ment. This relation is an angle- and energy-independent statement. In the case of Ξ^* production we obtained a result, not in agreement with experiment, such as that derived from $SU(6)_W$. There are many more results which we have not been able to analyze because of a lack of experimental information.

If group theory is to compete in any serious way with dispersion or field theories then one must be able to make comments on the momentum dependence of form factors (or in our language, matrix elements). In our approach, this is equivalent to the statement that the invariance group must include the Poincaré group, and the noninvariant generators (belonging to a suitable NIG) must mix states of the different representations of the invariance group (hence, different momenta). If we identify the noninvariant generators as, say, the isovector current, then the analytic expressions of the matrix elements of this operator will be our isovector form factors, and their momentum dependence will be known. Even though the problem can be formulated so easily, the task of solving it is a difficult one.

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Relativistic Harmonic Oscillator and Superconvergent Amplitudes*

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Current commutation relations and Fubini sum rules are saturated in a model that is a relativistic version of the three-dimensional harmonic oscillator. The model is essentially determined by the requirement that a local nonderivative relativistic coupling of the oscillator to an external electromagnetic field give rise to form factors that reduce to the usual result in the nonrelativistic limit. The relativistic form factors decrease as a power of the invariant momentum transfer, although they fall off exponentially in the limit $c \to \infty$. Vertex functions and scattering amplitudes are investigated, and it is found that (i) the Compton scattering amplitudes for current-particle interactions satisfy Fubini sum rules. (ii) All strong-interaction amplitudes are superconvergent in the Born approximation, in which an infinite/equal-mass multiplet is either exchanged or forms a set of intermediary states. (iii) Scattering amplitudes can be arranged in a hierarchy of increasing convergence (e.g., no spin flip, single spin flip, double spin flip), as suggested by de Alfaro *et al.* Finally, the problem of introducing the mass spectrum in the one-particle propagator is discussed.

I. INTRODUCTION

RECENT work on current algebras and superconvergence relations has been directed towards the construction of more or less realistic models, in which the attempt is made to saturate these relations with a number of idealized states.[1] In a nonrelativistic framework this is certainly possible; the success of the bootstrap in the static model may be cited as an example of saturation of a superconvergence relation with a small number of single-particle states. On the other hand, it has been discovered that saturation of superconvergence relations in a relativistic theory requires an infinite number of states.[1] As has been emphasized repeatedly, the technical complications that arise from an infinite number of single-particle states are compensated by unexpected dividends. In fact, it has been shown that what is originally introduced as a discrete set of one-particle states sometimes turn out to represent a continuum of two-particle states, in addition to a number (finite or infinite) of physical one-particle states.[2]

The three-dimensional harmonic oscillator seems to provide a suitable starting point for a relativistic model because the energy spectrum is not unlike that which might be expected for elementary particles. The absence

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[1] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Institute di Fisica Dell' Universita, Torino Report, December, 1966 (unpublished).

[2] C. Fronsdal, Phys. Rev. 156, 1665 (1967).

of ionization is also tantalizing; it suggests a picture of elementary particles as bound systems of quarks, without the need to discover free quarks in the laboratory, and it avoids the technical complication of a continuum of states in the completeness relation. This paper describes a model that reduces to the three-dimensional harmonic oscillator in the nonrelativistic limit. It has a countably infinite number of one-particle states, no ionization, and a number of physical features, including a current algebra and superconvergent amplitudes.

In Sec. II we construct a relativistic generalization of the three-dimensional harmonic oscillator. The model is essentially determined by the requirement that a local nonderivative relativistic coupling of the oscillator to an external electromagnetic field give rise to form factors that reduce to the usual result in the nonrelativistic limit. It is found that, for finite light velocity, the form factors decrease as a power of the invariant momentum transfer, although they fall off exponentially in the limit $c\rightarrow\infty$.

In Sec. III we investigate vertex functions and scattering amplitudes in the model and find that (i) The "Compton scattering" amplitudes for current-particle interactions satisfy Fubini sum rules, at least when the mass differences of the virtual particles are ignored. (ii) Strong-interaction scattering amplitudes are calculated in the lowest order of perturbation theory, in which an infinite tower is either exchanged or forms a set of intermediary states. If the mass differences within the virtual tower are ignored, then all scattering amplitudes are superconvergent. (iii) Alfaro *et al.*[1] have shown that scattering amplitudes that exhibit Regge behavior, and can be continued analytically from one channel to another, can be expanded in a hierarchy of increasingly convergent terms (no spin flip, spin flip, double spin flip, etc.). In the present model, amplitudes are found to have this property, independently of the assumptions of Regge poles or analytic continuation. (iv) The difficult problem of introducing the mass spectrum into the one-particle propagator is briefly discussed.

II. THE HARMONIC OSCILLATOR

A. The Suggestion of Local Field Theory

The three-dimensional harmonic oscillator shares with a number of other physical systems the property of possessing a symmetry algebra that includes the angular momentum. Suppose that a relativistic version of the oscillator has a current algebra S that includes the symmetry algebra $U(3)$; then S must have the following general structure. The generators form sets of covariant entities, scalars $s, s', \cdots$, vectors $s_\mu, s_\mu', \cdots$, tensors $s_{\mu\nu}, s_{\mu\nu}', \cdots$, and so on. The commutation relations between these generators and the generators $L_{\mu\nu}$, P_μ of the Poincaré group are

$$[L_{\mu\nu},s]=[P_\mu,s]=0\,,$$
$$[L_{\mu\nu},s_\lambda]=-ig_{\mu\lambda}s_\nu+ig_{\nu\lambda}s_\mu\,,\quad [P_\mu,s_\nu]=0\,,$$

and so on. The commutation relations between the elements of S are covariant, for example,

$$[s_{\mu\nu},s_{\lambda\rho}]=-ig_{\mu\lambda}s_{\nu\rho}+ig_{\nu\lambda}s_{\mu\rho}-ig_{\mu\rho}s_{\lambda\nu}+ig_{\nu\rho}s_{\lambda\mu}\,.$$

If S contains the symmetry algebra $U(3)$, and hence the generators of the internal (spin) angular momentum, then it contains a subalgebra that is isomorphic to the (homogeneous) Lorentz algebra, spanned by a set of six generators $s_{\mu\nu}=-s_{\nu\mu}$ with the commutation relations just written. The generator of Lorentz transformations

$$L_{\mu\nu}=i(p_\mu\partial/\partial p^\nu-p_\nu\partial/\partial p^\mu)+s_{\mu\nu}$$

includes an external part (p_μ is the total energy-momentum vector of the system), and an internal part which is $s_{\mu\nu}$.

In the case of the harmonic oscillator, the obvious choice of S is the algebra $U(3,1)$,[3] whose generators are the components of a tensor $C_{\mu\nu}$ with the following commutation relations:

$$[C_{\mu\nu},C_{\lambda\rho}]=-g_{\nu\lambda}C_{\mu\rho}+g_{\rho\mu}C_{\lambda\nu}\,.$$

The symmetry algebra $U(3)$ is the compact subalgebra, and

$$s_{\mu\nu}=i(C_{\mu\nu}-C_{\nu\mu})\,.$$

The states are described by an infinite set of fields $\psi_\sigma(x)$, where x_μ is the "center-of-mass" (c.m.) coordinate, that is, the coordinate whose conjugate momentum is the total momentum p_μ, and σ is an index that takes on an infinity of values. The operators of S act on the index only, not on the argument.

A *local, nonderivative interaction* between two or three or more fields is an S-invariant coupling of the form

$$\sum_{\sigma\lambda\tau}\psi_\sigma(x)\phi_\lambda(x)\chi_\tau(x)C_{\sigma\lambda\tau}\,,$$

where the $C_{\sigma\lambda\tau}$ are constants. More generally, a local interaction may include finite-order derivatives of the fields. Similarly, a *local current* is of the form

$$J(x)=\sum_\sigma\psi_\sigma^*(x)\phi_\sigma(x)\,,\tag{1}$$

or

$$J_\mu(x)=\sum_{\sigma\lambda}\psi_\sigma^*(x)(s_\mu)_{\sigma\lambda}\phi_\lambda(x)\,,$$

$$J_\mu'(x)=\sum_{\sigma\lambda}\psi_\sigma^*(x)(C_{\mu\nu})_{\sigma\lambda}\frac{\overleftrightarrow{\partial}}{\partial x_\nu}\phi_\lambda(x)\,,$$

and so on. If the fields $\psi_\sigma(x)$ satisfy canonical commutation relations, then it is clear that there exist currents that satisfy equal-time commutation relations related to the commutation relations of the $C_{\mu\nu}$.

Consider the simplest type of local current, the scalar density (1). In momentum space,

$$J(p,p')=\sum_\sigma\psi_\sigma^*(p)\phi_\sigma(p')\,.$$

[3] A. O. Barut (private communication); also, A. O. Barut, Phys. Rev. **139**, B1433 (1965). An alternative to $U(3,1)$ is the group $Sp(6)$; this group was used by G. Bisiachi and P. Budini, Nuovo Cimento 44, 418 (1966); and Nuovo Cimento (to be published).

The field $\phi(p')$ is related to $\phi(p)$ by a Lorentz transformation,

$$\phi(p')=\exp[iL(p,p')]\phi(p)\,,$$
$$L(p,p')=\theta^{\mu\nu}(p,p')L_{\mu\nu}\,.$$

Consequently, the matrix elements of the current $J(p,p')$ are related to the matrix elements of a finite Lorentz transformation;

$$\langle\psi|J(p,p')|\phi\rangle=\langle\psi|e^{iL(p,p')}|\phi\rangle\,.$$

If the states $|\psi\rangle$, $|\phi\rangle$ are identical, then this expression is a form factor for the state $|\phi\rangle$. Our principal point is this: because of the very attractive features of form factors of this type,[4] *it is proposed to obtain physical form factors and superconvergent amplitudes from currents and interactions that are local in the sense described above.* It is known that a relativistic description of the hydrogen atom is possible, in which the correct form factors and transition amplitudes are matrix elements of local currents. Here it will be shown that this idea also works very well in the case of the harmonic oscillator. In fact, we shall use it as the guiding principle for obtaining a relativistic generalization of that system.

B. The Harmonic Oscillator in a Relativistic Framework

In terms of canonical coordinates y_i and momenta q_i, satisfying the commutation relations[5]

$$[y_i,q_j]=i\delta_{ij}\,,$$

the Hamiltonian for the three-dimensional harmonic oscillator is

$$H=\tfrac{1}{2}w\sum_{i=1}^{3}(y_i^2+q_i^2)\,,$$

where w is the excitation frequency. This Hamiltonian is invariant under a group of transformations generated by an algebra of quantum-mechanical operators that satisfy the commutation relations of $U(3)$. A suitable basis in the algebra is

$$C_{ij}=a_i^\dagger a_j\,,$$
$$a_i=2^{-1/2}(y_i+iq_i)\,,\quad a_i^\dagger=2^{-1/2}(y_i-iq_i)\,. \tag{2}$$

The commutation relations are

$$[C_{ij},C_{kl}]=\delta_{jk}C_{il}-\delta_{il}C_{kj}\,,$$
$$[a_i,H]=wa_i\,,\quad [a_i,a_j^\dagger]=\delta_{ij}\,,$$

An obvious candidate for a relativistic extension of this algebra is $U(3,1)$, with basis elements satisfying the commutation relations

$$[C_{\mu\nu},C_{\lambda\rho}]=-g_{\nu\lambda}C_{\mu\rho}+g_{\mu\rho}C_{\lambda\nu} \tag{3}$$

and the Hermiticity condition

$$C_{\mu\nu}^\dagger=C_{\nu\mu}\,. \tag{4}$$

The question is, what are the extra generators C_{i0}, C_{0i}, and C_{00}? More precisely, we wish to determine the most convenient choice of these operators.

In the preceding section it was suggested that relativistic form factors should have the form of matrix elements

$$\langle\psi|e^{iL(p,p')}|\phi\rangle$$

of finite Lorentz transformation operators. If $|\psi\rangle$ is at rest ($\mathbf{p}=0$), then this is a matrix element of $\exp i\theta_iL_{i0}$. On the other hand, the form factors of the nonrelativistic harmonic oscillator are matrix elements of $\exp ik_iy_i$. We therefore try to relate L_{i0} to y_i.

Adjoining y_i to C_{ij}, we obtain a nonsemisimple algebra with 16 independent generators. It is spanned by C_{ij}, y_i, q_i, and I, the identity operator. This algebra does not contain the Lorentz algebra, but it does contain the Galilei algebra, spanned by

$$s_{ij}=i(C_{ij}-C_{ji})\,,\quad y_i\,.$$

This algebra is the nonrelativistic limit of the Lorentz algebra; it may be obtained by the contraction $c\rightarrow\infty$, where c is the velocity of light. In fact, the 16-parameter algebra that has just been introduced can be obtained by contraction of $U(3,1)$. We shall therefore try to choose the C_{i0}, C_{0i}, C_{00} in such a manner that

$$s_{i0}=i(C_{i0}-C_{0i})\xrightarrow[c\rightarrow\infty]{} y_i\,.$$

This is actually possible. The operators[6]

$$\begin{aligned}C_{0i}&=(\gamma^2+\tau)^{1/2}a_i\,,\\ C_{i0}&=a_i^\dagger(\gamma^2+\tau)^{1/2}\,,\\ C_{00}&=\gamma^2+\tau\,,\end{aligned} \tag{5}$$

where γ is a real constant and

$$\tau=w^{-1}H-\tfrac{3}{2}=0,\,1,\,2,\,\cdots\,,$$

together with the C_{ij} defined by (2), satisfy the commutation relations of (3) of $U(3,1)$, as well as the Hermiticity conditions (4). The Lorentz generators

$$L_{\mu\nu}=i(p_\mu\partial/\partial p^\nu-p_\nu\partial/\partial p^\mu)+s_{\mu\nu}$$

reduce to $s_{\mu\nu}$ in the static limit; in particular,

$$\begin{aligned}L_{i0}\rightarrow s_{i0}&=i(C_{i0}-C_{0i})\\ &=i(\gamma^2+\tau)^{1/2}a_i^\dagger-ia_i(\gamma^2+\tau)^{1/2}\,.\end{aligned}$$

In order to achieve our goal of having L_{i0} tend to y_i in the nonrelativistic limit, we must relate γ^2 to the

[4] C. Fronsdal and R. White, Phys. Rev. **151**, 1287 (1966); G. Cocho, C. Fronsdal, Harun Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966); G. Cocho, C. Fronsdal, Harun Ar-Rashid, and R. White, International Center for Theoretical Physics Report No. IC/66/84, Trieste (unpublished); A. O. Barut and Hagen Kleinert, Phys. Rev. **156**, 1546 (1967); C. Fronsdal, *ibid.* **156**, 1653 (1967).

[5] Throughout this paper we consistently use the letters p, x for c.m. coordinates. The nonrelativistic internal variables y and q are replaced by tensor indices in the covariant description.

[6] The fact that the following operators yield a representation of $SU(3,1)$ was discovered by Nambu and Rosen. We are very grateful to Professor Nambu for informing us of his results prior to publication.

velocity of light, so that γ^2 tends to infinity when c does. Now γ is one of the Casimir operators of $U(3,1)$; hence sending c to infinity takes the operators through a one-parameter sequence of representations of $U(3,1)$, a phenomenon that is not unusual in group contraction.

It is now clear that if $\psi_\sigma(x)$ forms the basis for a Hermitian representation $D(-\gamma^2)$ of $U(3,1)$, and if the relativistic form factors are the matrix elements of local currents, then these form factors reduce to the form factors of the harmonic oscillator in the nonrelativistic limit $\gamma^2 \to \infty$. The algebraic construction of unitary representations of $U(3,1)$, and the calculation of the form factors, is carried out in the next two sections.

C. Hermitian Representations of $U(3,1)$

Only a special kind of representations of $U(3)$ are realized by the three-dimensional harmonic oscillator; they are the "degenerate" representations, whose weight diagrams are triangular. The following construction gives all those representations of $U(3,1)$ that have the property that only degenerate representations of $U(3)$ occur.

We start with a representation in a function space, without specifying the definition of the inner product, pass quickly to an algebraic representation induced on a canonical basis, and construct the inner product so as to make the operators Hermitian.

Consider functions $f(z)$ of four complex variables z_μ, and put

$$C_{\mu\nu}f(z) = -z_\mu \frac{\partial}{\partial z^\nu} f(z). \tag{6}$$

These operators satisfy the commutation relations (3) of $U(3,1)$. They all commute with the operator $-g^{\mu\nu}C_{\mu\nu} = z_\mu \partial/\partial z_\mu$; hence this operator must be a multiple of the identity in an irreducible representation:

$$z_\mu \frac{\partial}{\partial z_\mu} f(z) = N f(z).$$

Equivalently, for every complex number λ,

$$f(\lambda z) = \lambda^N f(z).$$

The number N, the degree of homogeneity of $f(z)$, characterizes an irreducible representation, $D(N)$, say.

Next, we introduce a countable basis of monomials;

$$z_1^a z_2^b z_3^c z_0^\alpha, \quad a+b+c+\alpha = N.$$

For fixed α, these monomials span a representation of the $U(3)$ subalgebra $\{C_{ij}\}$. This is compact, and has only finite irreducible representations; therefore the exponents a, b, and c must be positive integers. When a, b, and c vary over all positive integers consistent with $a+b+c = N-\alpha =$ fixed, then the monomials span an irreducible, triangular representation of $U(3)$. It is convenient to replace the monomials by three-dimensional tensors;

$$\psi_{A_1\cdots A_\tau} = C(\tau) z_{A_1} \cdots z_{A_\tau} (z_0)^{N-t}$$
$$A_1, \cdots = 1, 2, 3; \quad \tau = 0, 1, 2, \cdots,$$

where $C(\tau)$ are complex normalizing coefficients to be chosen presently.

Direct application of the differential operators (6) gives

$$\begin{aligned} C_{0i}\psi_{A_1\cdots A_\tau} &= \tau \frac{C(\tau)}{C(\tau-1)} S\delta_{A_1 i}\psi_{A_1\cdots A_\tau}, \\ C_{i0}\psi_{A_1\cdots A_\tau} &= (\tau-N)\frac{C(\tau)}{C(\tau+1)}\psi_{iA_1\cdots A_\tau}, \\ C_{00}\psi_{A_1\cdots A_\tau} &= (\tau-N)\psi_{A_1\cdots A_\tau}. \end{aligned} \tag{7}$$

The Hermiticity condition (4) requires that $N=N^*$, and determines the $C(\tau)$;

$$|C(\tau)/C(\tau-1)|^2 = (\tau-N-1)/\tau, \quad \tau = 0, 1, \cdots.$$

This can be solved for $\tau = 0, 1, 2, \cdots$ if and only if $N<0$; hence $D(N)$ is a Hermitian representation for every negative real N, provided that we choose the normalization

$$C(\tau) = [(\tau-N-1)!/\tau!]^{1/2}. \tag{8}$$

Irreducibility of $D(N)$ is obvious by inspection.

D. Relativistic Form Factors

The prototype of relativistic local form factors is

$$F_{00}(p,p') = \sum_\sigma \psi_\sigma^*(p,0)\psi_\sigma(p',0),$$

where $\psi_\sigma(p,0)$ is the wave function for the ground state of the oscillator in the reference frame where the total momentum is p_μ. If $\mathbf{p}=0$, then this state is a $U(3)$ singlet; otherwise it is a singlet of $U(3)_p$, which is the subalgebra of $U(3,1)$ that is spanned by the "transverse generators";

$$\theta_\mu{}^\nu \theta_\lambda{}^\rho C_{\nu\rho},$$
$$\theta_\mu{}^\nu = \delta_\mu{}^\nu - p_\mu p^\nu/p^2.$$

To evaluate this form factor in the simplest possible way, we introduce the notation of generalized tensors.

We replace the monomial basis by a single four-dimensional tensor field;

$$\psi_{\mu_1\cdots\mu_N}(p) = z_{\mu_1}\cdots z_{\mu_N}.$$

The reduction of this tensor according to $U(3)_p$ is given by

$$\psi_{\mu_1\cdots\mu_N}(p) = S\sum_{\tau=0}^{\infty} C(\tau)\tilde{\psi}_{\mu_1\cdots\mu_\tau}(p)p_{\mu_{\tau+1}}\cdots p_{\mu_N}(p^2)^{(t-N)/2}, \tag{9}$$

where $C(\tau)$ is given by (8), and

$$\tilde{\psi}_{\mu_1\cdots\mu_\tau}(p) = C(t)\theta_{\mu_1}{}^{\nu_1}\cdots \theta_{\mu_\tau}{}^{\nu_\tau}p^{\nu_{\tau+1}}\cdots p^{\nu_N}(p^2)^{(\tau-N)/2}\psi_{\nu_1\cdots\nu_N}(p)$$

is an irreducible $U(3)_p$ tensor.

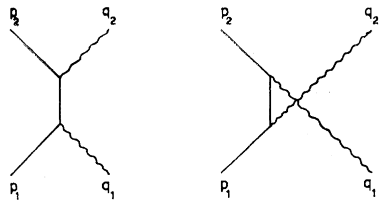

FIG. 1. Two contributions to Compton scattering. The invariant momentum variables are $s=(p_1+q_1)^2$, $t=(p_1-p_2)^2$, and $u=(p_1-q_2)^2$.

The invariant inner product is

$$\bar{\psi}(p)\phi(p')=\psi^{*\mu_1\cdots\mu_N}(p)\phi_{\mu_1\cdots\mu_N}(p')\,.$$

When $p=p'$, it reduces to

$$\bar{\psi}(p)\phi(p)=\sum_{\tau}\tilde{\psi}^{*\mu_1\cdots\mu_\tau}(p)\phi_{\mu_1\cdots\mu_\tau}(p)\,.$$

When $p\neq p'$ it defines the form factors and transition amplitudes;

$$\bar{\psi}(p)\phi(p')=\sum_{\tau,\tau'}F_{\nu_1\cdots\nu_\tau}{}^{\mu_1\cdots\mu_{\tau'}}(p,p')\tilde{\psi}^{*\nu_1\cdots\nu_\tau}(p)\phi_{\mu_1\cdots\mu_\tau}(p')\,.$$

In particular, the form factor for the ground state is found by inserting the first term in the expansions (9) of $\bar{\psi}(p)$ and $\phi(p')$:

$$\begin{aligned}F(p,p')&=p^{\mu_1}\cdots p^{\mu_N}\cdot p'_{\mu_1}\cdots p'_{\mu_N}(p^2)^{-N/2}(p'^2)^{-N/2}\\&=(pp'/m^2c^4)^N\,.\end{aligned}$$

In the last expression we have put

$$p^2=p'^2=m^2c^4\,,$$

where m is the rest mass and c is the velocity of light.

To evaluate the nonrelativistic limit, we first note that $\gamma^2=-N$; for

$$C_{00}=\gamma^2+\tau\,,$$

from Sec. IIB, Eq. (5), and

$$C_{00}=\tau-N$$

according to Sec. IIC, Eq. (7). In the nonrelativistic limit γ^2 tends to ∞, hence N tends to $-\infty$. Thus

$$\begin{aligned}F(p,p')&=(1-t/2m^2c^2)^N\to\exp k^2N/2m^2c^2\,,\\t&=(p-p')^2/c^2\to -k^2\,.\end{aligned}$$

If the nonrelativistic oscillator is a particle mass μ in a fixed-force field, then the form factor is $\exp(-k^2/4\mu w)$; hence the correct nonrelativistic result is obtained if

$$-N=\gamma^2=m^2c^2/2\mu w\,. \qquad (10)$$

III. IMPLICATIONS FOR HADRON PHYSICS

A. Form Factors

The simple three-dimensional harmonic oscillator considered here falls far short of describing actual hadrons and their interactions, for several reasons. Firstly, it must not be supposed that the symmetry group $U(3)$ of the oscillator have anything to do with the eightfold way. Instead there are strong analogies with Fock's $SO(4)$ symmetry of the nonrelativistic hydrogen atom, or with some type of subgroup of $SU(6)$, without isospin. If this analogy is stressed, then one could try to do particle physics by replacing $U(3)$ with $SU(6)$ and the relativistic extension $U(3,1)$ with $SL(6,C)$ or $SU(6,6)$. If it should turn out that $SU(6)$ is of less than fundamental importance, then one could investigate smaller groups in which the bond between isospin and angular momentum is less strong.[7] Secondly, only integer spins have been considered so far; the model comes closer to describing mesons than baryons. It is a simple matter to introduce half-integral spins, but if the three-quark picture of baryons contains an element of truth, then a more complicated oscillator model might be more appropriate.

Let us be permitted to stress, nevertheless, the physical features of the oscillator model. Firstly, the mass spectrum is at least reasonable. Secondly, the qualitative features of the form factors are quite attractive. Although Gaussian in the limit when the velocity of light (or the ratio of the rest energy to the excitation energy) goes to infinity, they decrease like a power of the invariant momentum transfer squared in the relativistic domain. Recent data on the proton magnetic form factor, which extended all the way from $-t=0$ to $-t=10$ $(\mathrm{GeV}/c)^2$,[8] may be fitted very well with the function $(1-t/2m^2)^N$ if $N=-2$ and $2m^2$ is taken as a free parameter.

It has recently been stressed by Gell-Mann[9] and by Fubini[1] that many physical requirements can be met by models with infinite numbers of one-particle states. The vast arbitrariness of such models can be greatly reduced by (i) relying on infinite irreducible representations of noncompact groups that include the spin, for the technical reason of rendering the algebraic structure manageable, and (ii) accepting the suggestions of infinite-component local-field theory for specific forms of currents and interactions. We now develop the local field-theoretic aspects, in order to show that current algebras and superconvergence are among the immediate consequences.

B. Current Algebras and Fubini Sum Rules

If a local Lagrangian density is constructed from the fields $\psi_\sigma(x)$, and if canonical equal-time commutation relations are postulated for the field components, then

[7] A. O. Barut, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965), pp. 689 ff; and in *Non-compact groups in Particle Physics* (W. A. Benjamin, Inc., New York, 1966), pp. 1–22; W. Rühl, Nuovo Cimento **46A**, 115 (1966). This possibility has been stressed repeatedly by M. Gell-Mann.

[8] S. D. Drell, *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

[9] M. Gell-Mann, Lecture notes, International School of Physics, "Ettore Majorana" Erice, Sicily, July, 1966 (unpublished).

it is superfluous to point out that local currents exist that satisfy equal-time commutation relations. Instead, we shall give an example of a Fubini sum rule that depends less intimately on the canonical formalism.

Consider the local interaction

$$A^{\mu\nu}(x)J_{\mu\nu}(x),$$

where

$$J_{\mu\nu}(x)=\psi^*(x)C_{\mu\nu}\psi(x).$$

The fields $A^{\mu\nu}(x)$ may be identified with the electromagnetic field strengths $F^{\mu\nu}(x)$ if it is antisymmetric in μ, ν and with the gravitational potentials $g^{\mu\nu}(x)$ if it is symmetric, but for simplicity we shall not assume either symmetry or antisymmetry. Relying on Feynman rules rather than the grandiose machinery of local-field theory, we notice that this interaction gives two contributions to Compton scattering, represented by the two Feynman diagrams in Fig. 1. The corresponding amplitude is

$$\psi^*(p_1)[C_{\mu\nu}(s-m^2)^{-1}C_{\lambda\rho}+C_{\lambda\rho}(u-m^2)^{-1}C_{\mu\nu}]\psi(p_2). \quad (11)$$

Here we have used the propagator $(s-m^2)^{-1}$ which would have been appropriate if the masses of all the levels of the oscillator were equal. Naturally, it would be much better to use a propagator that corresponds to the actual mass spectrum, but for the moment the best that we can do is to ignore the mass differences. The absorptive part of the amplitude (11) is

$$\pi\psi^*(p_1)[C_{\mu\nu}C_{\lambda\rho}\delta(s-m^2)-C_{\lambda\rho}C_{\mu\nu}\delta(u-m^2)]\psi(p_2) = t_{\mu\nu,\lambda\rho}(s,t,q_1^2,q_2^2). \quad (12)$$

Integrating over s, we obtain

$$\frac{1}{\pi}\int t_{\mu\nu,\lambda\rho}(s,t,q_1^2,q_2^2)ds = g_{\nu\lambda}J_{\mu\rho}(p_1,p_2)-g_{\rho\mu}J_{\lambda\nu}(p_1,p_2). \quad (13)$$

The right-hand side is a form factor, and the equation is a Fubini sum rule.

One may notice that the absorptive part, for fixed t, is independent of q_1^2 and q_2^2, and depends on s through the δ functions only. If a more realistic propagator had been used, it is possible that $(s-m^2)^{-1}$ in (11) would have been replaced by

$$\sum_{\tau=0}^{\infty} P_\tau(s)(s-m_\tau^2)^{-1},$$

where P_τ is an operator that, at least when $s=m_\tau^2$, is a projection operator for the τth level. Instead of $\delta(s-m^2)$, the absorptive part (12) would then have had the expression

$$\sum_\tau P_\tau(m_\tau^2)\delta(s-m_\tau^2).$$

Integration over s would then have given the same result (13) as before, since s occurs in the δ function only,

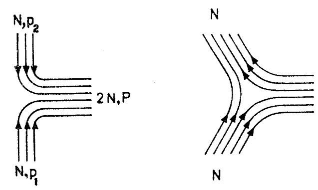

FIG. 2. River diagrams for the nucleon-nucleon-deuteron vertex and the nucleon-nucleon meson vertex. The quark number is N for nucleons, $-N$ for antinucleons, $2N$ for deuterons, and 0 for mesons.

and

$$\int ds \sum P_\tau(m_\tau^2)\delta(s-m_\tau^2)=\sum_\tau P_\tau(m_\tau^2)=1.$$

The other term, with $(u-m^2)^{-1}$, cannot be discussed with any confidence, since u is negative and the physical content of the propagator for spacelike momentum is less obvious. It is possible that $(u-m^2)^{-1}$ should be replaced by

$$\int d\tau\, P_\tau(u)(u-m_\tau^2)^{-1},$$

where $P_\tau(m_\tau^2)$ is a complete set of projection operators satisfying a completeness relation

$$\int d\tau\, P_\tau(m_\tau^2)=1.$$

In that case the sum rule could be derived without neglecting the mass differences. We shall return to the problem of propagators with mass spectra in Sec. IIIF.

C. Evaluation of a Vertex Function

Whereas the Compton scattering amplitude (12) decreases like s^{-1} for very large s (and constant t), the analogous "strong-interaction" scattering amplitudes have a quite different asymptotic form.[10] We shall calculate several scattering amplitudes, but first let us investigate invariant vertex functions.

Let $D(N,M)$ be an irreducible representation of $U(3,1)$, realized on a generalized tensor with N-covariant (lower) and M-contravariant (upper) indices. Up to now we have considered $D(N,0)$ and the contragredient $D(0,N)$; the corresponding states will be referred to as "nucleons" and "antinucleons." In addition, the states of $D(2N,0)$ will be called "deuterons," and the states of $D(M,M)$ will be called "mesons." Two kinds of Yukawa interactions may be considered,

$$\psi^{*\mu_1\cdots\mu_N}(x)\psi^{*\mu_{N+1}\cdots\mu_{2N}}(x)\chi_{\mu_1\cdots\mu_{2N}}(x), \quad (14)$$

and

$$\psi^{*\mu_1\cdots\mu_N}(x)\psi_{\nu_1\cdots\nu_N}(x)\phi_{\mu_1\cdots\mu_M}{}^{\nu_1\cdots\nu_M}\delta_{\mu_{M+1}}{}^{\nu_{M+1}}\cdots\delta_{\mu_N}{}^{\nu_N}. \quad (15)$$

[10] J. B. Bronzan *et al.*, Phys. Rev. Letters **18**, 32 (1967); Virendra Singh, *ibid.* **18**, 36 (1967).

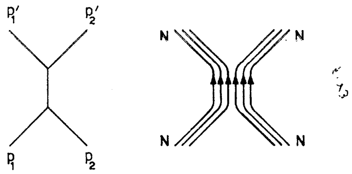

FIG. 3. Feynman diagram and river diagram for nucleon-nucleon scattering via deuteron intermediary states. The invariant variables are $s=(p_1+p_2)^2$, $t=(p_1-p_1')^2$, and $u=(p_1-p_2')^2$.

The corresponding vertices may be represented by the "river diagrams" of Fig. 2, in which each index (quark, as it were) is represented by a directed line. We shall evaluate the nucleon-nucleon-deuteron vertex in the case when all three momenta are positive timelike, the nucleons are in the ground state,

$$\psi^{*\mu_1\cdots\mu_N}\to\tilde{\psi}^*(p_1)p_1^{\mu_1}\cdots p_1^{\mu_N},$$
$$\psi^{*\mu_{N+1}\cdots\mu_{2N}}\to\tilde{\psi}^*(p_2)p_2^{\mu_{N+1}}\cdots p_2^{\mu_{2N}},$$

and the deuteron is in a τth excited state, represented by the $(\tau+1)$th term in the expansion (9) (replace N by $2N$):

$$\chi_{\mu_1\cdots\mu_{2N}}\to SC'(\tau)\tilde{\chi}_{\mu_1\cdots\mu_\tau}(P)P_{\mu_{\tau+1}}\cdots P_{\mu_{2N}}s^{(\tau-2N)/2},$$
$$P=p_1+p_2,\quad s=P^2,$$
$$C'(\tau)=[(\tau-2N-1)!/\tau!]^{1/2}.$$

Inserting these expressions into (14) we find, after a short calculation, using old tricks,[4] that the vertex reduces to

$$V^{\mu_1\cdots\mu_\tau}[\tilde{\psi}^*(p_1)\tilde{\psi}^*(p_2)\tilde{\chi}_{\mu_1\cdots\mu_\tau}(P)],$$
$$V^{\mu_1\cdots\mu_\tau}=V_\tau Q^{\mu_1}\cdots Q^{\mu_\tau},$$
$$V_\tau=\left[\frac{(\tau-1)!!(\tau-2N-2)!!}{\tau!!(\tau-2N-1)!!}\right]^{1/2}s^{N-\tau/2},\quad \tau\text{ even}\tag{16}$$
$$=0,\quad \tau\text{ odd}$$
$$Q=p_1-p_2.$$

Since N is negative, the amplitude $V^{\mu_1\cdots\mu_\tau}$ goes to zero as $s\to\infty$.

The nucleon-nucleon-meson vertex may be calculated too, but we shall postpone that to another paper.

D. Scattering of Spin-Zero Particles

We shall evaluate the amplitude for nucleon-nucleon scattering proceeding through deuteron intermediary states, as shown by the diagrams of Fig 3. Again the deuteron mass differences will be ignored, so that the propagator is just

$$(s-m^2)^{-1}S,\tag{17}$$

where S is the index symmetrizer. Then there is no need to reduce the amplitude to a sum over deuteron states. If all four nucleons are in the ground state, then the complete amplitude is

$$\psi^{*\mu_1\cdots\mu_N}(p_1)\psi^{*\mu_{N+1}\cdots\mu_{2N}}(p_2)\frac{S}{s-m^2}\times\psi_{\mu_1\cdots\mu_N}(p_1')\psi_{\mu_{N+1}\cdots\mu_{2N}}(p_2')$$
$$=p_1^{\mu_1}\cdots p_1^{\mu_N}p_2^{\mu_{N+1}}\cdots p_2^{\mu_{2N}}\frac{S}{s-m^2}\times p'_{1\mu_1}\cdots p'_{1\mu_N}p'_{2\mu_{N+1}}\cdots p'_{2\mu_{2N}}.\tag{18}$$

The symmetrizer S eliminates the contributions of all intermediary states except those that make up the irreducible representation $D(2N,0)$, i.e., the deuteron states.

The problem of carrying out the contractions on the indices in (18) is a combinatorial one. The solution is a special case of the formula

$$p_1^{\mu_1}\cdots p_1^{\mu_a}p_2^{\mu_{a+1}}\cdots p_2^{\mu_c}Sp'_{1\mu_1}\cdots p'_{1\mu_b}p'_{2\mu_{b+1}}\cdots p'_{2\mu_c}$$
$$=(p_1p_2')^a(p_2p_1')^b(p_2p_2')^{c-a-b}\times{}_2F_1(-a,-b;-c;z),\tag{19}$$

where

$$z=1-(p_1p_1')(p_2p_2')/(p_1p_2')(p_2p_1').$$

The amplitude for the elastic scattering of ground-state nucleons is thus, when deuteron mass differences are ignored,

$$A(s,t)=(1-u/2m^2)^{2N}(s-M^2)^{-1}\times{}_2F_1(-N,-N;-2N;z).\tag{20}$$

As long as p_1+p_2 is positive timelike, it is possible to reduce the amplitude to a sum over intermediary deuteron states, using the vertex functions calculated above, Eq. (16):

$$A(s,t)=\sum_{\tau/2=0,1,\cdots}V^{\mu_1\cdots\mu_\tau}(s-M^2)^{-1}V'_{\mu_1\cdots\mu_\tau},$$
$$=c\sum_{\frac{1}{2}\tau=0,1,\cdots}\frac{(\frac{1}{2}\tau-\frac{1}{2})!(\frac{1}{2}\tau-N-1)!}{(\frac{1}{2}\tau)!(\frac{1}{2}\tau-\frac{1}{2}-N)!}s^{2N}\times\left(\frac{t-u}{s}\right)^\tau(s-M^2)^{-1},$$
$$=\left(\frac{s}{4m^2}\right)^{2N}(s-M^2)^{-1}\times{}_2F_1\left[\tfrac{1}{2},-N;\tfrac{1}{2}-N;\left(\frac{t-u}{s}\right)^2\right].\tag{21}$$

By one of the standard quadratic transformations of hypergeometric functions, it may be shown that (20) and (21) are the same function.

Asymptotically, as $s\to\infty$, this amplitude tends to zero as $s^{2N-1}\ln s$, which is very fast if N is a reasonable number like -2, as is suggested by the form-factor results. The introduction of a more realistic deuteron propagator, reflecting more accurately the spectrum of resonances that is represented by the deuteron intermediary states, will modify this asymptotic behavior.

The amplitude for nucleon-antinucleon scattering with deuteron exchange is related to the amplitude for nucleon-nucleon scattering via deuteron intermediary states, by a simple substitution rule that may be called crossing, but *not* by analytic continuation.[11] All momenta are to be considered as physical positive timelike momenta; the direct evaluation of the complete amplitude is carried out as above, without regard to whether the external lines are entering or leaving the diagram. If $s^x=(p_1+p_1')^2$, $t^x=(p_1-p_2')^2$, and $u^x=(p_1-p_2)^2$ are the energy and momentum-transfer variables in the annihilation channel, then the amplitude is

$$\begin{aligned}A^x(s^x,t^x)&=(1-t^x/2m^2)^{2N}(u^x-M^2)^{-1}\\&\qquad\times{}_2F_1(-N,-N;-2N;z)\\&=((s^x-t^x)/4m^2)^{2N}(u^x-M^2)^{-1}\\&\qquad\times{}_2F_1[\tfrac{1}{2},-N;\tfrac{1}{2}-N;(u^x/(s^x-t^x))^2].\end{aligned}\tag{22}$$

Note that this result was obtained by a direct evaluation of the complete amplitude. It is also possible to obtain the second expression directly, by the type of calculation that led to the result (21), although the deuteron momentum is now spacelike, so that the symmetry group $U(3)p$ is isomorphic to $U(2,1)$.

The reduction of a unitary representation of a noncompact group to irreducible representations of a noncompact subgroup is usually continuous (direct integral), but in the present case it happens to be discrete. This is because the basis states $z_0^{N-\tau}z_1^az_2^bz_3^c$, which are eigenvectors of the Casimir operator C_{00} of $U(3)$, are eigenvectors of the Casimir operator C_{11} of $U(2,1)$ as well. The spectrum of eigenvalues of the latter operator is 0, 1, 2, $\cdots$. The operator τ is defined covariantly by

$$\tau=N-p^\mu p^\nu C_{\mu\nu}/p^2\tag{23}$$

for both spacelike and timelike p^μ; consequently the spectrum of τ is N, $N-1$, $N-2$, $\cdots$, for spacelike momentum. This means that the tensor-reduction formula (9) can be used for spacelike momentum, except that the sum now goes over the values N, $N-1$, $N-2$, $\cdots$, of τ.

The expression (16) for the vertex function is valid for spacelike P as well, except that now $P=p_1-p_2$, and $P^2=u_x$ instead of s. The scattering amplitude may be represented as follows:

$$\begin{aligned}A^x(s^x,t^x)&=\sum_{N-\frac{1}{2}\tau'=0,1,\cdots}V^{\mu_1\cdots\mu_\tau}(u^x-M^2)^{-1}V_{\mu_1\cdots\mu_\tau}\\&=c\sum_{\frac{1}{2}\tau'=0,1,\cdots}\frac{(\frac{1}{2}\tau'-1)!(\frac{1}{2}\tau'-N-1)!}{(\frac{1}{2}\tau')!(\frac{1}{2}\tau-N-\frac{1}{2})!}\\&\qquad\times(s^x-t^x)^{2N}\left(\frac{u^x}{s^x-t^x}\right)^{\tau'}(u^x-M^2)^{-1},\end{aligned}$$

where $\tau'=2N-\tau$. Although this formula represents the natural decomposition of the amplitude, it is of course possible to perform a Sommerfeld-Watson transformation and write

$$\begin{aligned}A^x(s^x,t^x)&=c(u^x-M^2)^{-1}(u^x)^{2N}\\&\quad\times\int_C\frac{(-\frac{1}{2}\tau-1)!(N-\frac{1}{2}-\frac{1}{2}\tau)!(\frac{1}{2}\tau-N-1)!}{(-\frac{1}{2}\tau-\frac{1}{2})!}\\&\qquad\times\left[-\left(\frac{s^x-t^x}{u^x}\right)^2\right]^{\tau/2}d(\tau/2),\end{aligned}\tag{24}$$

where the contour follows the line $\mathrm{Re}(\frac{1}{2}\tau)=N$ and loops the point $\tau=2N$ on the right. After a change of the variable of integration from $\frac{1}{2}\tau$ to $\frac{1}{2}\tau'=N-\frac{1}{2}\tau$ this becomes Barnes' integral representation[12] for the hypergeometric function that appears in (22).

E. Spin-1 and Superconvergence

It has been suggested that[13] if one of the two particles in an elastic scattering process has spin different from zero, a part of the amplitude, the double spin-flip amplitude, goes to zero as $s\to\infty$ faster than the other parts. It is almost obvious that our deuteron exchange amplitude has this property; we shall verify it by explicit evaluation.

In the Feynman diagram of Fig. 3, let p_1 and p_1' be the momenta of a nucleon-antinucleon initial state, and let the incoming and outgoing nucleons (momenta p_1 and p_2') be in the first excited state, while the antinucleons (momenta p_1' and p_2) are in the ground state. The amplitude is given by

$$\begin{aligned}T_{\mu\nu}^x(s^x,t^x)\tilde{\psi}^{*\mu}(p_1)\tilde{\psi}^\nu(p_2')(u^x-M^2)&\\=\tilde{\psi}^{*\mu_1}p_1^{\mu_1}\cdots p_1^{\mu_N}p_2^{\mu_{N+1}}\cdots p_2^{\mu_{2N}}Sp_{1\mu_1}'\cdots&\\p_{1\mu_N}'\tilde{\psi}_{\mu_{N+1}}p_{2\mu_{N+2}}'\cdots p_{2\mu_{2N}}'&\end{aligned}$$

or

$$\begin{aligned}T_{\mu\nu}^x(s^x,t^x)=4\alpha p_{1\mu}'p_{2\nu}+2\beta(p_{2\mu}'p_{2\nu}+p_{1\mu}'p_{1\nu})&\\+\gamma_1p_{2\mu}'p_{1\nu}+\gamma_2g_{\mu\nu},&\end{aligned}$$

where

$$\begin{aligned}\alpha&=(N/2)^2(p_1p_1')^{2N-2}{}_2F_1(1-N,1-N;2-2N;z),\\\beta&=\tfrac{1}{2}N(N-1)(p_1p_1')^{2N-3}(p_1p_2')_2F_1\\&\qquad\times(1-N,2-N;2-2N;z),\\\gamma_1&=(N-1)^2(p_1p_1')^{2N-4}(p_1p_2')^2{}_2F_1\\&\qquad\times(2-N,2-N;2-2N;z),\\\gamma_2&=(p_1p_1')^{2N-2}(p_1p_2')_2F_1(1-N,1-N;1-2N;z),\\z&=1-(p_1p_2'/p_1p_1')^2,\\s^x&=(p_1+p_1')^2,\\t^x&=(p_1-p_2')^2.\end{aligned}$$

[11] C. Fronsdal and R. White, Phys. Rev. (to be published).

[12] *Bateman Project Manuscript, Higher Transcendental Functions*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 62.

[13] V. de Alfaro, S. Fubini, C. Rossetti, and G. Furlan, Phys. Letters 21, 576 (1966).

The high-energy limits ($s^x \to \infty$, t^x fixed) of these invariant amplitudes are

$$\alpha \approx (1-2N)\frac{(s^x)^{2N-2}}{N^2}\left[-\gamma+\ln\frac{s^x}{2m^2-t^x}\right],$$

$$\beta \approx -(1-2N)\frac{(s^x)^{2N-1}}{N}(2m^2-t^x)^{-1},$$

$$\gamma_1 \approx 2(1-2N)(s^x)^{2N}(2m^2-t^x)^{-2},$$

$$\gamma_2 \approx (s^x)^{2N}(2m^2-t^x)^{-1}.$$

We notice that as $s^x \to \infty$ for fixed momentum transfer, β tends to zero faster than γ_1 and γ_2, and α faster than β. Identical results, except for the logarithmic term in α, were obtained by de Alfaro *et al.*[1] It is worth emphasizing that this superconvergent asymptotic behavior of the various amplitudes for spin-0—spin-1 scattering is completely independent, in our model, of the postulates introduced by these authors; we have used neither Regge hypotheses nor analytic continuation. The amplitudes β and α are superconvergent for any value of N for which the representation $D(N,0)$ is unitary. If N is of the order of -2, as suggested by the form-factor results, then γ_1 and γ_2 are also superconvergent, and all amplitudes fall off much faster than what appears reasonable. It is to be hoped that the introduction of mass breaking in the one-particle propagator will tend to temper the convergence of all amplitudes, while preserving the hierarchy of superconvergence.

F. Mass Splitting

The amplitudes that have been calculated so far are not true predictions of the harmonic oscillator model, since the operator $S(s-M^2)^{-1}$ is a poor substitute for the real propagator. Clearly, the propagator must have a pole at each of the points $s=m^2(\tau)$, $\tau=0, 1, 2, \cdots$, where $m(\tau)$ is the rest energy of the τth level of the oscillator. The precise form of the $m(\tau)$ might be $m+\tau\omega$, but the nonrelativistic theory allows $m(\tau)$ to be any increasing function of τ that tends to $m+\tau\omega$ in the limit $m/\omega \to \infty$. The relativistic theory places demands on the propagator for spacelike as well as timelike momenta, which restricts the functional dependence of $m(\tau)$ on τ. For example, there is a domain of s and τ where a singularity of the propagator would give a noncausal S matrix. In a local-field theory, questions of this type are related to the existence of unphysical solutions of the field equations. One natural way to select the propagator is to set up a complete local-field theory, and define the propagator as the Green's function of the relativistic field equation.

Unfortunately, we have not yet found a relativistic field equation that can be postulated with any degree of confidence. Part of the difficulty is uncertainty about the right criteria. If one insists, for example, on a *local* equation (in order to prepare for the introduction of a minimal electromagnetic interaction) then extra, unwanted solutions always spring up. Some would conclude from this that field theory, and particularly locality, is irrelevant, and abandon it, at the price of losing all predictive power; the form factors and the scattering amplitudes that were calculated above are suggested by the locality of the interactions. An alternative, that will be pursued a short way here, is to retain the idea of local interactions, and to postulate in addition that scattering amplitudes are defined by a set of Feynman rules, in which the one-particle propagator is some smooth function of p_μ and τ (and j, if we wish to introduce fine structure), that is to be determined phenomenologically, and into which one attempts to pack as much physics as possible.

Lacking a complete field theory, we have to postulate the Feynman rules of the theory with sufficient detail to make the amplitudes well defined. The following prescriptions seem to be reasonable, at least as far as simple diagrams are concerned.

1. In the region of timelike momentum transfers, $s>0$, the propagator is of the form

$$\sum_{\tau=0}^{\infty} P_\tau K(s,\tau),$$

where $K(s,\tau)$ is some reasonably smooth function of s and τ, and P_τ is the projection operator for the subspace defined by τ.

$$P_\tau P_{\tau'} = \delta_{\tau\tau'} P_\tau, \quad \sum P_\tau = 1.$$

2. In a region $s<s_0$, for sufficiently large negative s_0, the propagator is of the form

$$\int_C \frac{(-)^\tau d\tau}{2i\sin\pi\tau} P_\tau K^x(s,\tau),$$

where P_τ has the same interpretation as above. The value of s_0 is defined by the requirement that no singularity of $K^x(s,\tau)$ cross the contour C as s decreases from s_0 to $-\infty$.

3. In the region $s_0<s<0$, the propagator is defined by analytic continuation in s, from the region $s<s_0$, *not* from positive s. Hence it is not necessary to assume a direct analytic connection between different channels. As s approaches zero, it is possible that singularities of $K(s,\tau)$ migrate across the contour C, necessitating deformation of the contour of integration.

In Sec. IIID it was shown that these prescriptions give the correct answers in the equal-mass case. In that case the asymptotic behavior of the amplitude (23) is $(s^x)^{2N}\ln s^x$. In the general case the amplitude (24) is to be modified by replacing the factor $(u^x-M^2)^{-1}$ by $K(u^x,\tau)$. The manifold of singularities of this function

[14] See Ref. 12, p. 110. The constant γ is the Euler-Mascheroni constant.

plays the same role as Regge poles in determining the asymptotic behavior of the scattering amplitudes. Thus, if $K(u^x,\tau)$ has the form $\tau/(\tau-\tau_0-u^x)$, $0<\tau_0<1$, then in the asymptotic limit $s^x\to\infty$, $u^x\approx 0$, the amplitude is deminated by the Regge-pole contribution

$$(u^x)^{2N-\tau_0}(s^x)^{\tau_0}.$$

If the scattering particles (*external lines*) have different mass, then this is modified. Thus, if $m_1=m_2'=m$ and $m_1'=m_2=\mu$, then the first factor becomes

$$[u^x-(m^2-\mu^2)^2/2s^x]^{2N-\tau_0}.$$

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Some Features of Angular-Momentum Branch Points*

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Arguments for the existence of angular-momentum branch points based on the necessity for the Gribov-Pomeranchuk singularities to be fixed poles rather than essential singularities are reviewed. Sum rules for scattering amplitudes on the second sheet are then deduced. Reasons are given for expecting two kinds of branch points to be present, namely, those described as "Regge pole plus Regge pole" and "Regge pole plus elementary particle" (called types 1 and 2, respectively). It is argued that the latter must be concealed by the former in the scattering region, and from the requirement that the branch points are suitably positioned in general, an inequality on derivatives of Regge-pole trajectories is derived. A model of the Amati–Fubini–Stanghellini type is examined to indicate why type-2 branch points may be expected to occur in a theory without elementary particles.

1. INTRODUCTION

SEVERAL features of branch points in the angular-momentum plane for two-body scattering amplitudes are discussed in this paper. Recently, there has been significant progress[1] in understanding the way in which moving branch points eliminate the once-suspected need for essential singularities[2] at wrong-signature nonsense points and allow fixed poles to be present instead. We shall assume this mechanism to be generally valid and utilize it to make several deductions about the branch points. We emphasize the theme that while the branch points are extremely difficult to discuss in terms of Feynman diagrams, there is a great deal that can be learned about them from a liberal application of unitarity without getting involved with overwhelming complexities. Although not yet at the point of being able to present reliable formulas for their contributions to asymptotic behavior, we believe that there are grounds for optimism in this regard.

In Sec. 2 we review the arguments for the existence of angular-momentum branch points and the inferences that are drawn for the behavior of partial-wave amplitudes at wrong-signature nonsense points. Also, a sum rule for amplitudes on the second sheet of the elastic cut is deduced. In Sec. 3 we explain why two essentially different types of branch points are required, and indicate some interesting features of their sheet structure. In order to satisfy a requirement on the positioning of the two types of branch points in general, we conjecture an inequality involving derivatives that should be satisfied by *any* Regge-pole trajectory function. The structure of the branch points is explored further and illustrated in a simple model in Sec. 4. The model consists of an integral of the AFS type,[3] and while its relevance[4] may be questioned, it nevertheless offers an instructive example. Section 5 contains some comments about possible generalizations and applications and summarizes the conclusions.

2. NEED FOR BRANCH POINTS AND RESULTING SUM RULES

The existence of moving branch points in angular momentum can be most easily demonstrated by arguments given originally by Mandelstam.[5] Simply stated, the

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[1] C. E. Jones and V. Teplitz, Phys. Rev. **159**, 1271 (1967); S. Mandelstam and L. L. Wang, Phys. Rev. **160**, 1490 (1967).

[2] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Letters **2**, 239 (1962); Ya. I. Azimov, Phys. Letters **3**, 195 (1963); S. Mandelstam, Nuovo Cimento **30**, 1113 (1963).

[3] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962), hereafter AFS; Nuovo Cimento **26**, 896 (1962).

[4] S. Mandelstam, Nuovo Cimento **30**, 1127 (1963); H. J. Rothe, Phys. Rev. **159**, 1471 (1967). Most of the conclusions of Sec. 4 are also reached in Rothe's paper.

[5] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).