[\sigma^{(-)}(\nu)-\sigma^{(+)}(\nu)](\nu/m_K)^{1-\alpha_\rho}. \quad (9)$$

Using (9), we obtain $4\pi\beta_\rho=(12.9\pm0.3)(1/m_K)$, with $\alpha_\rho=0.54$. The error quoted is statistical. Within the

[5] For the (Y_1^*Kp) and (Y_0^*Kp) vertices we assume the following effective Hamiltonians:

$$H_1=(h_1/M)(\bar{Y}_1^*)_\mu p\partial_\mu K+\text{H.c.};$$
$$H_0=h_0\bar{Y}_0^*pK+\text{H.c.}$$

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[10] To simplify the numerical calculations, we have used instead of $P_\alpha(\nu)$, the simple power ν^α that has the right high-energy behavior. See also D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127 (unpublished).

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[12] In particular, we find that with a cutoff of 6 BeV/c, the value of $4\pi\beta_\rho$ from the sum rule is $12.9(1/m_K)$ and $12.7(1/m_K)$ corresponding to $f=0.2$ and 0.4, respectively.

TABLE I. The values of $4\pi\beta_\rho$ (in $1/m_K$ units) calculated from the sum rule (8) with the cutoff kaon laboratory momentum of 6 BeV/c and from the Regge fit (9). Each of the latter values of $4\pi\beta_\rho$ carries a statistical error of ± 0.3.

α_ρ	$4\pi\beta$ (Regge fit)	$4\pi\beta$ (sum rule)
0.52	13.7	13.4
0.53	13.3	13.1
0.54	12.9	12.9
0.55	12.5	12.6
0.56	12.1	12.4
0.57	11.7	12.2
0.58	11.4	11.9
0.59	11.0	11.7
0.60	10.7	11.5

accuracy of experimental data, the agreement between the values of β_ρ obtained from (8) and (9) with the choice of $\alpha_\rho = 0.54$ is excellent.

We have repeated this calculation of $4\pi\beta_\rho$ with slightly different values of α_ρ. The results are given in Table I.

It seems, therefore, that the sum rule (8) is satisfied by considering only one Regge term which we called the ρ term with $\alpha_\rho \sim 0.54$. In addition, however, we may have Regge terms corresponding to ω and ϕ. If $\alpha_\rho(0)$, $\alpha_\omega(0)$, and $\alpha_\phi(0)$ have approximately the same value, then ω and ϕ pole terms are automatically included in what we have called the ρ term.[13] In any case, once the values of α_ρ, α_ω, and α_ϕ are given from any phenomenological or theoretical analysis,[14] the sum rule (8) provides an independent test for the correctness of these values.

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[13] Actually β_ρ, β_ω, and β_ϕ can be related by unitary symmetry:

$$\beta_\rho : \beta_\omega : \beta_\phi = \tfrac{1}{2} : \tfrac{1}{2} : (2f' - 1),$$

where $(1+f')/f'$ is the D/F ratio for $(V\bar{N}N)$ vertex. We have used a nonet of vector mesons; see S. Okubo [Phys. Letters **5**, 165 (1963)]. If α_ρ, α_ω, and α_ϕ have a common value, we get $\beta_\rho + \beta_\omega + \beta_\phi = \beta$, where the values of β are given in Table I. We may now evaluate β_ρ, β_ω, and β_ϕ separately and obtain $\beta_\rho = \beta_\omega = \beta/4f'$ and $\beta_\phi = (2f'-1)\beta/2f'$. The value of f' is believed to be close to unity. See R. F. Sawyer, Phys. Rev. Letters **14**, 471 (1965).

[14] For instance, in the Regge-pole analysis of Kn scattering by Rarita and Philips (Ref. 11), the best fit is obtained with $\alpha_\rho = 0.54$ and $\alpha_\omega = 0.52$. For practical purpose, therefore, we may consider α_ρ and α_ω to have a degenerate value so that, in view of our preceding remarks, we may regard this set of values for α's as consistent with the sum rule (8).

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Normalization of Bethe-Salpeter Amplitudes*†

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Many different methods for normalizing Bethe-Salpeter amplitudes for bound states have been known. In this article, derivations of the normalization condition based on conservation of charge and of energy-momentum are studied in detail, and it is shown that the final result in both cases is identical with the one derived from other methods.

I. INTRODUCTION

In studying Bethe-Salpeter (BS) wave functions or amplitudes for bound states,[1–4] one of the fundamental problems is how to normalize them, since the BS amplitudes are not directly interpretable as being probability amplitudes. This problem is rather old and has been studied extensively in the literature, and there are, roughly speaking, two distinctive kinds of solutions. In earlier papers[5–7] this problem was solved by expressing the expectation value of a current, such as the electric or baryonic current, in terms of the BS amplitudes and putting it equal to the conserved quantity which is known *a priori*. In later papers,[8–11] methods were developed of normalizing the BS amplitudes without using conserved quantities.

In this paper we show that these two distinctive kinds of methods lead to identical results. It has been claimed that the electric or baryonic current cannot always be used for normalizing the BS amplitudes.[10] This problem

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