In Fig. 8, we show σ and γ as functions of the meson and baryon mass-splitting parameter δ. The plots are remarkably linear out to $\delta=0.7$. However, on closer examination we find that third-order effects and higher are important, even at $\delta=0.7$. This is because a strong second-order effect in s_c cancels the strong third-order effect in, for example, $K_{27-10}(Y^*)$.

The importance of the higher-order effects becomes obvious when we plot σ and γ as functions of the coupling symmetry-breaking parameter $\epsilon=\epsilon_1=\epsilon_2$. Here, the second-order terms of s_c are of the same sign and magnitude as the third-order effects in, for example, $K_{27-10}(Y^*)$, and the plot is quite nonlinear, as shown in Fig. 9.

These results are not very dependent on the value of s, if s is not near a pole of $K(s)$. The sum rules slowly become better satisfied as s is increased. For example, for $\delta=0.25$ and $\epsilon=0$, we have $\gamma=0.23$ at $s=14$ (BeV)^2; $\gamma=0.10$ at $s=20$ (BeV)^2, and $\gamma=0.04$ at $s=45$ (BeV)^2.

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Regge Daughter Trajectories in the Bethe-Salpeter Equation*

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Parent and daughter Regge trajectories are found numerically in the scalar Bethe-Salpeter equation with trilinear coupling in the ladder approximation. By using the ordinary partial-wave projection of the scattering amplitude, we obtain the trajectories over a range of energy below threshold; and by projecting the scattering amplitude into 4-dimensional spherical harmonics, we find the slope and intercept of the trajectories at $s=0$.

I. INTRODUCTION

RECENTLY, Freedman and Wang[1] have demonstrated that in order for unequal-mass scattering to be consistent with the Mandelstam representation Regge poles must occur in "families" where a "parent" trajectory at $l=\alpha_0(s=0)$ implies the existence of "daughter" trajectories with $\alpha_K(0)=\alpha_0(0)-K$, $K=1$, $2,\cdots$. Both the above paper and Domokos and Suranyi[2] have pointed out that daughter trajectories must exist for Bethe-Salpeter equations which display $O(4)$ symmetry. Freedman and Wang[1] and Domokos[3] have speculated about fitting experimental resonances to daughter trajectories. It is therefore important to examine the slopes and shapes of the trajectories of the Regge "family" in a relativistic model such as the Bethe-Salpeter (BS) equation, where the daughter trajectories were first found. We present the results of numerical computations of the trajectories of the scalar BS equation with trilinear coupling in the ladder approximation.

The calculations were done by two independent methods. The first method involves the direct solution for the poles of the partial-wave amplitude, and therefore does not make explicit use of $O(4)$ symmetry. The parent and first daughter trajectories were computed over a range of energy below threshold in the Regge-pole channel, and for different values of the coupling constant. The effect of renormalizing the propagator, and thus satisfying a form of three-particle unitarity, is also shown.

The second method, which gives only the intercepts and slopes of the trajectories at $s=0$, but to a higher degree of accuracy, uses an expansion of T in 4-dimensional spherical harmonics. This method of solution was used for several coupling constants, exchange masses, and different external masses.

II. THE BETHE-SALPETER EQUATION

Using the 4-momentum assignment shown in Fig. 1 and a $(-,+,+,+)$ Lorentz metric, the BS equation for 2 (nonidentical) particle scattering is

$$T(p,p',s)=K(p,p')-\int d^4p''\,K(p,p'')G(p'',s)T(p'',p',s)\,,$$

* Work supported in part by the U. S. Atomic Energy Commission.

[1] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967).
[2] G. Domokos and P. Suranyi, Nucl. Phys. **54**, 529 (1964).
[3] G. Domokos, Phys. Rev. **159**, 1387 (1967).

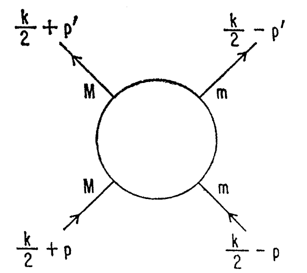

FIG. 1. Kinematics.

where

$$G(p,s)=[(p+\tfrac{1}{2}k)^2+M^2-i\epsilon]^{-1}[(p-\tfrac{1}{2}k)^2+m^2-i\epsilon]^{-1},$$

$$K(p,p')=\frac{(i/\pi^2)\lambda}{(p-p')^2+\mu^2-i\epsilon}$$

for the ladder approximation,

and λ is the coupling strength. In the center-of-mass system $k=(\sqrt{s},0,0,0)$. After making the Wick rotation,[4] and redefining K and T to absorb some i's, the BS equation becomes

$$T(p,p',s)=-K(p,p')+\int d^4p''\,K(p,p'')\times G(p'',s)T(p'',p',s)\,,\quad (1)$$

where the p space is now Euclidean, $k=(0,0,0-i\sqrt{s})$, and

$$K(p,p')=\frac{(\lambda/\pi^2)}{(p-p')^2+\mu^2}.$$

III. METHOD I: SOLUTION OF THE PARTIAL-WAVE EQUATION

Let

$$T(p,p',s)=\sum_{l,m}Y_l^m(\theta,\varphi)T_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';s)Y_l^{m*}(\theta',\varphi'),$$

and (2)

$$K(p,p')=\sum_{l,m}Y_l^m(\theta,\varphi)V_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega')Y_l^{m*}(\theta',\varphi'),$$

where the $|\mathbf{p}|$ is the magnitude of 3 momentum and ω is the 4th component. Then the BS equation for the lth partial-wave amplitude is

$$T_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';s)=V_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega')+\int_{-\infty}^{\infty}d\omega\int_0^{\infty}|\mathbf{p}''|^2d|\mathbf{p}''|\times V_l(|\mathbf{p}|,\omega;\,|\mathbf{p}''|,\omega'')G(|\mathbf{p}''|,\omega'')\times T_l(|\mathbf{p}''|,\omega'';\,|\mathbf{p}'|,\omega';s)\,,\quad (3)$$

where

$$G(|\mathbf{p}|,\omega)=[|\mathbf{p}|^2+(\omega+\tfrac{1}{2}i\sqrt{s})^2+M^2]^{-1}\times[|\mathbf{p}|^2+(\omega-\tfrac{1}{2}i\sqrt{s})+m^2]^{-1},$$

and

$$V_l=\frac{2\lambda}{\pi}\frac{1}{pp'}Q_l\left(\frac{|\mathbf{p}|^2+|\mathbf{p}'|^2+\mu^2+(\omega-\omega')^2}{2|\mathbf{p}|\,|\mathbf{p}'|}\right).$$

[4] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962); G. C. Wick, *ibid.* **96**, 1124 (1952).

The partial-wave scattering amplitude T_l and the potential V_l can be separated into even and odd parts under the exchange $\omega\rightarrow-\omega$:

$$T_l^{e,o}(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';s)=\tfrac{1}{2}[T_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';s)\pm T_l(|\mathbf{p}|,-\omega;\,|\mathbf{p}'|,\omega';s)],$$

$$V_l^{e,o}(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';s)=\tfrac{1}{2}[V_l|\mathbf{p}|,\omega';\,|\mathbf{p}'|,\omega';s)\pm V_l(|\mathbf{p}|,-\omega;\,|\mathbf{p}'|,\omega';s)],$$

with the plus signs for e and the minus for o.

Then T_l^e and T_l^o separately satisfy Eq. (3) with potentials V_l^e and V_l^o, respectively. In operator form, we have

$$T_l^e=V_l^e+V_l^eGT_l^e,\quad (4)$$

$$T_l^o=V_l^o+V_l^oGT_l^o.\quad (5)$$

Solving Eqs. (4) and (5) gives us

$$T_l=T_l^e+T_l^o=(1-V_l^eG)^{-1}V_l^e+(1-V_l^oG)^{-1}V_l^o\quad (6)$$

Regge poles are obtained after continuing Eq. (6) to noninteger l. In practice, integrations are done by the method of Gaussian quadratures. The kernels are then approximated by matrices, and the poles of the scattering amplitude found by searching for the value of l for which one of the Fredholm determinants equals zero, i.e., either

$$\det[1-V_l^eG]=0\,,\quad (7)$$

or

$$\det[1-V_l^oG]=0.\quad (8)$$

We note the following two points. For the scattering of identical particles, Eqs. (7) and (8) would correspond to the condition for locating the even- and odd-signature trajectories, respectively, if exchange of a particle of mass μ in the u channel were included in the potential.

Secondly, for equal-mass scattering the odd potential V_l^o vanishes when the final-state particles are put on the mass shell. Thus the odd trajectory given by Eq. (8) does not contribute a pole to the scattering amplitude in Eq. (6). Nevertheless, the trajectory computed by Eq. (8) has relevance in our model, since a small splitting in the external masses would perturb Eq. (8) only continuously, while allowing a nonzero residue.

In relation to this last point, note that at $s=0$ in unequal-mass scattering the momenta on the mass shell are unphysical.[5] Nevertheless, our determination of the location of the trajectory is still relevant since the Fredholm determinant does not depend on these momenta. This dependence would, however, be reflected in the residue of the pole.

Since we are calculating the Fredholm determinant of a two-dimensional integral equation, accuracy is a problem. (The size of the mesh is severely limited by the speed and the memory of the computer, CDC 3600.) The accuracy was checked by changing the number and distribution of mesh points.

[5] See Ref. 1, Eq. (21).

In order to judge the usefulness of the ladder approximation, a "higher-order" solution was computed by renormalizing the propagators, following Levine *et al.*[6] The free-particle propagators in G are replaced by the propagators obtained by summing all numbers of self-energy bubbles as shown in Fig. 2. This is done by replacing the factors $(|\mathbf{p}|^2+m^2)^{-1}$ in G by $(|\mathbf{p}|^2+m^2)^{-1}$

$$[1+(|\mathbf{p}|^2+m^2)A(|\mathbf{p}|^2)]^{-1},$$

where

$$A(|\mathbf{p}|^2)=\lambda\int_{(\mu+m)^2}^{\infty} ds\frac{\{[s-(\mu+m)^2]/s\}^{1/2}}{(s-m^2)^2(s+|\mathbf{p}|^2)}. \quad (9)$$

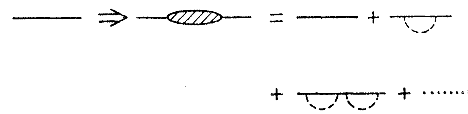

FIG. 2. Sum of self-energy bubbles that renormalize the propagator.

IV. METHOD II: EXPLOITATION OF $O(4)$ SYMMETRY

Starting from Eq. (1) and letting

$$T'(p,p',s)=G(p,s)T(p,p',s),$$

we get[7]

$$G^{-1}(p,s)T'(p,p',s) =-K(p,p')+\int d^4p''\,K(p,p'')T'(p,p',s). \quad (10)$$

(In what follows, we drop the primes on T.)

In the 4-dimensional Euclidean space

$$p_4=P\cos\Psi,$$
$$p_3=P\sin\Psi\cos\theta,$$
$$p_1=P\sin\Psi\sin\theta\cos\varphi,$$
$$p_2=P\sin\Psi\sin\theta\sin\varphi,$$

and

$$G^{-1}(p,s)=(P^2-\tfrac{1}{4}s+M^2)(P^2-\tfrac{1}{4}s+m^2) +P^2s\cos^2\Psi+iP(M^2-m^2)(\sqrt{s})\cos\Psi \equiv F_0(P^2,s)+P^2s\cos^2\Psi+iP\Delta(\sqrt{s})\cos\Psi.$$

We now expand T and K in 4-dimensional spherical harmonics:

$$Z_{nl}^{m}(\Psi,\theta,\varphi)=2^{l+1/2}\Gamma(l+1)\left[\frac{(n+1)\Gamma(n-l+1)}{\pi\Gamma(n+l+2)}\right]^{1/2} \times(\sin\Psi)^lC_{n-l}^{l+1}(\cos\Psi)Y_l^m(\theta,\varphi) =p_n^l(\Psi)Y_l^m(\theta,\varphi), \quad (11)$$

which are nonzero for $n=0, 1, \cdots, l=0, 1, 2, \cdots, n, m=-l, l+1, \cdots, l$. $C_n^m(x)$ is a Gegenbauer polynomial.[8] The Z's are orthogonal over the surface of a 4-dimensional sphere, i.e., over

$$\int_0^\pi \sin^2\Psi d\Psi\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi\equiv\int d\Omega$$

and obey (for integers n, l, and m) the following vector addition theorem:

$$Z_{n0}^{0}(\gamma,0,0)=\frac{2\pi}{\sqrt{2}(n+1)}\sum_{l=0}^{n}\sum_{m=-l}^{l} Z_{nl}^{m}(\Psi_1,\theta_1,\varphi_1) \times Z_{nl}^{m*}(\Psi_2,\theta_2,\varphi_2),$$

where

$$\cos\gamma=\cos\Psi\cos\Psi_2+\sin\Psi_1\sin\Psi_2 \times[\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)].$$

Now let

$$K(p,p')=\sum_n K_n(P,P')\frac{(n+1)\sqrt{2}}{2\pi}Z_{n0}^{0}(\hat{p}\cdot\hat{p}',0,0) =\sum_{nlm} Z_{nl}^{m}(\hat{p})K_n(P,P')Z_{nl}^{m*}(\hat{p}'),$$

$$K_n(P,P')=\frac{\lambda[z+(z^2-1)^{1/2}]^{-n-1}}{8\pi^2(n+1)PP'},$$

$$z=\frac{P^2+P'^2+\mu^2}{2PP'},$$

$$T(p,p',s)=\sum_{nn'lm} Z_{nl}^{m}(\hat{p})T_{nn'}^{l}(P,P',s)Z_{n'l}^{m*}(\hat{p}'). \quad (12)$$

From Eqs. (2), (11), and (12) we see that the relation of 4-dimensional projected T and the ordinary 3-dimensional projected T is

$$T_l(|\mathbf{p}|,\omega;\,|\mathbf{p}'|,\omega';\,s) =\sum_{n=l,n'=l}^{\infty} p_n^l(\Psi)T_{nn'}^{l}(P,P';\,s)p_n^l(\Psi'), \quad (13)$$

where $|\mathbf{p}|=P\sin\Psi$ and $\omega=P\cos\Psi$.

Using the orthonormality of the Z's, the BS equation becomes [with the argument (P,P') suppressed]

$$F_0T_{nn'}^{l}+P^2s(f_{nn}^{l}T_{nn'}^{l} +f_{n,n+2}^{l}T_{n+2,n'}^{l}+f_{n,n-2}^{l}T_{n-2,n'}^{l}) +iP\Delta(\sqrt{s})(g_{n,n-1}^{l}T_{n-1,n'}^{l}+g_{n,n+1}^{l}T_{n+1,n'}^{l}) =-\delta_{nn'}K_n+\int P''^3dP''\,K_n(P,P'')T_{nn'}^{l}(P'',P'), \quad (14)$$

[6] M. J. Levine, J. Wright, and J. A. Tjon, Phys. Rev. **157**, 1416 (1967).

[7] Except for the trivial generalization to unequal-mass scattering the following derivation follows Domokos and Suranyi. [See Ref. (2)]. Note that our n equals their $n-1$.

[8] *Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, Sec. 3.15.1.

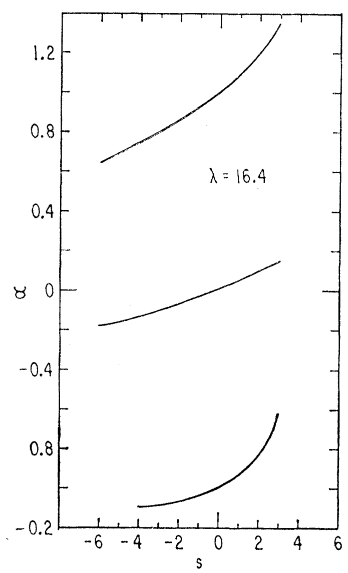

FIG. 3. Regge tragectories for Bethe-Salpeter euqation computed by Method I. Note integer spacing at $s=0$, $m=\mu=M=1$.

where

$$f_{nn'}{}^{l}\equiv\int d\Omega\, Z_{nl}{}^{m}(\Omega)\cos^2\Psi Z_{n'l}{}^{m*}(\Omega)$$

$$\begin{aligned} &=A_n{}^lA_{n+1}{}^l, && n'=n+2\\ &=(A_n{}^l)^2+(A_{n-1}{}^l)^2, && n'=n\\ &=A_{n-1}{}^lA_{n-2}{}^l, && n'=n-2\\ &=0, && \text{otherwise}\end{aligned}$$

and

$$g_{nn'}{}^{l}=\int d\Omega\, Z_{nl}{}^{m}(\Omega)\cos\Psi Z_{n'l}{}^{m*}(\Omega)=A_n{}^l,\quad n'=n+1$$

$$\begin{aligned} &=A_{n-1}{}^l, && n'=n-1\\ &=0, && \text{otherwise}\end{aligned}$$

where

$$A_n{}^l=\left(\frac{(n-l+1)(n+l+2)}{4(n+1)(n+2)}\right)^{1/2}.$$

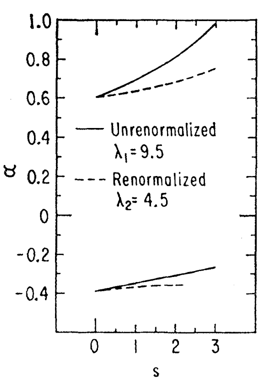

FIG. 4. Effect of propagator renormalization on the Regge trajectories computed by Method I. $m=\mu=M=1$. $\lambda_1(\lambda_2)$ is the coupling constant using unrenormalized (renormalized) propagator.

Equations (13) and (14) are now continued to noninteger n and l, keeping $K\equiv(n-l)$ an integer.[1,3] The parent trajectory has $K=0$; the Nth daughter has $K=N$. At $s=0$ the BS equation and hence $T_{nn}{}^l(P,P')$ are independent of l. Therefore by Eq. (13) a pole in $T_{nn}{}^l$ at n_0 will produce poles in T_l at $l=n_0$, n_0-1, n_0-2, $\cdots$.

Since the inhomogeneous term is only in Eq. (14) for $T_{nn}{}^l$, it is evident that for s near zero $T_{n,n+m}{}^l\propto s^{m/2}$. Thus to find $T_{nn}{}^l$ to order s it is only necessary to solve the equations with $n'=n$, $n\pm1$ (only $n'=n$ for equal external masses) in which one sets $T_{n,n\pm m}=0$ for $m=2$, $3,\cdots$. But to find the Regge poles it is not necessary to find T, so that instead of solving the three (one for equal external masses) simultaneous integral equations we only searched for what n and l would make the Fredholm determinant zero. This was done at $s=0$ and small values of s to determine both the intercept and slope of the trajectories.

There was no difficulty obtaining any desired accuracy with this method, since the integral equations are only one dimensional.

V. DISCUSSION AND RESULTS

The results of calculation by Method I with $M=m=\mu=1.0$ for three different coupling strengths are shown in Figs. 3–5. Figures 4 and 5 also show trajectories with and without the renormalized propagators. Here the coupling strengths were chosen to make the trajectories coincide at $s=0$. The intercepts and slopes at $s=0$ calculated by Method II (with unrenormalized propagators) are given in Table I. In two cases, two families were found for the same constants. Note that where the two methods were applied to the same problem, the same results were obtained. Also our results agree with Schwartz's bound states[9] when l is an integer.

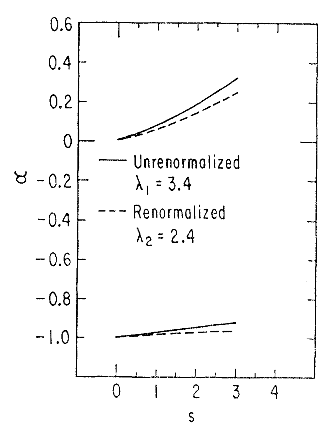

FIG. 5. Effect of propagator renormalization on the Regge trajectories computed by Method I. $m=\mu=M=1$. λ_1 (λ_2) is the coupling constant using unrenormalized (renormalized) propagator.

[9] C. Schwartz, Phys. Rev. 137, 717 (1965).

TABLE I. Intercepts and slopes at $s=0$ of Regge trajectories from the Bethe-Salpeter equation in the ladder approximation. These results were obtained by Method II in the text. The mass of one external particle, M, is 1.00, m is the mass of the other external particle, μ is the exchange mass, and λ the coupling strength.

				Slopes			
m	μ	λ	$l_{(\text{parent})}$	Parent	First daughter	Second daughter	Third daughter
1.0	1.0	16.38	1.000	0.0861	0.0401	0.0397	0.0840
			−0.027	0.0921	0.0422		
1.0	1.0	3.417	0.000	0.0688	0.0219		
1.0	1.0	7.320	0.430	0.0787	0.0309	0.091	0.251
			−0.629	0.0792	0.0287	−0.096	−0.29
1.0	0.150	2.191	0.028	0.128	0.0616		
1.0	0.146	5.661	0.823	0.206	0.127	0.141	0.246
0.150	0.820	1.463	1.001	0.040	0.0209	0.0204	0.039

According to Domokos[3] the slopes of the trajectories at $s=0$ of different members of a family are given by

$$[\text{slope}]=\alpha+\beta l(l+1)\,, \tag{15}$$

where α and β are the same for all members of a family and l is the $s=0$ intercept of the trajectory. In particular, this says that if a family intercepts $s=0$ at integer or half-integer l then the slope of the trajectory at l equals the slope of the one at $-l-1$. Our results agree with this except where $n=0(l_{(\text{parent})}=0)$, where Domokos's Eq. (6.4) should be modified slightly to explicitly exhibit the poles in the slope formula.[3,10]

$$[\text{slope}]=\alpha+\frac{n(n+1)-l(l+1)}{n(n+2)}\,.$$

Note that the perturbation formula, Eq. (14), cannot be applied at this point because some of the $f_{nn'}{}^l$ and $g_{nn'}{}^l$ are singular.

In Figs. 4 and 5 we see that "renormalization" does not affect the slopes of all the trajectories equally; it reduces the slope of the daughter trajectory by a greater factor than the parent, but this effect is less marked for smaller coupling constants. Although we did not obtain any second daughter trajectories with the renormalized propagators, their slopes would presumably have agreed with Domokos's relation since his derivation can be easily extended to that case.

The following observations are probably model-dependent, but it is interesting to note that the slope of the first daughter is always less than that of the parent (by a factor of about 0.3 to 0.5) and that a linear approximation to a trajectory is sometimes not good. Thus, our results do not support either of the conjectures that daughter trajectories are straight or approximately parallel to the parent. However, an improved model could presumably change these results.

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[10] V. Chung and J. Wright, this issue, Phys. Rev. **162**, 1716 (1967).