plays the same role as Regge poles in determining the asymptotic behavior of the scattering amplitudes. Thus, if $K(u^x,\tau)$ has the form $\tau/(\tau-\tau_0-u^x)$, $0<\tau_0<1$, then in the asymptotic limit $s^x\to\infty$, $u^x\approx 0$, the amplitude is deminated by the Regge-pole contribution

$$(u^x)^{2N-\tau_0}(s^x)^{\tau_0}.$$

If the scattering particles (*external lines*) have different mass, then this is modified. Thus, if $m_1=m_2'=m$ and $m_1'=m_2=\mu$, then the first factor becomes

$$[u^x-(m^2-\mu^2)^2/2s^x]^{2N-\tau_0}.$$

ACKNOWLEDGMENTS

We gratefully acknowledge fruitful conversations with Y. Nambu. The discovery, by Y. Nambu and P. Rosen, of the representation (5) of $U(3,1)$ was crucial to this work.

PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Some Features of Angular-Momentum Branch Points*

JOHN H. SCHWARZ
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 21 March 1967; revised manuscript received 14 June 1967)

Arguments for the existence of angular-momentum branch points based on the necessity for the Gribov-Pomeranchuk singularities to be fixed poles rather than essential singularities are reviewed. Sum rules for scattering amplitudes on the second sheet are then deduced. Reasons are given for expecting two kinds of branch points to be present, namely, those described as "Regge pole plus Regge pole" and "Regge pole plus elementary particle" (called types 1 and 2, respectively). It is argued that the latter must be concealed by the former in the scattering region, and from the requirement that the branch points are suitably positioned in general, an inequality on derivatives of Regge-pole trajectories is derived. A model of the Amati–Fubini–Stanghellini type is examined to indicate why type-2 branch points may be expected to occur in a theory without elementary particles.

1. INTRODUCTION

SEVERAL features of branch points in the angular-momentum plane for two-body scattering amplitudes are discussed in this paper. Recently, there has been significant progress[1] in understanding the way in which moving branch points eliminate the once-suspected need for essential singularities[2] at wrong-signature nonsense points and allow fixed poles to be present instead. We shall assume this mechanism to be generally valid and utilize it to make several deductions about the branch points. We emphasize the theme that while the branch points are extremely difficult to discuss in terms of Feynman diagrams, there is a great deal that can be learned about them from a liberal application of unitarity without getting involved with overwhelming complexities. Although not yet at the point of being able to present reliable formulas for their contributions to asymptotic behavior, we believe that there are grounds for optimism in this regard.

In Sec. 2 we review the arguments for the existence of angular-momentum branch points and the inferences that are drawn for the behavior of partial-wave amplitudes at wrong-signature nonsense points. Also, a sum rule for amplitudes on the second sheet of the elastic cut is deduced. In Sec. 3 we explain why two essentially different types of branch points are required, and indicate some interesting features of their sheet structure. In order to satisfy a requirement on the positioning of the two types of branch points in general, we conjecture an inequality involving derivatives that should be satisfied by *any* Regge-pole trajectory function. The structure of the branch points is explored further and illustrated in a simple model in Sec. 4. The model consists of an integral of the AFS type,[3] and while its relevance[4] may be questioned, it nevertheless offers an instructive example. Section 5 contains some comments about possible generalizations and applications and summarizes the conclusions.

2. NEED FOR BRANCH POINTS AND RESULTING SUM RULES

The existence of moving branch points in angular momentum can be most easily demonstrated by arguments given originally by Mandelstam.[5] Simply stated, the

* Work supported by the U. S. Air Force Office of Research, Air Research and Development Command under Contract No. AF49(638)-1545.

[1] C. E. Jones and V. Teplitz, Phys. Rev. **159**, 1271 (1967); S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).

[2] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962); Ya. I. Azimov, Phys. Letters **3**, 195 (1963); S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

[3] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962), hereafter AFS; Nuovo Cimento **26**, 896 (1962).

[4] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); H. J. Rothe, Phys. Rev. **159**, 1471 (1967). Most of the conclusions of Sec. 4 are also reached in Rothe's paper.

[5] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).

idea is that in the absence of moving branch points, the arguments of Gribov and Pomeranchuk for essential singularities at wrong-signature nonsense points would be valid, since they depend only on analytically continued elastic unitarity and noncontroversial aspects of the Mandelstam representation. Such a picture is unsatisfactory, however, because it cannot be reconciled with the Froissart limit[6] (which is a consequence of unitarity and very weak analyticity assumptions) when there is sufficient helicity for nonsense points to occur above $J=1$. Mandelstam showed for a certain class of diagrams having third double-spectral functions that there is a moving branch point and that the Gribov-Pomeranchuk singularity does not contribute to the asymptotic behavior.

The mechanism by which moving branch points enable the Gribov-Pomeranchuk singularities not to appear in asymptotic behavior has been clarified recently.[1] The critical step in the reasoning of Gribov and Pomeranchuk that fails is their claim that the existence of a pole in the left-hand-cut discontinuity of the analytically continued partial-wave amplitude generates an essential singularity (consisting of an accumulation of moving Regge poles) as a consequence of unitarity. They did not consider the possibility that the region in which elastic unitarity holds shrinks to a point as J is decreased to the nonsense point in question. In other words, the existence of a moving cut, with trajectory given by $J=\alpha_c(t)$, and having the property that $\alpha_c(t_0)=J_N$, where t_0 is the normal threshold and J_N is the nonsense point, can make it possible for a fixed pole in J, located at $J=J_N$, to be consistent with the analytically continued elastic-unitarity equation

$$b^{\pm}(J,t)-b^{\pm}(J,t_{\mathrm{II}})=2i\rho(J,t)b^{\pm}(J,t)b^{\pm}(J,t_{\mathrm{II}}). \quad (1)$$

The branch points studied by Mandelstam possess the necessary properties for this to be the case. What happens is that $b^{\pm}(J,t)$ has the fixed pole while $b^{\pm}(J,t_{\mathrm{II}})$, the amplitude contained to the corresponding point on the second sheet of the elastic-unitarity cut, takes a determined finite value at $J=J_N$. It is consistent for $b(J,t_{\mathrm{II}})$ not to be an analytic continuation of $b(J,t)$ at $J=J_N$ if the cut through which one continues to reach the second sheet shrinks to a point as J approaches J_N. Furthermore, since fixed poles at wrong-signature nonsense points are easily shown not to contribute to asymptotic behavior, only the contributions of the moving poles and cuts remain. In fact there does not appear to be any obstacle, in principle, to deforming the background-integral contour in the Mandelstam-Sommerfeld-Watson representation[7] arbitrarily far into the left half-plane.

While the above reasoning rigorously establishes the existence of fixed poles and the nonexistence of essential singularities as a consequence of branch points only for wrong-signature nonsense points above the Froissart limit, there seems to be little question of its general validity. Therefore, we assume the presence of branch points passing through nonsense points at energies corresponding to normal thresholds and the absence of essential singularities.

In a recent paper[8] we have explored the possibility that the fixed poles may have weak residues, citing as evidence the experimental observation of dips in certain differential cross sections, thereby obtaining sum rules of the form

$$\int_{s_0}^{\infty} \mathrm{Im}T(s,t)ds\approx 0. \quad (2)$$

Equation (2) resembles the usual superconvergence formulas,[9] but it pertains to those cases having the wrong crossing symmetry to give a superconvergence formula. It is interesting to observe that (2) can be reexpressed in an exact version. The key observation is that the second-sheet function $b^{\pm}(J,t_{\mathrm{II}})$ does not contain the fixed pole at the nonsense point, as indicated above. Therefore, by setting equal to zero the residue of the apparent fixed pole given by the Froissart-Gribov formula, one obtains

$$\int_{s_0}^{\infty} \mathrm{Im}T(s,t_{\mathrm{II}})ds=0, \quad (3)$$

where

$$\mathrm{Im}T(s,t_{\mathrm{II}})=\mathrm{Im}T(s,t)-2i\rho_{st}^{\mathrm{el}}(s,t).$$

$\rho_{st}^{\mathrm{el}}(s,t)$ is the elastic part (with respect to t) of the double-spectral function, analytically continued in t. Equation (3) should be valid whenever the integral converges. If the moving branch point is the leading singularity controlling the asymptotic behavior of $\mathrm{Im}T(s,t_{\mathrm{II}})$, the integral would converge for t below threshold. However, we should point out in this connection that in all probability a moving pole (Regge pole) on the second-sheet asymptotes to the nonsense point at infinite t.[10] It remains to be clarified what bearing, if any, this pole will have on the convergence of (3).[11]

3. TWO TYPES OF BRANCH POINTS

Let us consider the elastic scattering of particles A and B, having masses m_A and m_B, spins J_A and J_B, and lying on Regge trajectories $\alpha_A(t)$ and $\alpha_B(t)$, respectively. An analytically continued partial-wave amplitude for this scattering contains a moving branch point

[6] M. Froissart, Phys. Rev. **123**, 1053 (1961); Y. Hara, Phys. Rev. **136**, B507 (1964).

[7] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).

[8] J. Schwarz, Phys. Rev. **159**, 1269 (1967).

[9] V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters **21**, 576 (1966).

[10] I am grateful to V. Teplitz for bringing this point to my attention.

[11] It may be worth noting that by saturating with resonances alone one does not discern any difference between (2) and (3), because the pole contributions to the two of them are the same.

due to exchange of the two Regge poles A and B, in particular. To give formulas for its location, it is helpful to express the trajectories as functions of $x=t^{1/2}$ (even functions for boson trajectories) and to define $\alpha_{AB}(x,y)$ by

$$\alpha_{AB}(x,y)=\alpha_A(x-y)+\alpha_B(y)-1 \quad (4)$$

and $y=f(x)$ as a solution of the equation

$$\frac{\partial}{\partial y}\alpha_{AB}(x,y)=0. \quad (5)$$

Then the branch-point trajectory is given by

$$\alpha_{AB}{}^{(1)}(x)=\alpha_{AB}(x,f(x)). \quad (6)$$

The function $\alpha_{AB}{}^{(1)}(x)$ is symmetrical in A and B.

In the case that A and B are the same trajectory, a solution of (5) is $y=\frac{1}{2}x$, so that

$$\alpha_{AA}{}^{(1)}(x)=2\alpha_A(x/2)-1. \quad (7)$$

Thus at the AA threshold the branch point has the property stipulated in Sec. 2 to be the crucial one. However, this desirable circumstance no longer holds at a threshold corresponding to two different particles. One can readily verify that the condition $\alpha_{AB}{}^{(1)}(m_A+m_B)=J_A+J_B-1$ implies $\alpha_A'(m_A)=\alpha_B'(m_B)$. The latter condition surely fails in general for many reasons. In particular, if it were interpreted to imply that trajectories are linear in energy, it would require the mass of the first recurrence of the $\Delta(1236)$ to equal the average of the masses of the $\Delta(1236)$ and its second recurrence, which is not the case. It follows, in view of the considerations of Sec. 2, that additional branch points are required.

In his paper on branch points, Mandelstam studied diagrams involving exchange of a ladder (Regge pole) and an elementary particle. The positions of the branch points determined in this way are

$$\alpha_{AB}{}^{(2)}(x)=\alpha_A(x-m_B)+J_B-1=\alpha_{AB}(x,m_B), \quad (8a)$$

$$\alpha_{BA}{}^{(2)}(x)=J_A+\alpha_B(x-m_A)-1=\alpha_{AB}(x,x-m_A). \quad (8b)$$

These functions do have the desired property

$$\alpha_{AB}{}^{(2)}(m_A+m_B)=\alpha_{BA}{}^{(2)}(m_A+m_B)=J_A+J_B-1. \quad (9)$$

This suggests that such branch points should, in fact, be present. In Sec. 4 we will indicate the way in which the type-2 branch points are to be understood in a theory without elementary particles. For the time being, let us just accept that such branch points are present, and that this has nothing to do with the question of whether or not there are elementary particles among the hadrons.

Attempting to seek maximum simplicity, one might at this point suspect that the type-2 branch points alone are sufficient to give a consistent theory. This is wrong, however, for the simple reason that in cases with enough spin the type-2 branch points can rise above the Froissart limit. For example, combining two identical boson trajectories we find $\alpha_{AA}(0)=2J_A-1$, which exceeds the Froissart limit for $J_A>1$. The most natural way to overcome this difficulty is for the type-2 branch point to move behind another one as x is decreased, and for this other one to remain below the Froissart limit. Apparently, this role is played by the type-1 branch points. Notice that they respect the Froissart limit, assuming the Regge poles do, since

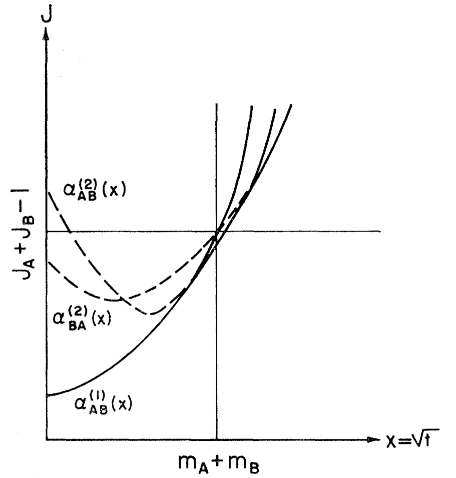

FIG. 1. A typical branch-point configuration. Nonsingular portions are shown as dashed lines.

$$\alpha_{AB}{}^{(1)}(0)=\alpha_A(0)+\alpha_B(0)-1, \quad (10)$$

at least for two boson poles. This formula may fail for fermion poles, because they can contain terms linear in x.

Let us next show that the type-1 and type-2 branch points have a point of tangency. (A typical configuration is shown in Fig. 1.) We accomplish the proof by showing that when $\alpha_{AB}{}^{(1)}(x)$ and $\alpha_{AB}{}^{(2)}(x)$ meet, they also have the same slope. Therefore we compute the derivatives using Eqs. (4)–(6), (8a) and obtain

$$\frac{d\alpha_{AB}{}^{(1)}(x)}{dx}=\left(\frac{\partial\alpha_{AB}(x,y)}{\partial x}+\frac{\partial\alpha_{AB}(x,y)}{\partial y}f'(x)\right)_{y=f(x)} \quad (11a)$$

$$=\partial\alpha_{AB}(x,y)/\partial x\,|_{y=f(x)}=\alpha_A'(x-f(x)),$$

and

$$d\alpha_{AB}{}^{(2)}(x)/dx=\alpha_A'(x-m_B). \quad (11b)$$

These two slopes agree for $f(x)=m_B$, which is precisely the condition for $\alpha_{AB}{}^{(1)}(x)$ to equal $\alpha_{AB}{}^{(2)}(x)$. Therefore, the branch point trajectories are tangent (assuming they meet at all)—a very typical behavior when one singularity moves behind another one. In order to respect the Froissart limit in general, we conclude that on the "first" sheet there is no singularity at the position described by the type-2 trajectory for x to the left of the tangency. The nonsingular portions of the trajectories are shown as dashed lines in Fig. 1. This analysis indicates that the type-1 branch points control asymptotic behavior for physical momentum transfer (besides the poles, of course), while the type-2 branch points play the crucial role described by Eq. (9).

The type of behavior discussed above for two-pole branch points can be generalized to n-pole branch points. In this case the "one Regge $n-1$ elementary" branch point goes through the n-body normal threshold at a nonsense point and is subsequently covered up by the "two Regge $n-2$ elementary" branch point at a point of tangency. Such tangencies are repeated until finally the "n Regge" branch point is leading. It appears that the necessity for such behavior as a consequence of unitarity could only be understood by studying n-body scattering.

Returning to the study of two-particle channels, notice that as drawn in Fig. 1 only one of the two type-2 branch points is singular when passing through the normal threshold. This is because one of them has its tangency with the type-1 branch point to the right of the normal threshold and the other to the left. Clearly, if both tangencies were to the right of the threshold, we would be in trouble. Requiring that this should never happen leads us to a weak, but not entirely trivial, restriction on the shape of Regge-pole trajectories. The positions of the two tangencies, x_1 and x_2, are given by

$$\alpha_A'(m_A)=\alpha_B'(x_1-m_A) \tag{12a}$$

and

$$\alpha_B'(m_B)=\alpha_A'(x_2-m_B). \tag{12b}$$

In the case that A and B are stable particles, the condition $\alpha_A''(x)\geqslant 0$ for *every* Regge pole below threshold is sufficient to guarantee that x_1 and x_2 lie on opposite sides of m_A+m_B. This condition, restated in terms of t, becomes

$$\alpha'(t)+2t\alpha''(t)\geqslant 0 \quad \text{for} \quad 0\leqslant t\leqslant t_0. \tag{13}$$

Notice that assuming a once-subtracted dispersion relation

$$\alpha(t)=\alpha(0)+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\text{Im}\alpha(t')}{t'(t'-t)}dt', \tag{14}$$

and $\text{Im}\alpha(t)\geqslant 0$ on the right-hand cut is sufficient to ensure that (13) holds for boson trajectories. For fermion trajectories one has to work in the x plane. The corresponding assumptions in that case also yield (13) without any difficulty, however. There must also be a restriction on the trajectory functions along the rest of the curve in the t plane on which $\alpha(t)$ is real, in order to include the case of unstable particles. The condition is awkward to state in an exact form, but when $\text{Im}\alpha(t)$ is small, we would expect (13) to hold for $\text{Re}\alpha(t)$, along the real t axis.

In the sense just described, the known trajectories are apparently consistent with (13). A word of caution should be given in this regard, however: It is not entirely correct to say that the nucleon recurrences lie very nearly along a straight line in t and hence the nucleon trajectory satisfies (13). Each time t is increased past a new inelastic threshold, the resonance region corresponds to a new sheet, the sheet immediately adjacent to the physical region. Therefore, strictly speaking, each recurrence of the nucleon belongs to a *different* Regge pole. Only to the extent that the inelastic thresholds are weak singularities is it meaningful to speak of one nucleon Regge pole. It is even conceivable that the "trajectory" obtained by connecting up all the nucleon recurrences extends to infinity while any single Regge pole asymptotes to a finite point. The condition (13) should apply to each Regge pole separately and not necessarily to a curve joining the resonances.

4. SINGULARITY STRUCTURE IN A SIMPLE MODEL

As stated in Sec. 3, a consistent theory without elementary particles should contain the type-2 branch points, which are ordinarily described as arising from exchange of a Regge pole and an elementary particle. In the following we attempt to show in terms of a simple model the way in which their occurrence can be understood.

Before getting into the details of the model, a few words of apology are required. The formulas to be considered are elastic-unitarity integrals, i.e., formulas of the type studied by Amati, Fubini, and Stanghellini.[3] It is well known[4] that any diagram that contributes to the two-particle discontinuity cannot contain third double-spectral functions in a suitable fashion to generate angular-momentum branch points. Therefore, the resulting singularities are known to be canceled, while the "true" singularities arise only from more complicated diagrams. On the other hand, our experience indicates that the more complicated diagrams always end up giving back essentially the same singularities. Therefore, perhaps there is some justification for counting in a different fashion. Namely, if we are interested in the behavior of $T(s,t)$ at large s we can regard it as consisting of a sum of contributions $\sum T_n(s,t)$, where $T_n(s,t)$ arises from dispersing the elementary discontinuity across an n-particle cut. From this point of view there would seem to be reason to study $T_2(s,t)$, especially since it is by far the simplest term. This suggestion only applies to investigations of singularity structure and not to quantitative matters. We shall find, in fact, that even the singularities obtained in this way have some unsatisfactory features.

We now consider

$$\text{disc}T(s,t)\sim\int f(s,t',t'')d\Omega, \tag{15}$$

where

$$f(s,t',t'')=T(s,t')T(s_{\text{II}},t'').$$

In the formulas of this section we neglect all but leading asymptotic powers of s, and we also drop unimportant constant factors. Thus we may rewrite (15) in the form

$$\mathrm{disc}T(s,t)\sim\frac{1}{s}\int_{-s}^{0}dt'\int_{t+t'-2(tt')^{\frac{1}{2}}}^{t+t'+2(tt')^{\frac{1}{2}}}dt''\frac{f(s,t',t'')}{[K(t,t',t'')]^{1/2}},\quad(16)$$

where

$$K(t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'').\quad(17)$$

Next, we make the charge of variables

$$t'=\tfrac{1}{4}t(\eta+\zeta)^2\quad\text{and}\quad t''=\tfrac{1}{4}t(\eta-\zeta)^2,\quad(18)$$

so that (16) becomes

$$\mathrm{disc}T(s,t)\sim\frac{t}{s}\int_{1}^{\infty}\frac{d\eta}{(\eta^2-1)^{1/2}}\int_{-1}^{1}\frac{d\zeta}{(1-\zeta^2)^{1/2}}\times(\eta^2-\zeta^2)f(s,t',t'').\quad(19)$$

As part of our asymptotic approximations we have set the upper limit of the η integration to infinity. The change of variables in (18) is singular for $t=0$, but is otherwise all right. For this reason, the t in front of the integral in (19) does not imply a vanishing of the function at $t=0$.

For the integrand of (19) we substitute Regge asymptotic behavior. A suitably descriptive diagram is given in Fig. 2. Let us take

$$f(s,t',t'')\sim\frac{s^{\alpha_A(x')}}{x'-m_A}\frac{s^{\alpha_B(x'')}}{x''-m_B},\quad(20)$$

where we have used x's for square-root variables as in Sec. 3, i.e., $x^2=t$ and

$$x'=\tfrac{1}{2}x(\eta+\zeta)\quad\text{and}\quad x''=\tfrac{1}{2}x(\eta-\zeta).\quad(21)$$

The familiar $1/\sin\pi\alpha$ factors have been replaced by simple poles and signature has been neglected in (20). These simplifications are permissible for present purposes. Thus we now have

$$\mathrm{disc}T(s,t)\sim\int_{1}^{\infty}\frac{d\eta}{(\eta^2-1)^{1/2}}\int_{-1}^{1}\frac{d\zeta}{(1-\zeta^2)^{1/2}}\times\frac{(\eta^2-\zeta^2)s^{\alpha_A(x')+\alpha_B(x'')-1}}{(x'-m_A)(x''-m_B)}.\quad(22)$$

The contribution to the partial-wave amplitude can be obtained by substituting (22) into the Froissart-Gribov formula. In the approximation of keeping only leading asymptotic behaviors, one finds

$$b_2(J,t)\sim\int_{1}^{\infty}\frac{d\eta}{(\eta^2-1)^{1/2}}\int_{-1}^{1}\frac{d\zeta}{(1-\zeta^2)^{1/2}}\times\frac{\eta^2-\zeta^2}{\alpha_A(x')+\alpha_B(x'')-1-J}\frac{1}{x'-m_A}\frac{1}{x''-m_B}.\quad(23)$$

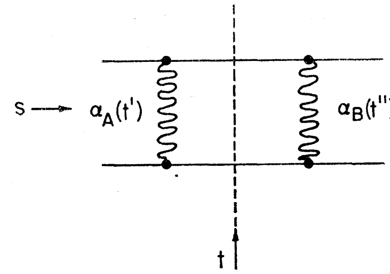

FIG. 2. Diagrammatic description of the integral in Eq. (22).

The subscript 2 indicates that we have only considered two-particle intermediate states in the s channel. The remaining project is to study the analytic structure of $b_2(J,t)$, given by (23).

As a first observation, we point out that for physical x (imaginary), the poles of the integrand of (23) at $x'=m_A$ and $x''=m_B$ are well separated from the integration contour and therefore cannot participate in a singularity-producing pinch. On the other hand, it is easy to show that the factor $[\alpha_A(x')+\alpha_B(x'')-1-J]^{-1}$ pinches with the contour to produce a singularity when $J=\alpha_{AB}^{(1)}(x)$, with $\alpha_{AB}^{(1)}(x)$ as defined in (4)–(6). Let us next consider continuing x to positive values so that additional pinches can occur. One finds a pinch involving $[\alpha_A(x')+\alpha_B(x'')-1-J]^{-1}$ and $(x'-m_A)^{-1}$ for $J=\alpha_{BA}^{(2)}(x)$, and analogously $(x''-m_B)^{-1}$ participates in a pinch for $J=\alpha_{AB}^{(2)}(x)$. There is also a pinch involving all three poles for $J=J_A+J_B-1$, and another one involving $(x'-m_A)^{-1}$ and $(x''-m_B)^{-1}$ for $x=m_A+m_B$. We therefore see that there are a number of possible singularities, but none really too surprising. The point we wish to emphasize most is the occurrence of the type-2 branch points in the positive x region and their nonoccurrence in the physical-scattering region. This is precisely the behavior deduced in Sec. 3. Despite the previously discussed deficiencies of the model, this feature is believed to be quite general.

In the present model one can work out the type of singularities that arise as well as determine their location and sheet structure. The type-1 branch points are found to be of the kind $\ln(J-\alpha_{AB}^{(1)}(x))$, while the type-2 branch points behave as $[J-\alpha_{AB}^{(2)}(x)]^{-1/2}$. Both of these behaviors are wrong for the complete partial-wave amplitude because they violate the condition, deduced by Bronzan and Jones,[12] that the branch points cannot be infinite singularities. A simplified version of their argument runs as follows. Since the branch-point trajectories do not contain the normal threshold at $t=(m_A+m_B)^2$, they occur at the same position in $b(J,t)$ and $b(J,t_{\mathrm{II}})$, the amplitudes on the first and second sheets of the elastic cut, respectively. Consequently, it is evident that an infinite singularity cannot be reconciled with the elastic unitarity equation,

[12] J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

Eq. (1). It may be a reasonable guess for the singularities of the complete amplitude that they are one power less singular than found in this model, i.e., $[J-\alpha_{AB}^{(1)}(t)] \times \ln[J-\alpha_{AB}^{(1)}(t)]$ and $[J-\alpha_{AB}^{(2)}(t)]^{1/2}$, respectively.

5. CONCLUSIONS

It has been shown that there are two essentially different types of angular-momentum branch points, which can be expected to occur. The reasoning that led to the necessity for their existence and certain details of their sheet structure involved three applications of unitarity. First, we required that there be fixed poles rather than essential singularities so as not to violate the Froissart bound. Second, fixed poles could only be reconciled with analytically continued elastic unitarity by requiring that branch points having the property of Eq. (9) be present. Third, the branch points satisfying (9) can rise above the Froissart limit, and therefore additional branch points were required to conceal them.

Branch points having the required positioning and sheet structure were shown to be contained in integrals of the AFS type. We argued that although such integrals are in a certain sense not relevant, they nevertheless serve as a useful guide in showing that the postulated mechanism is plausible. If an analysis of this kind is to be extended to multiparticle unitarity integrals, it will be necessary to have suitable asymptotic expressions for the production amplitudes[13] that enter into the integrals.

The optimism expressed in the Introduction regarding the possibility of obtaining expressions for asymptotic behavior arising from branch points is based on the following considerations. The cut discontinuities may be dominated by the wrong-signature nonsense poles, whose residues can in turn be determined by saturating integrals of the form

$$\int \operatorname{Im}T(s,t)ds$$

with resonances, as one often does in the case of superconvergence. Another possibility is to use Eq. (3) in conjunction with some theoretical knowledge of the double spectral function. The complications that would arise in carrying out such a program are numerous, however. First, in order to study a branch point not associated with the Regge poles of the external particles, a many-channel treatment is required. Second, it is difficult to introduce the Pomeranchuk trajectory into such a calculation because of the uncertainty about which particles, if any, lie on it. Third, other singularities than the fixed pole may give appreciable contributions to the cut discontinuities.

ACKNOWLEDGMENTS

I wish to acknowledge stimulating correspondence with Professor C. E. Jones, Professor V. Teplitz, and Professor S. Mandelstam. I am also grateful to Professor M. L. Goldberger, Professor S. B. Treiman, Professor G. Tiktopoulos, and Dr. S. Nussinov for interesting discussions.

[13] Some progress in understanding production amplitudes has been made by V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martiroysan, Phys. Rev. **139**, B184 (1965); A. A. Anselm, Ya. I. Azimov, G. S. Danilov, I. T. Dyatlov, and V. N. Gribov, Ann. Phys. (N.Y.), **37**, 227 (1966).