Extrapolation of Nucleon Form Factors. II*

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The magnetic isoscalar form factor is fitted with two poles: the ω and the ϕ resonance. The magnetic isovector form factor G_{MV} for spacelike and timelike momentum transfers is fitted using the conformal-transformation techniques of Levinger, Peierls, and Wang, and these techniques are tested using artificial data. If we do not assume a contribution from the ρ resonance, we find a spectral function with a broad peak some 100 MeV below the position of the ρ, and with a negative dip around 1200 MeV. If we assume that the ρ resonance contributes additively at its known position and width, but with an adjustable coefficient for its strength, we argue that the ρ contributes about 90% of the static moment. The complete spectral function shows a shoulder around 500 MeV, and again a negative dip around 1200 MeV. If we assume that G_{MV} is a product of the form factor for the ρ and an adjustable function, we find that the spectral function has a high peak near the ρ resonance, and a marked negative dip at 900 MeV. The ρ again contributes some 90% of the static moment. The different phenomenonological fits are in semiquantitative agreement with each other and with the recent field-theoretical calculations of Furuichi *et al.*

I. INTRODUCTION

EXPERIMENTS on electron scattering at several different laboratories[1–9] have recently added to our knowledge of nucleon form factors, particularly at high negative (spacelike) momentum transfers. Meanwhile, two experiments[10,11] on lepton pair production in proton-antiproton annihilation are starting to give reliable results for form factors for high positive (timelike) values of momentum transfer.

If we assume that the form factors are analytic functions of the momentum transfer t, in the complex-cut t plane, we can try to extrapolate from the measured form factors to find the spectral function on the cut.[12–14] This spectral function, in turn, is expected to have peaks at the positions of vector (1^-) meson resonances of strangeness zero. The experimental knowledge of these mesons has improved recently, so that one can assert with some confidence that only three such mesons have been found[15]: the isovector ρ at 765 MeV, and the isoscalar ω and ϕ at 783 and 1020 MeV, respectively. The isovector resonance has an appreciable width; but the isoscalar resonances are sufficiently narrow that one can treat them as providing poles in the spectral function.

It is convenient to extrapolate each of the four nucleon form factors separately: the isovector V magnetic G_{MV} and electric G_{EV}, and the isoscalar S magnetic G_{MS} and electric G_{ES}. These are simply related to the measured proton p and neutron n magnetic and electric form factors. Here we face the difficulty[16] that while the proton magnetic form factor G_{Mp} is known accurately, and the neutron magnetic form factor G_{Mn} and the proton electron form factor G_{Ep} are known fairly well, the neutron electric form factor G_{En} continues to be peculiarly difficult to measure.[17] Indeed, except for its static value, and its first derivative $G_{En}'=dG_{En}/dt$ at the static limit, G_{En} is only constrained by the rather broad limits $|G_{En}|<0.15$. However, $G_{En}'(0)$ is known with very good accuracy.[18] Wilson's remark[19] still holds that it "is probably the best determined of all nucleon-form-factor data."

In view of difficulties in the determination of G_{En}, we wish to primarily emphasize fits to the magnetic isovector form factor G_{MV}, and secondarily fits to the magnetic isoscalar form factor G_{MS}. The electric isoscalar form factor G_{ES} might be calculated, following

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Dudelzak,[20] rather than taking it from experiment. We shall pay no attention to the electric isovector form factor G_{EV}. Alternatively, one can concentrate on fits[21] only to proton data.

Several different types of fits have been made to the isovector magnetic form factor.[22] One cannot fit the data using only a single pole at the observed position of the ρ resonance. Shifting[23] the pole to a lower value of t to account for the effect of the width of the resonance in the dispersion relation helps, but still does not achieve a satisfactory fit. One can fit[6,24] by shifting the resonance from 765 MeV all the way to 550 MeV; but this means giving up the program of relating the spectral function for form factors to experiments on multipion resonances.

If one-pole fits to G_{MV} prove unsatisfactory, a natural alternative is to try two-pole fits.[6,24,25] For instance, one[24] can combine an (unshifted) ρ with an (unobserved) ρ' at 875 MeV. Or one can use[26] a "dipole" expression by having the ρ and ρ' close together, with strengths such that the "monopole" term cancels, leaving a form factor proportional to $(1-t/m_\rho^2)^{-2}$. Orman[27] uses (ρ-wave-modified) Lorentzian resonances instead of poles, and fits the G_{MV} data with broad resonances at 800 and 1200 MeV, respectively.

Recently, different groups,[28,29] have fitted G_{MV} and other nucleon form factors with expressions which, from a mathematical point of view, amount to introducing another pole into the spectral function, but which are quite different from a physical point of view. For instance, Massam and Zichichi[28] use the form

$$G_{MV}=2.353/(1-t/m_\rho^2)(1-t/\Lambda^2), \tag{1}$$

where Λ is a parameter, which is adjusted to 980 MeV in fitting the data. But the product of two Clementel-Villi factors is trivially rewritten as a sum of two Clementel-Villi factors:

$$G_{MV}=a/(1-t/m_\rho^2)+b/(1-t/\Lambda^2). \tag{2}$$

Corresponding to the two terms on the right of (2), we have a spectral function with poles at m_ρ and at Λ; that is, Λ merely takes the place of the hypothetical ρ' resonance discussed above. But Massam and Zichichi give Eq. (1) quite a different interpretation. They argue that while the factor $(1-t/m_\rho^2)$ is due to a pole in the spectral function corresponding to a virtual photon "materializing" into a virtual ρ, we should allow for "deviations from pointlike couplings," in particular in the coupling of the ρ to the nucleon, thus justifying the $(1-t/\Lambda^2)^{-1}$ factor in Eq. (1).

The above is a special case of the relation between fitting the form factor G_{MV} as a product of two terms and fitting the same data as the sum of two terms. We might try a form

$$G_{MV}=G_rG_\rho, \tag{3}$$

where r is the ratio, and the subscript ρ implies that the second term on the right could be a shape[27] appropriate for the ρ resonance. Alternative, we might try

$$G_{MV}=G_d+AG_\rho, \tag{4}$$

where the real coefficient A represents the strength of the contribution of the ρ resonance, and d is the difference. We relate the spectral functions by equating the imaginary parts of the right-hand sides of these two equations:

$$\begin{aligned}\mathrm{Im}G_{MV}&=\mathrm{Re}G_r\,\mathrm{Im}G_\rho+\mathrm{Im}G_r\,\mathrm{Re}G_\rho\\&=\mathrm{Im}G_d+A\,\mathrm{Im}G_\rho.\end{aligned} \tag{5}$$

If the function G_r has an imaginary part that varies slowly over the region where the ρ resonance is appreciable, then

$$A=\mathrm{Re}G_r \tag{6}$$

evaluated at the position of the ρ, and

$$\mathrm{Im}G_d=\mathrm{Re}G_\rho\,\mathrm{Im}G_r.$$

From a purely phenomenological point of view, there are advantages in the use of the product form, Eq. (3). First, the contribution of the ρ resonance to the static magnetic moment is directly determined from Eq. (5), while with Eq. (4) the constant A must be determined separately. Second, in Eq. (6) the term $\mathrm{Re}G_\rho$ has considerable structure near the ρ resonance (positive below, changing rapidly to negative above the resonance), allowing the possibility of fitting a complicated function G_d with a relatively simple function G_r.

Phenomenological fits of forms (3) or (4) have been made by Kane and Zdanis[30] and by Chilton and Uhrhane,[31] and in a preliminary form by the present author.[17] Different theoretical fits of the product form (3) have been proposed by Furuichi *et al.*,[32,33] Signell

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and Durso,[34] Antoniou and Bowcock,[35] Cocho *et al.*,[36] and Hoehler *et al.*[37]

We shall not attempt here a review of or comparison among these different fits; we shall concentrate on a comparison between our phenomenological fit, of product form, with the fit very recently achieved by Furuichi *et al.*,[33] which is based on an analysis of structure of the ρNN vertex following the early dispersion-theoretical treatment of Chew *et al.*[38] on the scattering of π mesons by nucleons. Note that Signell's and Antoniou's arguments follow the same general lines as Furuichi's. A "correlated pion state" is treated as a ρ resonance, with point coupling both to the virtual photon and to the nucleon line. Uncorrelated pion states[38] are approximated by letting the (virtual) ρ disintegrate into pions each of which joins the baryon line. (However, Cocho introduces the factor we call G_r as due to "damping at the vertex," while Hoehler introduces it as due to the Regge-pole behavior of the ρ resonance.) Also note that the phenomenological work of this paper is complementary to a field-theoretical approach such as Furuichi's, since he is limited to not-too-high-mass intermediate states, while herein the function G_r is fitted over the whole range. In particular, we pay attention to the upper limits for the cross section for proton-antiproton annihilation into lepton pairs[10,11] which Furuichi neglects. On the other hand, Furuichi's fit permits a direct theoretical interpretation, in particular concerning the values of certain adjustable parameters.

It is also possible[14] to follow Paper I and to fit the magnetic isovector spectral function without assuming prior knowledge of the ρ resonance. Hopefully, the ρ resonance and other possible structure would appear naturally as a result of the extrapolation procedure. We shall continue to pursue this attempt in the present paper.

In the next section we shall re-examine our conformal transformation[12–14] technique of determining spectral functions. We pay special attention to the sensitivity of our results to the assumption made concerning the value of the arbitrary parameter called b that enters in the particular conformal transformation chosen. In Sec. III we present the input data used for nucleon electromagnetic form factors: This is a selective compilation of experimental data as of April, 1966. We then give statistical tests of Dudelzak's fits[20] to the isoscalar magnetic form factor. In Sec. IV we determine the isovector magnetic spectral function without prior assumptions concerning the ρ resonance, while in Sec. V we assume knowledge of the position and width of the ρ resonance, and use the sum form (4) to find the remaining part of the spectral function. We use the product form (3) in Sec. VI. In the final section we compare the results of Sec. IV, Sec. V, and Sec. VI and compare with Furuichi's recent results.[33]

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TABLE I. Artificial data for "smooth spectral function." These artificial data were used in Ref. 12. The variable η is related to t/t_0 using $b=1.00$. See Eqs. (7) and (9).

η	G	Error
−0.04	0.972	0.0097
−0.08	0.948	0.0095
−0.12	0.922	0.0092
−0.16	0.877	0.0088
−0.20	0.843	0.0084
−0.24	0.805	0.0080
−0.28	0.784	0.0078
−0.32	0.739	0.0074
−0.36	0.685	0.0068
−0.40	0.633	0.0063
−0.44	0.600	0.0060
−0.48	0.535	0.0054
−0.52	0.480	0.0048
−0.56	0.435	0.0044
−0.60	0.379	0.0038
−0.64	0.324	0.0032
−0.68	0.270	0.0027
−0.72	0.222	0.0022
−0.76	0.167	0.0017
−0.80	0.118	0.0012

II. FURTHER TEST OF OUR EXTRAPOLATION TECHNIQUE

The conformal transformation technique of extrapolation used, for instance, by Levinger and Peierls,[12] transforms the cut t plane into the interior of a unit circle, and expands the form factors as a power series in a new variable called η. (Here the Mandelstan variable t is the squared momentum transfer, chosen negative for spacelike values.) We test our technique by the use of artificial data as follows. We assume a spectral function $g(t')$, with assumed threshold t_0, and use it in a nonsubtracted dispersion integral to find the form factor $G(t)$ for different spacelike momentum transfers. We then add random errors, of an assumed root-mean-square value and of normal distribution, to $G(t)$ to give artificial data, which we then feed into the computer. The computer extrapolates, and prints a spectral-function output, which we then compare with the spectral function initially assumed. We found that our extrapolation technique worked well for an assumed smooth spectral function: The favorable result is illustrated in Fig. 12 of Ref. 12 for a "constrained quartic" fit. On the other hand, the extrapolation technique is much less successful with artificial data of the Clementel-Villi form, based on a single pole. Of course we cannot reproduce an arbitrarily narrow resonance, and we also have the problem of tending to produce spurious side resonances, as illustrated in Fig. 9 of Ref. 12. Finally, the position of the main resonance in the output spectral function shifts when we choose different values for a parameter

TABLE II. Smooth spectral function coefficients $a_n(b)$. Coefficients for quartic fits to the data of Table I; see Eqs. (7) and (10). The χ^2 value shows the goodness of fit for 17 degrees of freedom. The coefficient a_5 is for a quintic fit, while the last column gives a cubic fit for $b=2.5$.

$n \backslash b$	1.01	1.5	2.0	2.5	3.0	4.0	6.0	2.5 (cubic)
a_0	0.997	0.849	0.714	0.604	0.514	0.380	0.227	0.602
a_1	0.558	0.870	0.968	0.987	0.967	0.868	0.607	0.981
a_2	−0.970	−0.523	−0.224	0.009	0.202	0.488	0.554	0.026
a_3	−0.055	−0.256	−0.349	−0.371	−0.343	−0.139	0.997	−0.345
a_4	0.388	0.243	0.132	0.026	−0.086	−0.356	−1.18	0.000
χ^2	17.4	18.1	17.9	19.2	19.3	16.4	19.8	20.8
a_5	−0.135	0.116	0.215	0.252	0.253	0.163	−0.482	

b in the conformal transformation

$$\eta=[b-(1-t/t_0)^{1/2}]/[b+(1-t/t_0)^{1/2}]. \qquad (7)$$

In this section the same artificial data used earlier for a smooth spectral function are used; we examine the dependence of the output spectral function on the choice of b. Our purpose is to learn what criteria should be used in choosing b, so that we can apply these criteria in the extrapolation of real data in the following sections.

Levinger and Peierls[12] choose a spectral function for $t'\geq t_0$,

$$g(t')=2.315[1-\exp(t'/2t_0-\tfrac{1}{2})]^2\exp(-t'/20t_0). \qquad (8)$$

Here t_0 is the threshold; numerical values were chosen to give a broad peak at t' of $7t_0$, and the coefficient 2.315 was chosen to give a form factor $G(0)$ equal to unity.

We assumed that the form factor $G(t)$ obeys a nonsubtracted dispersion relation

$$K(\eta)=G(t)=\frac{1}{\pi}\int_{t_0}^{\infty} g(t')dt'/(t'-t). \qquad (9)$$

Random errors of Gaussian distribution and of rms

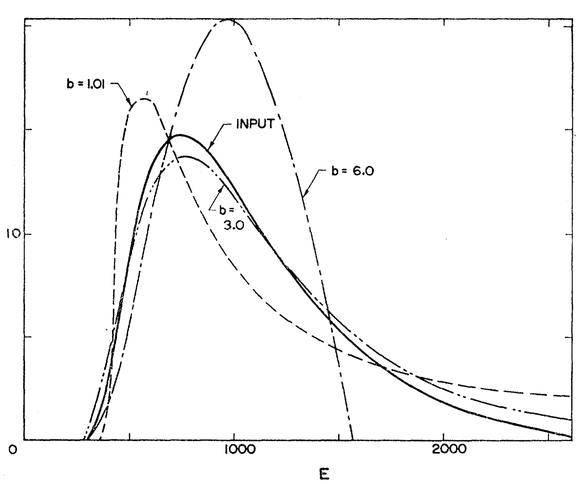

FIG. 1. The input spectral function, and three output functions, for smooth artificial data, from Table III. The abscissa is the energy in MeV.

value 1% of G are folded in, giving the artificial data presented in Table I.

For seven choices of b, from 1.01 to 6.0, the "data" $K(\eta)$ are fitted for least squares with power series:

$$K(\eta)=\sum_{n=0}^{N} a_n\eta^n. \qquad (10)$$

We use two constraints on the truncated polynomial: (i) that we obtain the exact static value of unity for $G(0)$, and (ii) that the spectral function have zero slope at threshold. [This second constraint is physically due to the p-wave behavior of intermediate states in electron-nucleon scattering. It is also built in to the assumed spectral function (8).]

For each choice of degree N of the polynomial, we looked at the χ^2 value of the power-series fit. Table II shows that the value $N=4$ gives a χ^2 acceptable by statistical criteria; i.e., close to 17, the number of degrees of freedom for a quartic fit with two constraints. (One should not go to a higher value of N, as discussed previously,[12] since the errors increase rapidly when N is chosen too high.) Table II also gives the coefficients $a_n(b)$ for these seven different quartic fits: Note that a cubic fit is also satisfactory for $b=2.5$.

The constrained least-squares-fitting program of Peierls[12] evaluates the error matrix for the coefficients a_n, and uses the values a_n and their (diagonal and correlated) errors to evaluate the spectral function $g(t)$ and its statistical error:

$$g(t)=\sum_{n=1}^{N} a_n\sin[n\xi_b(t)]. \qquad (11)$$

Here the angle ξ_b depends both on t and on :

$$\cos\xi_b(t)=(b^2+1-t/t_0)/(b^2-1+t/t_0). \qquad (12)$$

These seven spectral functions, and associated standard errors, are presented both in Table III and in Fig. 1.

We see from this table and figure that the five values of b in the range $1.5\leq b\leq 4.0$ reproduce the main features of the input spectral function (8) rather well. On the other hand, the extreme choice $b=1.01$ shifts the peak position from $7t_0$ down to $4t_0$, while the other extreme choice $b=6.0$ shifts the peak up to $11t_0$. Both extreme choices also introduce spurious dips around $1.2t_0$ and $80t_0$, respectively. The three choices $b=2.0$,

TABLE III. Smooth spectral function g for quartic fits using coefficients $a_n(b)$ from Table II [See Eq. (11)]. The errors in g are found using the complete error matrix. The input is Eq. (8).

		$b=1.01$		$b=1.5$		$b=2.0$	
E (MeV)	Input	g	Error	g	Error	g	Error
289	0.00	−0.24	0.03	−0.02	0.01	0.00	0.00
308	0.02	−0.67	0.08	−0.08	0.03	0.03	0.01
337	0.08	−0.54	0.09	−0.07	0.05	0.11	0.02
373	0.22	0.14	0.05	0.11	0.05	0.26	0.04
418	0.44	0.93	0.01	0.47	0.04	0.48	0.04
459	0.66	1.37	0.04	0.81	0.02	0.70	0.03
509	0.92	1.61	0.06	1.15	0.02	0.94	0.02
571	1.19	1.65	0.06	1.41	0.04	1.18	0.01
622	1.34	1.58	0.06	1.50	0.05	1.31	0.02
683	1.44	1.45	0.04	1.52	0.05	1.40	0.04
718	1.47	1.37	0.04	1.49	0.05	1.42	0.04
756	1.47	1.28	0.03	1.45	0.05	1.42	0.04
800	1.46	1.19	0.03	1.39	0.05	1.40	0.05
903	1.35	0.99	0.01	1.21	0.04	1.29	0.05
1037	1.16	0.79	0.01	0.98	0.02	1.11	0.04
1217	0.90	0.61	0.01	0.73	0.01	0.85	0.02
1473	0.58	0.45	0.01	0.50	0.01	0.57	0.01
1867	0.25	0.31	0.02	0.30	0.02	0.31	0.02
2154	0.12	0.26	0.02	0.22	0.02	0.20	0.02
2545	0.04	0.21	0.01	0.15	0.02	0.11	0.02
3111	0.01	0.16	0.01	0.10	0.02	0.05	0.03
4000	0.00	0.12	0.01	0.06	0.02	0.01	0.02
5600	0.00	0.08	0.01	0.04	0.01	−0.01	0.02

	$b=2.5$		$b=3.0$		$b=4.0$		$b=6.$		Cubic $b=2.5$	
E (MeV)	g	Error	g	Error	g	Error	g	Error	g	Error
289	0.01	0.000	0.01	0.001	0.01	0.000	0.004	0.000	0.01	0.000
308	0.05	0.005	0.05	0.003	0.04	0.001	0.02	0.001	0.06	0.001
337	0.16	0.013	0.15	0.01	0.12	0.004	0.08	0.001	0.17	0.001
373	0.31	0.022	0.30	0.02	0.25	0.01	0.16	0.003	0.33	0.003
418	0.51	0.030	0.51	0.02	0.44	0.01	0.30	0.01	0.55	0.004
459	0.70	0.03	0.70	0.03	0.63	0.02	0.46	0.01	0.74	0.01
509	0.91	0.03	0.91	0.03	0.86	0.02	0.67	0.01	0.94	0.01
571	1.11	0.02	1.12	0.02	1.12	0.02	0.95	0.02	1.13	0.01
622	1.23	0.01	1.24	0.02	1.29	0.02	1.18	0.02	1.23	0.01
683	1.32	0.01	1.33	0.01	1.44	0.02	1.44	0.02	1.30	0.01
718	1.35	0.02	1.36	0.01	1.50	0.02	1.58	0.02	1.32	0.01
756	1.36	0.03	1.37	0.01	1.54	0.01	1.71	0.02	1.33	0.01
800	1.35	0.04	1.37	0.02	1.56	0.01	1.84	0.02	1.31	0.005
903	1.29	0.05	1.29	0.04	1.49	0.01	2.02	0.02	1.23	0.003
1037	1.13	0.05	1.13	0.05	1.26	0.03	2.00	0.01	1.08	0.001
1217	0.90	0.04	0.88	0.05	0.87	0.05	1.56	0.02	0.86	0.004
1473	0.62	0.02	0.60	0.04	0.40	0.06	0.52	0.06	0.60	0.01
1867	0.33	0.01	0.32	0.01	0.05	0.04	−0.89	0.09	0.34	0.01
2154	0.21	0.02	0.20	0.02	−0.01	0.02	−1.37	0.08	0.23	0.01
2545	0.11	0.03	0.11	0.03	0.01	0.02	−1.38	0.05	0.14	0.01
3111	0.03	0.03	0.04	0.04	0.08	0.04	−0.71	0.02	0.07	0.01
4000	−0.02	0.03	0.00	0.04	0.17	0.06	0.51	0.08	0.02	0.01
5600	−0.04	0.03	−0.02	0.04	0.22	0.06	1.66	0.13	−0.01	0.01

2.5, and 3.0 give particularly favorable results: The output spectral function is in general only about one of its standard errors away from the input spectral function, and the three output curves are so close together that we have drawn only one in the figure.

This check with artificial data shows that, at least under favorable circumstances, our extrapolation technique is quite successful. The problem remains how to recognize, in advance, that the circumstances are indeed favorable; i.e., how we should choose b in (7). Levinger and Peierls[12] suggest that it is desirable that the peak in the spectral function lie near $\xi_b=90°$, since a truncated Fourier series can easily reproduce a peak near 90°, but has difficulty reproducing a peak close to 0° or 180°. The peak position is at $7t_0$, so Eq. (12) shows that $b=1.5$ corresponds to a peak position near 120°, while $b=4.0$ corresponds to a peak position near 60°. We can restate our result above to read that a broad peak will be reproduced rather well if the peak position is between 60° and 120°.

Another criterion suggested earlier[12] is that the choice of b should be such that Eq. (7) would spread the data almost symmetrically about the origin in the η plane. With our artificial data in the spacelike range from $-80t_0$ to zero, this criterion gives a range for b similar to that in the paragraph above. This desirable agreement between two criteria persists in our work below, since the main structure in electromagnetic spectral functions is in the region of $7t_0$, and the real data lies in the range $-80t_0\leq t\leq 0$.

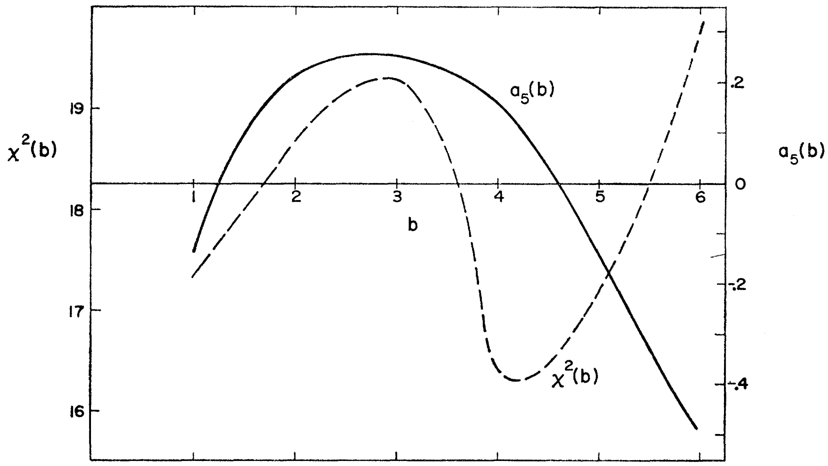

FIG. 2. The goodness of fit χ^2 for quartic fits is drawn as the dotted curve, with the ordinate on the left. The coefficient a_5 for the quintic fit is the solid curve and right ordinate. The abscissa b is the adjustable parameter used in the conformal transformation.

Two other criteria for choosing b were subsequently suggested[13]: that satisfactory values for the annihilation form factor [$G(t)$ for $t>4M^2$] be obtained, and that $X^2(b)$ have a minimum for fixed N. In this section we ignore annihilation form factors; they are used for the real data treated below, by putting the annihilation form factors directly in the fit, rather than indirectly by the choice of b. The argument below suggests that the choice of a minimum for $X^2(b)$ is *not* a reliable criterion. Consider Fig. 2, in which X^2 (b) is plotted for the seven quartic fits given in Table II; the last row of that table, namely $a_5(b)$ for the quintic fits, is also plotted. We observe, as could have been predicted, that the minima of $X^2(b)$ fall very nearly at the positions of the zeros of $a_5(b)$. That is, when $a_5(b)$ goes through a zero, a quartic fit is just as good as a quintic; since X^2 for a quintic can be expected to be small we should expect a minimum in the X^2 value for a quartic fit.

We can also understand our one success with a cubic fit, namely at $b=2.5$, by the same argument. As seen from Table II, $a_4(b)$ goes through zero very near 2.5. [In fact $a_4(2.5)$ is not significantly different from zero: $a_4=0.026\pm0.024$, quoting the diagonal (uncorrelated) error.] Since the quartic fits have an acceptable X^2, so does the cubic fit in the case of $b=2.5$. We conclude that the X^2 criterion should be used *only* to decide if a fit is statistically acceptable. Thus for 17 degrees of freedom we have a 10% probability of finding a X^2 greater than 24, so all the quartic fits presented are well within the 90% confidence limit. In this context[39] we should *not* distinguish between "statistically acceptable" and "statistically more acceptable."

Here an additional criterion for choosing b is proposed; namely, that the output spectral function $g_b(t)$ have zero variation with respect to b. Figure 3 illustrates $\Delta g/\Delta b$ for values of t near the peak of the spectral function and at t values giving $g(t)$ some $\frac{2}{3}$ of the maximum value. We observe that all three curves have nodes for b near 2.5, thus confirming our use above of the criterion that the peak of the spectral function should occur near $\xi=90°$.

III. THE ISOSCALAR FORM FACTOR

Leaving artificial data we return to actual data on nucleon form factors which were compiled in April, 1966. Table IV gives the two magnetic form factors: the isoscalar G_{MS} and the isovector G_{MV}. Since the main source of error is the neutron magnetic form factor G_{Mn}, we give specific references only for the neutron measurements; we interpolate and average proton measurements[1–3] when necessary. Recent Deutsche Synchrotron and Cambridge Electron Accelerator proton measurements[4,9] are not included. This omission is not serious since the limiting factor is our knowledge of G_{Mn}, which has not changed greatly during the past 14 months.[40]

In this section we examine three different fits to the magnetic isoscalar form factor, each fit assuming that the spectral function contains two poles, one located at the position of the ω isoscalar resonance. In the first fit we allow the position of the second isoscalar pole to vary, in an attempt to "discover" the ϕ resonance. In the second fit, we use the ϕ resonance at its known position of 1020 MeV and adjust the residues

[39] In the usual least-squares fit, one has already decided on the mathematical form used to fit the data, and is using the least-squares criterion for optimum choice of the parameters to put into the preordained form. But in our present context we do *not* have a definite mathematical form. We are making phenomenological fits, which means that we are wandering more or less arbitrarily among different hyperplanes in Hilbert space. Once we have decided to stay on a given hyperplane, we should determine the optimum position by a least-squares criterion; but we should not argue in favor of the choice of one hyperplane rather than another by saying that while both χ^2 values are acceptable, one is more acceptable than the other.

[40] The work of R. E. Rand, R. F. Frosch, C. E. Littig, and M. R. Yearian [Phys. Rev. Letters 18, 469 (1967)] supports the significance of meson-exchange effects in inelastic electron-deuteron scattering in cases where both recoil nucleons receive high momentum. This work calls into question the usual neglect (Refs. 5–8) of such exchange effects in inelastic scattering in cases where a single nucleon receives almost all the momentum.

TABLE IV. Magnetic form factors. The proton data are taken from Refs. 1–3, and in general contribute only small errors.

t (BeV/c)^2	G_{MS}	G_{MV}	Standard error	Reference
−0.0389		2.55	0.35	a
−0.0584		2.04	0.11	a
−0.0972		1.97	0.08	a
−0.179		1.524	0.07	a
−0.292	0.18	1.134	0.05	a
−0.389	0.14	0.980	0.02	b, c
−0.486	0.11	0.842	0.044	a
−0.564	0.11	0.746	0.027	b
−0.583	0.10	0.732	0.041	a
−0.623	0.10	0.696	0.022	c
−0.778	0.09	0.569	0.048	a
−0.857	0.11	0.526	0.018	c
−0.972	0.05	0.478	0.033	a
−1.17	0.05	0.384	0.013	c
−1.75	0.01	0.240	0.013	c
−2.92	0.018	0.118	0.007	c, d
−3.89	0.011	0.080	0.008	c, d
−6.81	0.006	0.032	0.010	c, d

[a] Hughes *et al.*, Ref. 6.
[b] Stein *et al.*, Ref. 5.
[c] Dunning *et al.*, Ref. 8.
[d] Treating upper limits on absolute value of G_{Mn} as data, assuming negative values for the neutron's magnetic form factor.

at the two poles, with the static constraint $G_{MS}(0)=0.44$, to give a least-squares fit to the data of Table IV. This approach has been used by other workers.[6,24] In the last fit we follow Dudelzak[20] in first fitting the electric isoscalar form factor G_{ES}, and then using $G_{ES}(4M^2)=G_{MS}(4M^2)$ to determine $G_{MS}(t)$ with no further adjustable parameters

In each case,

$$G_{MS}(t)=0.44[am_\omega^2/(m_\omega^2-t)+(1-a)m_\phi^2/(m_\phi^2-t)]. \quad (13)$$

We first allow m_ϕ to vary and for each choice of m_ϕ determine a by a least-squares fit to the G_{MS} data of Table IV. The resulting $\chi^2(m_\phi)$ has a minimum value of 17.1 at $m_\phi=980$ MeV. The variation of χ^2 with m_ϕ gives us an error of 60 MeV in determining the ϕ mass. That is, we have "discovered" an isoscalar resonance at 970±70 MeV, i.e., consistent with the ϕ position of 1020 MeV. Our best χ^2 of 15 for 13 degrees of freedom is well within the 90% confidence limit.[41]

In the second fit, the coefficient a and its standard error are determined using $m_\phi=1020$ MeV. We find

$$a=2.76\pm0.09. \quad (14)$$

The slightly poorer χ^2 value of 16 for 13 degrees of freedom has a probability of about 20%. This fit is also statistically acceptable, using the 90% confidence limit.

Dudelzak[20] first fits $G_{ES}(t)$ with a form similar to that of (13), using the additional constraint $G_{ES}'(0)=1.23\pm0.05$ (BeV/c)$^{-2}$. (See also I.) This constraint

[41] Note that the data of Table IV is good enough to support two adjustable parameters, a and m_ϕ, but *not* three. See J. S. Levinger and M. W. Kirson, *Eastern Theoretical Physics Conference* (Gordon and Breach Science Publishers, Inc., New York, 1963), p. 175.

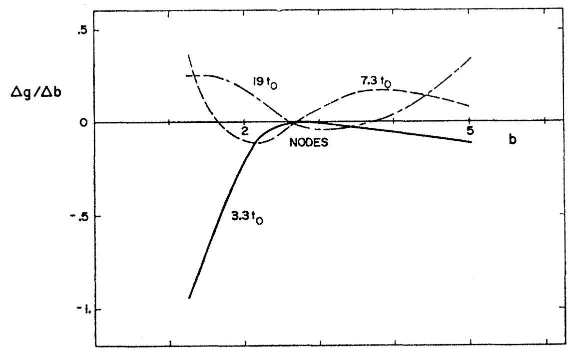

FIG. 3. Slope $\Delta g/\Delta b$ versus b for spectral functions from Table III. The curves are for three values of t: $3.3t_0$, $7.3t_0$, and $19t_0$.

on the slope of the isoscalar electric form factor at the static limit is based on the very accurately determined electron-neutron scattering length,[18] and on Dudelzak's determination[20] of $G_{Ep}'(0)$, or the proportional proton "mean-square radius." The error in $G_{ES}'(0)$ comes almost entirely from the proton measurements. Dudelzak then uses his expression for $G_{ES}(t)$, and the constraint that the complex G_{ES} and G_{MS} must be equal at $4M^2$ (to give nonsingular Dirac and Pauli form factors F_1 and F_2):

$$G_{ES}(4M^2)=G_{MS}(4M^2)=-0.05. \quad (15)$$

This second constraint on G_{MS} gives the value of a with no further adjustment. The error in a comes from the error in $G_{ES}'(0)$, which propagates by means of $G_{ES}(4M^2)$. Dudelzak finds

$$a=2.32\pm0.18. \quad (16)$$

We see that the values of a given by Eqs. (14) and (16), respectively, disagree by two standard errors. This disagreement indicates, but does not establish, a failure of either (13) or (15). For instance, if we keep the value of $G_{ES}(4M^2)$ from (15), but use the value of $G_{MS}(4M^2)$ from Eqs. (13) and (14), we find that $G_{ES}(4M^2)$ and $G_{MS}(4M^2)$ differ by only 0.02; i.e., Eq. (13) can be extrapolated into the far timelike region with quite small inaccuracies. We conclude that both isoscalar spectral functions can be fitted quite well using two poles, at the known positions of the two isoscalar resonances. There is no evidence for other terms in the isoscalar spectral functions.

Similar conclusions have been reached by Hughes *et al.*,[6] and by Chan *et al.*[24] The former group fits G_{MS} allowing a "core term" (i.e., a pole at very high t), and find that the core is zero within its statistical error. Their value for a is 2.42, in agreement with (14) and (16). The latter group makes four different pole fits; their "fit 4" corresponds to use of Eq. (15), and gives $a=2.42$, in agreement with (16). Their "fit 5" does not use the constraint at $4M^2$, and obtains $a=2.69$, in good agreement with our result (14).

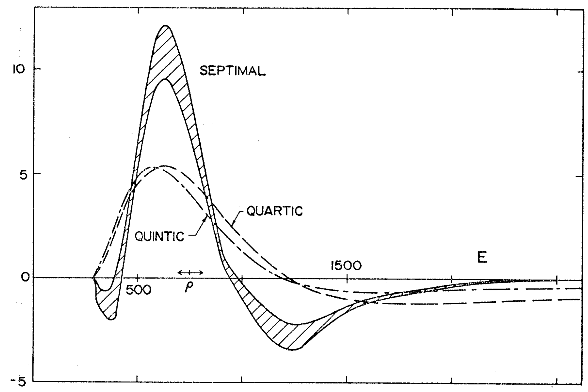

FIG. 4. Three magnetic isovector spectral functions from Table VII: quartic with three constraints, dashed curve; quintic with four constraints, dash-dot curve; and a spetimal with five constraints, shaded to show statistical error. The abscissa is the energy in MeV.

However, Massam and Zichichi[28] argue that we are *not* free to adjust the coefficient a; instead, the value of a is given by SU_3 symmetry as

$$a \simeq \tfrac{1}{2}. \tag{17}$$

With this choice of a between zero and unity, a two-pole fit (13) must fail. They conclude that the two-pole form (13) should be multiplied by the same factor $G_r = (1-t/\Lambda^2)^{-1}$ as was done for the magnetic isovector form factor in Eq. (1).

At present, we believe it is impossible to be sure if either Dudelzak's procedure (followed above) or Zichichi's argument is correct. Besides its conflict with SU_3, we have no assurance that Dudelzak is correct in assuming pointlike coupling for ωNN and ϕNN; e.g., we are willing to make a different assumption in Sec. VI for the ρNN vertex. On the other hand, we know that SU_3 arguments on coupling constants are not only approximate; they may fail badly as in the case of lepton pair production from photoproduced ϕ mesons.[42]

IV. THE MAGNETIC ISOVECTOR FORM FACTOR

In this section, the magnetic isovector form-factor data of Table IV are fitted using the conformal transformation technique discussed above in Sec. II. As has been argued in the Introduction and elsewhere,[18] one-pole and two-pole fits are unsuccessful in fitting isovector form factors and simultaneously the properties of meson resonances, so it seems desirable to go to the other extreme and use a technique which works well for a smooth spectral function. The present section is a continuation of Paper I.[14] The data used now (April, 1966 instead of June, 1964) extends over a larger range of t, and are changed in value and in quoted errors in the range where they overlap. [For instance, $G_{MV}(1.17)$ has been changed from 0.340 ± 0.020 to read 0.384 ± 0.013.] Also, in the present work use is made of the recent small upper limits on the form factors[10,11] in the annihilation region $t = 6.8(\mathrm{BeV}/c)^2$.

The cross section for proton-antiproton annihilation into lepton pairs is a linear combination of the squared moduli of the complex form factors $|G_{Ep}|^2$ and $|G_{Mp}|^2$. Zichichi's measurement of an upper limit for the cross section gives

$$|G_{Ep}|^2 + 1.92|G_{Mp}|^2 \leq 0.05. \tag{18}$$

If we use $G_E \simeq G_M$ we have an upper limit of 0.15 for the modulus of G_{Mp}. (Drell[22] quotes $|G_{Mp}| \leq 0.1$.)

We need the value $G_{MS}(6.8)$ to obtain an upper limit on G_{MV} to use in our fit. Equations (13) and (14) give the real part $G_{MS}(6.8) = 0.02$ with an imaginary part of zero. These numbers are small enough that we can use the upper limits measured for $|G_{Mp}|$ directly as upper limits for the real and imaginary parts of $G_{MV}(6.8)$. As a matter of convenience, we use the annihilation data as constraints on the complex form factor; namely, $G_{MV}(6.8) = 0.0$.

Computer runs have been made for different combinations of the following: (i) values of $b = 2$, 3, or 4 [see Eq. (7)]; (ii) use of constraints or no constraints at $t = 6.8(\mathrm{BeV}/c)^2$; (iii) the value of the quantity $g_{MV}(6.8)$ chosen for the constraint on the imaginary part of G_{MV}. We have selected only part of the results, emphasizing the choice of $b = 3$. This choice of b satisfies the three criteria verified in Sec. II: (i) The data runs from $-0.52 < \eta < 0.42$, so it is nearly symmetrical about the origin; (ii) the peak in the spectral function around 620 MeV corresponds to an angle $\xi_b = 65°$, which is within the favorable 60° to 120° range found above; (iii) comparisons of spectral functions found for $b = 3$ with those for $b = 4$ show small variation with the choice of b.

In all our runs at least three constraints were imposed on the coefficients in the power-series fit: the two constraints for the static value and the zero slope of the spectral function at its threshold t_0, used in Sec. II; and a third constraint that the form factor $G(-\infty) = 0$. Occasionally use was made of a fourth constraint

TABLE V. Coefficients for fits to magnetic isovector data of Table IV. See Eq. (11), Eq. (20), and discussion above it. All fits use $b = 3$. See Table VI for error matrix for septimal fit.

	3 constraints	4 constraints	5 constraints	
	Quartic	Quintic	Sextic	Septimal
a_0	0.70	0.69	0.70	0.70
a_1	2.50	2.41	2.42	2.64
a_2	2.30	2.26	2.33	2.81
a_3	−0.73	−0.26	−0.25	−1.08
a_4	−1.23	−1.13	−1.50	−3.66
a_5		−0.32	−0.38	−1.14
a_6			0.26	1.48
a_7				0.92
χ^2	14.0	21.3	22.8	8.3

[42] R. C. Chase, P. Rothwell, and R. Weinstein, Phys. Rev. Letters 18, 710 (1967).

TABLE VI. Error matrix for septimal fit. $\mathbf{H}^{-1}$ of Appendix, Ref. 12, used with septimal coefficients of Table V to give errors in septimal spectral function, Table VII.

$j \backslash i$	0	1	2	3	4	5	6	7
0	0.00006	0.00022	−0.00009	−0.00088	−0.00087	0.00013	0.00061	0.00026
1	0.00022	0.00400	0.00680	−0.0150	−0.0341	−0.0105	0.0197	0.0140
2	−0.00009	0.00680	0.0170	−0.0251	−0.0712	−0.0268	0.0398	0.0307
3	−0.00088	−0.0150	−0.0251	0.0562	0.127	0.0389	−0.0736	−0.0523
4	−0.00087	−0.0341	−0.0712	0.127	0.324	0.112	−0.184	−0.137
5	0.00013	−0.0105	−0.0268	0.0389	0.112	0.0425	−0.0622	−0.0483
6	0.00061	0.0197	0.0398	−0.0736	−0.184	−0.0622	0.105	0.0772
7	0.00026	0.0140	0.0307	−0.0523	−0.137	−0.0483	0.0772	0.0580

designed to keep factors small in the annihilation region; namely, that the slope of the spectral function also be zero at infinite negative values of t, i.e.,

$$\sum_n (-1)^n n a_n = 0. \tag{19}$$

Finally, a run with five constraints was made by removing the fourth constraint (19) and using two additional constraints, namely, that both the real and imaginary parts of the complex form factor be zero at $6.8(\mathrm{BeV}/c)^2$.

$$\begin{aligned} \sum a_n \sin n\xi_a &= 0, \\ \sum a_n \cos n\xi_a &= 0. \end{aligned} \tag{20}$$

Here the angle ξ_a is given by Eq. (12), using $b=3$ and $t/t_0=87$. (Of course, t_0 is equal to $4m^2$, where m is the pion mass.)

In each case the data of Table IV could be fitted with two adjustable parameters. (Good χ^2 values were obtained with a quartic fit for three constraints, and within the 90% confidence limit for a quintic with four constraints and for a sextic with five constraints.) However, in the case of the latter two fits, the χ^2 values decreased markedly when an extra adjustable parameter was added. Table V gives the coefficients a_n, and the χ^2 values for four different polynomial fits: a quartic with three constraints, a quintic with four constraints, and sextic and septimal with five constraints. Table VI gives the error matrix for the septimal fit. The spectral functions, and their statistical errors (found using the complete error matrix) are given for these four fits in Table VII, and are also illustrated in Fig. 4.

TABLE VII. Magnetic isovector spectral functions using coefficients of Table V, Eq. (11), and complete error matrix; e. g., Table VI.

	3 constraints Quartic		4 constraints Quintic		5 constraints Sextic		5 constraints Septimal	
E (MeV)	g	Δg	g	Δg	g	Δg	g	Δg
289	0.06	0.001	0.07	0.003	0.05	0.01	−0.18	0.06
308	0.32	0.005	0.41	0.02	0.32	0.04	−0.80	0.30
337	0.93	0.01	1.15	0.04	0.93	0.09	−1.50	0.64
373	1.80	0.03	2.16	0.07	1.87	0.14	−1.28	0.84
418	2.90	0.04	3.36	0.10	3.11	0.16	0.68	0.66
459	3.80	0.05	4.26	0.11	4.17	0.14	3.49	0.23
509	4.67	0.06	4.99	0.10	5.16	0.09	7.03	0.50
571	5.29	0.06	5.33	0.06	5.80	0.03	10.1	1.12
622	5.44	0.05	5.21	0.03	5.82	0.09	10.8	1.32
682	5.26	0.04	4.75	0.03	5.37	0.16	9.93	1.21
718	5.02	0.03	4.38	0.06	4.94	0.18	8.76	1.02
757	4.68	0.02	3.92	0.08	4.36	0.19	7.16	0.76
800	4.23	0.01	3.38	0.10	3.67	0.20	5.22	0.45
903	3.01	0.03	2.15	0.14	2.02	0.16	1.03	0.30
1037	1.54	0.06	0.91	0.13	0.40	0.08	−2.08	0.66
1217	0.11	0.07	−0.06	0.10	−0.69	0.02	−2.81	0.56
1474	−0.87	0.07	−0.56	0.04	−0.89	0.04	−1.57	0.18
1647	−1.09	0.06	−0.62	0.02	−0.71	0.04	−0.81	0.04
1867	−1.12	0.05	−0.58	0.02	−0.45	0.03	−0.26	0.05
2154	−0.98	0.03	−0.47	0.03	−0.19	0.01	−0.02	0.05
2545	−0.72	0.02	−0.34	0.03	−0.01	0.001	0.00	0.01
3111	−0.42	0.01	−0.20	0.03	0.05	0.003	−0.06	0.03
4000	−0.16	0.01	−0.10	0.02	0.02	0.001	−0.08	0.02
5600	0.01	0.01	−0.04	0.01	−0.04	0.004	−0.03	0.004
	Re G	ΔRe G	Re G	ΔRe G	Re G	ΔRe G	Re G	ΔRe G
1867	−0.69	0.04	−0.47	0.06	−0.04	0.01	0.42	0.12
2154	−0.21	0.04	−0.22	0.04	0.05	0.003	0.19	0.04
2545	0.13	0.04	−0.06	0.02	0.01	0.001	0.01	0.000
$\lvert G(2.6)\rvert$	0.7	0.02	0.34	0.03	0.00	0.00	0.00	0.00

TABLE VIII. Coefficients, χ^2 and $\sum_n a_n^2$ for quintic and sextic fits to $G_d(t)$ using Eq. (22) and data of Table IV. All fits use $b=3$ and have five constraints, given below Eq. (24).

	$A=0.6$		$A=0.87$		$A=1.0$		$A=1.1$	$A=2.0$
	$N=5$	$N=6$	$N=5$	$N=6$	$N=5$	$N=6$	$N=6$	$N=6$
a_0	−0.07	−0.06	−0.387	−0.396	−0.541	−0.559	−0.685	−1.81
a_1	0.97	1.01	0.383	0.349	0.10	0.03	−0.218	−2.35
a_2	2.09	2.05	1.86	1.90	1.75	1.83	1.77	1.35
a_3	0.66	0.50	0.88	1.01	0.99	1.26	1.45	2.74
a_4	−1.01	−1.09	−0.87	−0.81	−0.80	−0.67	−0.56	0.03
a_5	−0.62	−0.53	−0.66	−0.72	−0.67	−0.81	−0.88	−1.23
a_6		0.07		−0.06		−0.12	−0.17	−0.42
χ^2	18.1	15.3	18.5	16.6	26.2	17.8	17.0	12.6
$\sum_n a_n^2$	7.14	6.96	5.72	6.08	5.42	6.37	6.88	19.82

We see from the last row in Table VII that we need to make use of the annihilation data; e.g., the quartic fit with three constraints gives $|G(6.8)|=0.7$, with a small statistical error, which is some five times the measured upper limit. Use of four constraints gets us closer to the measured upper limit, but it is desirable to use the annihilation data explicitly. (Of course, it need not be used as a constraint, but could, in future work, be used as additional data points, with associated standard errors.) However, we also see from Table VII and Fig. 4 that the main features of the spectral function are unchanged when we compare the first three fits, each with two adjustable parameters. The spectral function has a peak of height about 5.5 at about 620 MeV, and also dips down to about -1 at about 1600 MeV. The full width at half-maximum Γ of the main peak decreases slowly from 500 to 450 MeV as we increase the degree of the polynomial.

The last fit, a septimal (five constraints) with three adjustable parameters, shows the main peak at the same position of 620 MeV (with much larger errors in determining the peak position). The peak is higher and narrower ($\Gamma=300$ MeV) than those for fits with polynomials of lower order, but remains much broader than the observed ρ resonance ($\Gamma=120$ MeV). However, the half-width in angle of 32° is almost as small as could be expected for a truncated Fourier series[12] with seven terms. The septimal fit has a dip at 1200 MeV.

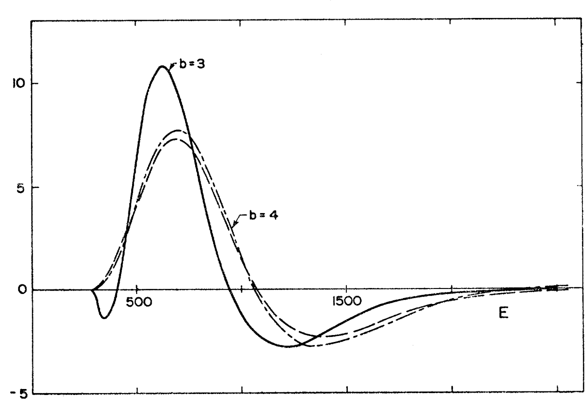

FIG. 5. Three magnetic isovector spectral functions for different choices of the parameter b, and of the spectral function $g(6.8)$ at $6.8(\text{BeV}/c)^2$ timelike: solid curve, the septimal from Table VII and Fig. 4, $b=3$, and $g(6.8)=0.0$; dash-dot curve $b=4$ and $g(6.8)=0.1$; dashed curve, $b=4$ and $g(6.8)=-0.1$, both sextics.

I regard this septimal fit, with five constraints, as the most satisfactory fit yet achieved using these conformal transformation techniques. The χ^2 value is excellent, so that we can certainly use this fit as a convenient means to interpolate in the spacelike region. The accuracy of the extrapolation to the near timelike region is not certain, but at least the statistical errors claimed are not small, so that our spectral function may be about as accurate as it claims to be.

In Fig. 5 a comparison is made between the spectral function of Table VII (septimal for $b=3.0$) with sextic fits using $b=4.0$, and different values (0.1 and -0.1) for the spectral function at $t=6.8(\text{BeV}/c)^2$. [The results for $b=4$ and $g(6.8)=0.00$ fall between the dashed and dash-dot curves for $g(6.8)=\pm 0.1$.]

V. MAGNETIC ISOVECTOR: SUM FORMULATION

In the preceding section we found that the data of Table IV, combined with knowledge of the threshold, threshold behavior, and other properties of the isovector spectral function, determined a spectral function similar, but not identical, to the experimentally determined position and shape of the ρ resonance. In particular, Table VII and Figs. 4 and 5 show spectral functions with the main peak about 100 MeV below the ρ resonance, and also with a dip around 1.2 BeV. This partial agreement suggests a different approach, discussed in the Introduction: Assume that the isovector spectral function is dominated by the ρ resonance with a determined position and shape, and fit the data to find the (hopefully small) remaining spectral function. In this section we try the "sum formulation" $G_{MV}=G_d+AG_\rho$; in the next section the product formulation $G_{MV}=G_rG_\rho$.

I shall assume that the shape of the ρ resonance is given by a Lorentzian modified, in the numerator, by the p-wave behavior of the two-pion system, for $t'>t_0$:

$$g_\rho(t')=0.358(t'-t_0)^{3/2}/[(t_\rho-t')^2+\Gamma^2]. \tag{21}$$

This shape (for two isovector resonances with adjusta-

TABLE IX. Magnetic isovector spectral functions (sum formulation). The residual spectral function g_d is found using the coefficients for the quintic fits, $A=0.87$ from Table VIII. The function g_0 is given in Eq. (21), and the isovector spectral function is found from Eq. (23). The four columns on the right give g_{MV} for four other choices of A: 0.6, 1.0, 1.1, and 2.0.

E (MeV)	g_d	Error	$0.87g_\rho$	g_{MV} (0.87)	g_{MV} (0.6)	g_{MV} (1.0)	g_{MV} (1.1)	g_{MV} (2.0)
289	0.07	0.001	0.00	0.07	0.08	0.09	0.10	0.12
308	0.40	0.004	0.00	0.40	0.43	0.48	0.51	0.61
337	1.07	0.01	0.01	1.08	1.18	1.28	1.35	1.55
373	1.94	0.02	0.03	1.97	2.16	2.30	2.34	2.53
418	2.83	0.02	0.06	2.89	3.22	3.18	3.24	3.14
459	3.34	0.03	0.11	3.45	3.92	3.64	3.63	3.04
509	3.52	0.03	0.23	3.75	4.36	3.70	3.58	2.27
571	3.16	0.02	0.57	3.73	4.37	3.37	3.12	1.08
622	2.53	0.02	1.28	3.81	4.32	3.29	3.00	0.98
682	1.58	0.01	3.85	5.43	5.21	5.09	6.13	5.06
718	1.02	0.01	7.89	8.91	7.44	9.10	9.44	13.5
757	0.42	0.01	12.7	13.1	10.2	14.0	14.9	23.8
800	−0.20	0.01	9.83	9.63	7.53	10.1	10.6	16.7
903	−1.30	0.02	2.86	1.56	1.50	1.13	0.97	0.27
1037	−2.00	0.02	1.17	−0.83	−0.55	−1.19	−1.37	−2.87
1217	−2.04	0.02	0.62	−1.42	−1.22	−1.52	−1.60	−2.54
1474	−1.49	0.01	0.37	−1.12	−1.08	−1.02	−1.01	−1.32
1647	−1.10	0.01	0.29	−0.81	−0.82	−0.70	−0.66	−0.77
1867	−0.72	0.01	0.23	−0.49	−0.54	−0.40	−0.35	−0.37
2154	−0.42	0.001	0.18	−0.24	−0.29	−0.19	−0.14	−0.12
2545	−0.22	0.001	0.15	−0.07	−0.12	−0.04	−0.02	−0.01
3111	−0.12	0.001	0.11	−0.01	−0.02	−0.01	0.00	0.00
4000	−0.10	0.001	0.09	−0.01	0.00	−0.01	−0.02	−0.01
5600	−0.09	0.002	0.06	−0.03	−0.01	0.00	−0.04	−0.02
	ReG_d	Error	0.87 ReG	ReG_{MV} (0.87)	ReG_{MV} (0.6)	ReG_{MV} (1.0)	ReG_{MV} (1.1)	
1867	0.40	0.004	−0.38	0.02	0.01	0.04	0.05	0.23
2154	0.34	0.002	−0.28	0.06	0.09	0.04	0.03	0.13
2545	0.22	0.001	−0.19	0.03	0.08	0.00	−0.02	0.03

ble parameters) was used by Orman.[27] In principle, the width Γ in the denominator should be allowed to vary with t'; but since we are using this form only as a first approximation it does not seem necessary to take account of this effect, which is rather small. The form factor $G_\rho(t)$ is determined[27] using (21) in an unsubtracted dispersion relation, giving

$$G_\rho(t)=0.358\frac{-(0.078-t)^{3/2}+2.17-3.64t}{(0.57-t)^2+0.0085}. \qquad (22)$$

Here t is in $(\mathrm{BeV}/c)^2$ and we have used the position 0.57 and the width $\Gamma=0.0922$ for the ρ corresponding[15] to an energy of 765 MeV and a width of 140 MeV. The coefficient A determines what fraction of the static isovector magnetic moment of 2.353 magnetons is contributed by G_ρ; i.e., $A=1.0$ means that the ρ resonance accounts completely for the static moment.

For a given choice of A, the data of Table IV are fitted for G_{MV} by first defining

$$G_d(t)=G_{MV}(t)-AG_\rho(t). \qquad (23)$$

We then fit $G_d(t)$ by our conformal transformation technique, and determine its spectral function $g_d(t')$ for $t'>t_0=4m^2$. Finally, we determine the spectral function $g_{MV}(t')$ using (21) and (23):

$$g_{MV}(t')=g_d(t')+Ag_\rho(t'). \qquad (24)$$

Again, we face the question of how to use the upper limits on the annihilation cross sections; let us answer this as in the preceding section by placing two additional constraints on the fit to $G_d(t)$. That is, we use five constraints: (i) $G_d(0)=2.353\,(1-A)$; (ii) $g_d'(t_0)=0$; (iii) $G_d(-\infty)=0$; (iv) $g_d(6.8)=-0.2$; (v) $\mathrm{Re}G_d(6.8)=0.2$. The last two constraints use the assumed shape of G_ρ and the value of A to keep the real and imaginary parts of $G_{MV}(6.8)$ close to zero.

We propose the tentative criterion that the spectral function $g_d(t')$ be small. That is, we assume that the resonance does in fact dominate the magnetic isovector spectral function, and choose A so that the non-ρ part is as small as possible. We interpret "as small as possible" to mean a minimization of $\sum_n a_n^2$, where we sum the squared coefficients from 0 to the order N of the polynomial that fits G_d. (This criterion corresponds to minimizing the area under the squared spectral function.)

Table VIII presents the coefficients a_n found for different choices of the parameter A. In each case $b=3$ has been chosen and use has been made of the χ^2 criterion to determine the degree N of the polynomial that fits satisfactorily. We find good fits with a quintic (with five constraints) for $A=0.6$ and $A=0.87$; but we must go to a sextic for $A=1.0$, 1.1, and 2.0. [Note that $a_0(A)$ goes through zero for $A=0.8$.] The value of $\sum_n a_n^2$ does not vary rapidly with N; but nevertheless it seems inconsistent to minimize this quantity using different values of N. If we choose sextic fits throughout, the minimum in $\sum_n a_n^2$ occurs at $A=0.87$, as illustrated in Fig. 6. (The value $\sum_n a_n^2=28.5$ for $A=0.0$ has been taken from the coefficients for the sextic fit with five constraints, Table V.) However,

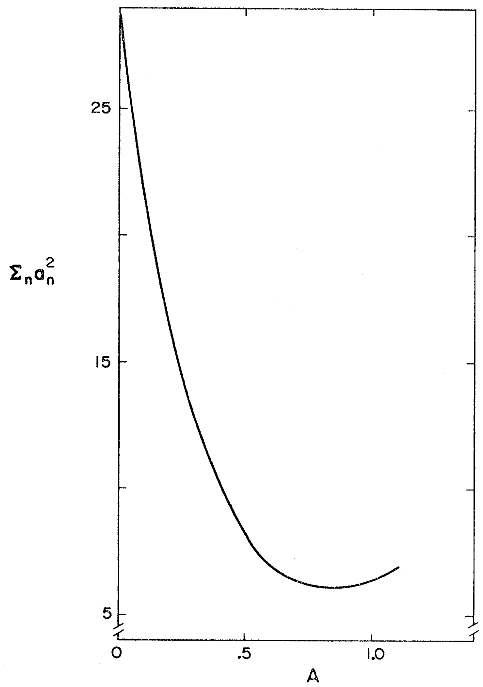

FIG. 6. Sum of squared coefficients for polynomial fit to residual G_d [Eq. (23)] versus parameter A. The coefficients are for sextic fits, Tables VIII and V (for $A=0$). An A of zero means no contribution of the ρ resonance; $A=1$ means that the ρ resonance gives the static isovector magnetic moment.

with this value of A a quintic fit with five constraints is chosen, since it gives an acceptable χ^2 and a slightly smaller $\sum_n a_n^2$. The spectral function g_d, and $0.87g_\rho$ and g_{MV} are given in Table IX and illustrated in Fig. 7. Note that $|G_{MV}(6.8)|=0.08$ is within the limit set by annihilation experiments. *If* errors in g_ρ and in A were to introduce no additional errors, the small error Δg_d of Table IX would give us an equally small error in determining the magnetic isovector spectral function g_{MV}.

Of course, our results for the spectral function g_{MV} do depend on our choice of A; this dependence is illustrated in Fig. 8, in the four columns on the right of Table IX, and in the results without the ρ resonance ($A=0$) of Sec. IV. We see that there is qualitative agreement among all six fits: All show a sizeable positive spectral function around 500 MeV, a high peak at or near the ρ resonance (620 to 760 MeV), and an appreciable dip at about 1200 MeV. The height of the peak at the ρ of course increases as we increase the value of A. Also as we increase A we "split off" the low-energy peak from the ρ peak. For $A=0$, we see a single broad shifted peak; for $0.6\leq A\leq 1.1$ we see a shoulder near 500 MeV; for $A=2$ we see a distinct dip between the peaks at 450 and 750 MeV.

TABLE X. Coefficients and χ^2 values for fits to $G_r(t)$, Eq. (25), using data of Table IV. All fits have three constraints, given below Eq. (25).

	$b=2$	$b=3$ Cubic	$b=3$ Quartic	Error
a_0	0.755	0.549	0.556	0.006
a_1	0.920	0.963	0.987	0.018
a_2	−0.471	0.056	−0.015	0.050
a_3	−0.360	−0.358	−0.391	0.024
a_4	0.275		0.054	0.038
χ^2	7.94	10.8	8.8	

VI. MAGNETIC ISOVECTOR: PRODUCT FORMULATION

Again we shall assume that the ρ resonance, with spectral function g_ρ and form factor G_ρ given by Eqs. (21) and (22), dominate the magnetic isovector form factor $G_{MV}(t)$. We replace Eq. (23) by

$$G_r(t)=G_{MV}(t)/G_\rho(t). \tag{25}$$

We then fit $G_r(t)$ for $t\leq 0$ by our conformal transformation technique. We use three constraints: (i) $G_r(0)=1.0$; (ii) $dG_r/dt=0$ at $t=t_0$; (iii) $G_r(-\infty)=0$. The first two constraints for the static value and the p-wave behavior at threshold are the same as used throughout this paper. The third constraint, that G_r obeys an unsubtracted dispersion relation, is adopted to fit the low upper limits[10,11] on the annihilation form factors, and is consistent with the $1/t^2$ behavior of the form factors for large spacelike momentum transfers.

The computer then gives us $\mathrm{Re}G_r(t')$ and $\mathrm{Im}G_r(t')$ for $t'>t_0$ which we use together with the real and imaginary parts of G_ρ from Eq. (22) to find the desired spectral function $\mathrm{Im}G_{MV}$, Eq. (5).

For $b=3$, a cubic with three constraints gives an excellent χ^2 value of 10.8 for 17 degrees of freedom. Table X gives the coefficients for this cubic fit, which has only one adjustable parameter. Table XI and Fig. 9 give the corresponding spectral function g_{MV}.

Note that at 2545 MeV, $|G_{MV}|=0.10$ is in agreement

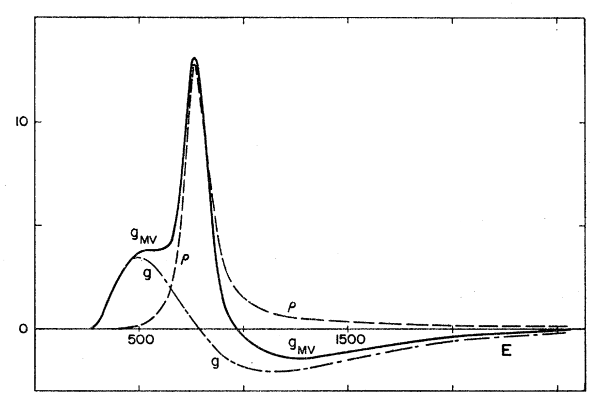

FIG. 7. Magnetic isovector spectral function versus energy in MeV, from Table IX. The dashed curve ρ is the ρ resonance, chosen with a coefficient 0.87, to contribute 87% of the static moment. The dash-dot curve g is a quintic fit to the residual spectral function. The solid curve g_{MV} is the sum, the magnetic spectral function.

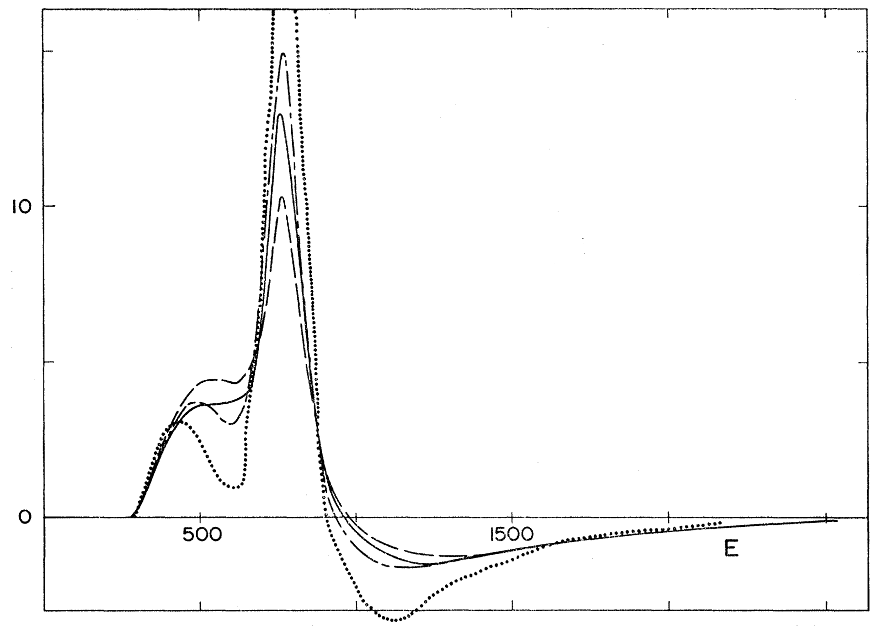

FIG. 8. Dependence of magnetic isovector spectral function versus energy in MeV on the choice of the parameter A, the assumed contribution of the resonance; data taken from Table IX. The dashed curve is $A=0.6$; the solid curve $A=0.87$; the dash-dot curve $A=1.1$; and the dotted curve $A=2.0$.

with Zichichi's upper limit. At 2154 MeV, $|G_{MV}|=0.19$ (see Tollestrup *et al.*[11]).

Fits were also made with $b=2.0$ and $b=2.5$; these two runs demanded use of a quartic, and gave somewhat different spectral functions. The results for $b=2$ are included in the tables. Table X shows that $a_4(b)$ goes through zero for b near 3; hence the good fit for a cubic for this choice of b. This cubic fit has a very small statistical error (not given in the table since it is deceptive), but the true error is much larger since b could easily be 2 or 4, instead of 3. The values of $g_{MV}(b)$ for 2 and 3 are more or less consistent within the appreciable errors of the former.

Table XI and Fig. 9 also give the very recent results of Furuichi *et al.*,[33] which they have determined by adjusting subtraction constants to fit magnetic isovector data for small spacelike momentum transfers $[-1(\mathrm{BeV}/c)^2 \leq t \leq 0]$. They also use electric isovector data (since they make their analysis in terms of Dirac and Pauli form factors F_1 and F_2); this reintroduces the perennial problem of what to choose for the electric form factor of the neutron.

The two curves shown in Fig. 9 are in qualitative agreement only. Both show a peak near the 760-MeV position of the ρ resonance, but shifted slightly towards lower energy. They both show a low-energy tail, without structure, and a dip in the region of 1 BeV. The Furuichi dip is much shallower, while we find a much deeper dip than his or than in others of our fits. Compensating for the deep dip, our fit has a very high peak at the position of the ρ resonance—much higher than Furuichi's or than any others of our fits except for the sum-formulation fit with $A=2.0$.

We note that $\mathrm{Re}G_r$, evaluated at the ρ resonance, is 0.85 for a cubic fit with $b=3$. From Eq. (6), the corresponding value of A in the sum formulation would also be 0.85, in remarkably good agreement with the choice $A=0.87$ for the sum formulation argued for in the previous section.

VII. DISCUSSION

It has been found that using a conformal transformation, it is possible to extrapolate measurements of isovector magnetic form factors in order to determine the spectral function. (The isoscalar fit was discussed in Sec. III and is ignored here.) Significantly different spectral functions are obtained by making different assumptions concerning the contribution of the ρ resonance (neglect, sum formulation, or product formulation) and by neglecting or including the annihilation data, as shown in Figs. 4, 5, 8, and 9. For instance, if we neglect the annihilation measurements, as in the quartic fit (three constraints) of Fig. 4, our extrapolated spectral function is much too large in absolute value in the annihilation region. Also Fig. 8 shows that as stronger ρ resonance contributions are assumed, we obtain higher and higher peaks in the vicinity of the ρ resonance. However, three main qualitative features persist in all our fits as well as in Furuichi's fit[33] shown in Fig. 9. First, the spectral function is large in the 400–600-MeV region; i.e., it is much larger than would be expected from the low-energy tail of a ρ resonance of Lorentzian shape. Second, the ρ resonance, or something similar, either appears without prior assumptions in the spectral function, as in Figs. 4 and 5, or can be assumed and still give a good fit to the data, as in Figs. 7–9. Third, all our spectral functions show appreciable dips in the region near 1 BeV. If the B resonance were known[15] to have spin 1 and negative parity, we would identify this

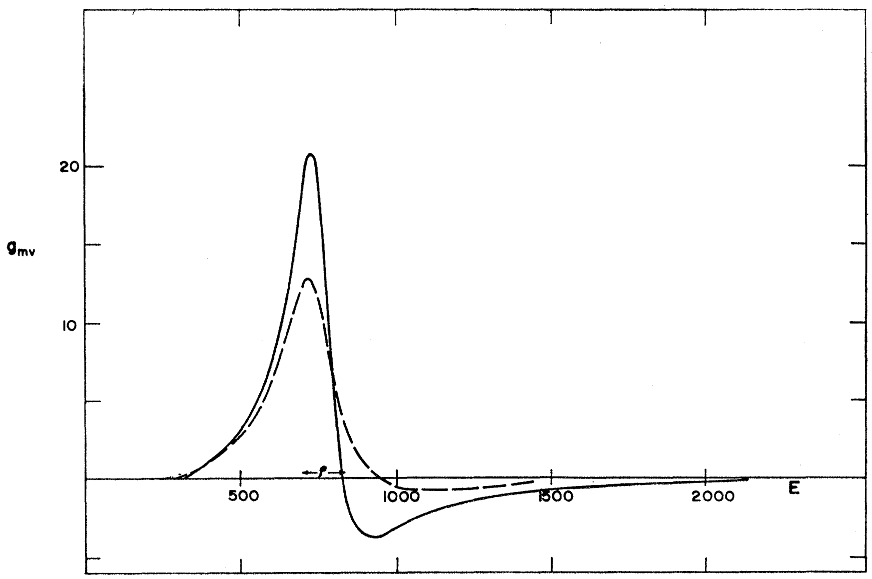

FIG. 9. The magnetic isovector spectral function versus energy in MeV from Table XI, for $b=3$, shown as a solid curve. The dashed curve shows the theoretical fit of Furuichi *et al.*, Ref. 33.

dip with the B. On the other hand, our extrapolation by itself is not good enough to assert whether the dip represents a narrow resonance, or a broader continuum. Given the present uncertainty about the quantum numbers of the B resonance, we cannot make any firm identification of the dip.

This qualitative agreement makes us believe that there is indeed significance in these qualitative features of our phenomenological fits, and also of the more theoretical fits.[33–35] It must be made clear that these successful fits do not in themselves deny the possible significance of various two-pole fits.[24,25,41] The two-pole fits are rejected on the physical grounds that (i) the nucleon form factors, we believe are dominated by mesonic intermediate states; and (ii) only one isovector 1^- meson is established at present. If either of these two assumptions proves to be invalid, then the two-pole fits would have as much (or more) physical significance as the work presented here.

TABLE XI. Spectral functions for product formulation.

E (MeV)	$b=2$ g_{MV}	$b=2$ Error	$b=3$ g_{MV}	Furuichi *et al.*[a]
289	−0.03		0.02	0.1
308	−0.14	0.06	0.10	0.3
337	−0.25	0.16	0.32	0.5
373	−0.13		0.70	0.8
418	0.45		1.35	1.4
459	1.4	0.3	2.13	2.0
509	3.2		3.37	2.9
571	6.4		5.63	4.9
622	10.4	0.2	8.67	8.0
682	18.6		15.1	11.3
718	25.1	1.5	20.6	12.7
757	18.8		16.7	9.5
800	1.3	0.9	2.6	5.4
903	−5.4		−4.0	0.4
1037	−3.3	0.2	−2.70	−0.7
1217	−1.74		−1.54	−0.8
1474	−0.78	0.04	−0.76	
1647	−0.50		−0.50	
1867	−0.30	0.02	−0.31	
2154	−0.17		−0.19	
2545	−0.09	0.01	−0.10	
3111	−0.04		−0.04	
4000	−0.02	0.003	−0.02	
5600	−0.01		−0.01	
	Re G_{MV}		Re G_{MV}	
1867	0.10		0.05	
2154	0.08		0.04	
2545	0.05		0.03	

[a] See Ref. 33. The spectral functions for $b=2$ uses a quartic and that for $b=3$ uses a cubic. Both have three constraints; see Table X and Eqs. (5) and (22).

We group the quantitative disagreements among the results shown in Figs. 4, 5, 8, and 9 under three main headings. (i) What is the quantitative value of the contribution of the ρ resonance? (ii) In the region below the main peak, is the spectral function monotonic, or does it have a shoulder, or even structure? (iii) What is the magnitude and shape of the dip above the main peak?

Let us note, concerning the first of these questions, that for a broad resonance it is hard to find an unambiguous procedure to separate the resonance contribution from the remainder of the continuum. For brevity, we shall consider the value of the coefficient A of Sec. V. (Other workers[34,43,44] treat the value of the ρNN coupling.) The septimal fit of Sec. IV gives a value $A=1.3$ (when we integrate from $3t_0$ to $9t_0$ to find the contribution to the static magnetic moment). *If* we have confidence in the criterion used in Sec. V, namely, minimizing the summed squares of the coefficients to G_d, we choose an A of about 0.9; but we note that much larger or smaller values of A give good χ^2 values. The product formulation of Sec. VI gives $\mathrm{Re}G_r=A=0.85$

[43] K. J. Barnes, Phys. Rev. 150, 1331 (1966).
[44] J. J. Sakurai, Phys. Rev. Letters 17, 1021 (1966).

(for the choice $b=3$). Furuichi's spectral function[33] integrated from $3t_0$ to $9t_0$ gives $A=1.0$. We feel that A is in the region from 0.8 to 1.3; but it is hard to obtain a firm value, or to estimate the error of the determination of A, since subjective factors enter so strongly. Consequently, we find it hard to have confidence in the values quoted[43,44] for the ρNN coupling constant, found from analysis of form factors.

The bulk of the present evidence favors a monotonic rise or perhaps a shoulder in the region below the main peak. The only case treated giving pronounced structure is the sum formulation, with the "high" value of $A=2$. Also, there is evidence from measurements of the mass spectrum for the leptonic decay of photoproduced vector mesons[45] in favor of a substantial contribution below the main peak at the ρ; but there is no compelling evidence at present in favor of structure in this region.

My phenomenological fits give a dip going down to only -1.5 (for the sum formulation with $A=0.87$) or as far as down to -4.0 (for the product formulation). While Furuichi[33] finds an even less pronounced dip, we are now considering high-mass states around 1 BeV, where his approximations become less reliable.

How can one decide which phenomenological fit is preferable? Also, what are the relative merits of the present fits compared to, say, Furuichi's spectral function? The former question may be answered in favor of the product formulation of Sec. VI. If we do not introduce the ρ resonance into our fit (as in the treatment of Sec. IV) we cannot find a spectral function anything near as narrow as the ρ: Hence it seems desirable to introduce the ρ before fitting, rather than to expect it (and other contributions) to come directly from the form-factor data with no additional assumptions. The product formulation of Sec. VI is preferred over the sum formulation of Sec. V since the sum formulation has the additional difficulty of the determination of the constant A.

As stated above, there are complementary advantages to our fit, and to Furuichi's. In both, use is made of about two adjustable parameters; e.g., we use two in our quartic fit with three constraints in Sec. VI. (As pointed out in that section, the fit with a single adjustable parameter for the constrained cubic is an accident.) Furuichi's two parameters are subtraction constants in his fits to the helicity amplitudes for πNN coupling. (In considering the fit to G_{MV} alone, it might be possible to combine these two constants into a single adjustable constant.) The present work has the advantage that it can determine the spectral function for high-mass intermediate states (such as those that dominate the annihilation experiments[10,11]) while Furuichi's approximations become dubious at these high energies. On the other hand, Furuichi's work has the marked advantage of having a theoretical interpretation in terms of definite dispersion-theory diagrams. This should permit testing Furuichi's analysis by consideration of other experiments, such as those on vector-meson decays into lepton pairs,[45] or on measurements[46] on the form factor of the meson.

The discussion above was limited to the isovector magnetic form factor. Recent proton data[4,9] is more extensive and more accurate than that for the neutron data, and has not been used in this paper. Further, it is stated[21] that "the best present theoretical predictions are not adequate" to provide a satisfactory fit to G_{Mp}. That statement was made in the context of fits with poles. It is clear that a fit using conformal transformations can at this time determine a proton spectral function giving a satisfactory χ^2 value, just as it did several years ago[12,13] for data then current. This treatment would be analogous to that in Sec. IV. Alternatively, one could introduce the ρ resonance, using, the product formulation, and use a palatable parametrization of the magnetic isoscalar form factor (perhaps that of Sec. III). A paper on the proton form factor is in preparation.

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