Vector-Meson Sum Rules from Current Commutators. I*

C. W. KIM AND MICHAEL RAM†
Department of Physics, The Johns Hopkins University, Baltimore, Maryland
(Received 5 May 1967)

Vector-meson sum rules are obtained from a charge commutator and from three mixed charge-current commutators using the noncovariant approach. In deriving these results, we use the most general form for the matrix elements of the weak hadron currents and for strong-interaction vertices, consistent with Lorentz covariance. Our results are compared with those obtained by dispersion-relation, covariant techniques, and with available experimental data.

I. INTRODUCTION

In this paper we shall consider the following commutation relations conjectured by Gell-Mann[1]:

$$[Q_A^{(+)}(t),Q_A^{(-)}(t)]=2I^{(3)}, \tag{1}$$

$$[Q_A^{(+)}(t),A_\mu^{(-)}(\mathbf{x},t)]=2V_\mu^{(3)}(\mathbf{x},t), \tag{2a}$$

$$[Q_A^{(+)}(t),V_\mu^{(3)}(\mathbf{x},t)]=-A_\mu^{(+)}(\mathbf{x},t), \tag{2b}$$

$$[Q_A^{(+)}(t),V_\mu^{(-)}(\mathbf{x},t)]=2A_\mu^{(3)}(\mathbf{x},t). \tag{2c}$$

In the above, $V_\mu^{(j)}(\mathbf{x},t)$ are the components of the unitary spin current ($j=1, 2, \cdots 8$; $\mu=1, \cdots 4$), and $A_\mu^{(j)}(\mathbf{x},t)$, the corresponding set of axial-vector currents associated with the weak interactions. Also,

$$Q_A^{(\pm)}(t)=-i\int A_4^{(\pm)}(\mathbf{x},t)d\mathbf{x}, \tag{3}$$

where $A_\mu^{(\pm)}(\mathbf{x},t)=A_\mu^{(1)}(\mathbf{x},t)\pm iA_\mu^{(2)}(\mathbf{x},t)$. $I^{(3)}$ is the third component of the isotopic spin.

Since their conjecture by Gell-Mann, the applications of these, and other, commutation relations have been very numerous. One very exhausted type of application has been the derivation of sum rules. Two methods have essentially been used to derive sum rules. The first method is not covariant and requires the use of states of infinite momentum. This is the method originally suggested by Fubini and Furlan,[2] and very successfully applied by Adler and Weisberger.[3] The second approach is due to Fubini, Furlan and Rossetti,[4] and is manifestly covariant. We shall refer to these two approaches as the noncovariant and covariant methods, respectively. It is generally thought that both methods give identical results, but no general proof of this exists. This has been shown to be so for the case of the Adler-Weisberger sum rule which has been derived by both methods.[3,5] In the absence of a general proof, it may be very useful, and certainly interesting, to compare the predictions of the two methods in other cases as well. This is just the purpose of the present paper.

In Sec. II we consider the matrix element of relation (1) between identical ρ^+ states of nonzero momentum $\mathbf{p}_\rho$ and polarization $\xi^{(M)}(\rho)$.[6] Using the noncovariant method, and in the limit when $|\mathbf{p}_\rho|\rightarrow\infty$, we derive two independent sum rules for $M=1$, 2 and $M=3$. In deriving these sum rules, we have assumed the most general form for the matrix elements of weak hadron currents and for strong-interaction vertices, consistent with Lorentz covariance. This requires, for example, that we introduce two strong-coupling constants $g_{A_1\rho\pi}$ and $g'_{A_1\rho\pi}$ associated with the decay $A_1\rightarrow\rho\pi$.[7] The sum rules we obtain are the same as those derived by Gasiorowicz and Geffen[8] using the covariant method, except for the fact that they assigned a specific value[9] to the coupling constant $g'_{A_1\rho\pi}$. The sum rules are quadratic in the coupling constants $g_{A_1\rho\pi}$ and $g'_{A_1\rho\pi}$. Assuming the Gell-Mann, Sharp, and Wagner[10] estimate for the strong-coupling constant $g_{\omega\rho\pi}$, our sum rules yield two possible sets of values for $g_{A_1\rho\pi}$ and $g'_{A_1\rho\pi}$. Both sets are not inconsistent with the experimental decay width of the A_1, within the limits of the approximations introduced.

In Sec. III we consider the matrix element of relation (2a) between the vacuum state and a ρ^0 state of momentum $\mathbf{p}_\rho$ and polarization $\xi^{(M)}(\rho)$. Using the noncovariant approach, and in the limit when $|\mathbf{p}_\rho|\rightarrow\infty$, we obtain *three* (two nontrivial) independent sum rules. The corresponding covariant calculation has been per-

* Work supported in part by the National Science Foundation.
† Present address: Physics Department, State University of New York at Buffalo, Buffalo, New York.

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).

[2] S. Fubini and G. Furlan, Physics **1**, 229 (1965).

[3] S. Adler, Phys. Rev. Letters **14**, 1051 (1965); W. I. Weisberger, *ibid.* **14**, 1047 (1965).

[4] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).

[5] W. I. Weisberger, Phys. Rev. **143**, 1302 (1966); S. Adler, *ibid.* **140**, B736 (1965).

[6] We use a representation of the polarization vectors $\xi^{(M)}(\rho)$ such that the three unit vectors $\xi^{(1)}(\rho)$, $\xi^{(2)}(\rho)$, and $\hat{p}_\rho=\mathbf{p}_\rho/|\mathbf{p}_\rho|$ form a right-handed orthogonal triad and $\xi^{(3)}(\rho)=(0,0E_\rho/m_\rho, i|\mathbf{p}_\rho|/m_\rho)$.

[7] Throughout this paper the notation A_1 will refer to the $I^G=1^-$ "enhancement" observed at a mass of 1080 MeV (I=isotopic spin; G=G parity). We shall assume this enhancement to correspond to a $\pi\rho$ resonance with $J^P=1^+$ (J=spin; P=parity). See A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

[8] S. Gasiorowicz and D. A. Geffen, Phys. Letters **22**, 344 (1966).

[9] The matrix element Gasiorowicz and Geffen used to describe the decay $A_1\rightarrow\rho\pi$ corresponds to the choice

$$g_{A_1\rho\pi}'(k^2)=(m_{A1}^2-m_\rho^2)/(m_{A1}^2+m_\rho^2+k^2)(m_{A1}^2+m_\rho^2).$$

This gives $g_{A_1\rho\pi}'(0)=0.12$.

[10] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962). This paper will be referred to as GSW.

formed by Renner[11] and Geffen,[12] and results in *only one* sum rule, which is the same as one of our sum rules, provided we assume a certain form factor satisfies an unsubtracted dispersion relation. The reason why Renner and Geffen obtain only one sum rule is due to the fact that they took the limit $q_\mu \to 0$.[13] We show that if this limit is not taken, additional sum rules can be obtained.

Similar results are obtained when we consider the matrix element of relation (2b) between the vacuum state and a single A_1^+ state.

We have also considered the matrix element of commutator (2c) between a single ω state and the vacuum. If a certain form factor is known, independently, to satisfy an unsubtracted dispersion relation, then our sum rule reduces to the trivial result $0\equiv 0$. The covariant method gives directly the result $0\equiv 0$ simply from covariance.

Throughout this paper we use natural units ($\hbar = c = 1$).

II. SUM RULES FROM CHARGE-CHARGE COMMUTATOR

A. Derivation of Sum Rules

Let $|\rho^+\rangle = |\rho^+;\, \mathbf{p}_\rho,\, \xi^{(M)}(\rho)\rangle$ be a ρ^+-meson state of momentum $\mathbf{p}_\rho$ and polarization $\xi^{(M)}(\rho)$. Consider the matrix element of Eq. (1) between identical ρ^+ states of nonzero momentum, i.e.,

$$\langle\rho^+|[Q_A{}^{(+)},Q_A{}^{(-)}]|\rho^+\rangle = 2. \tag{4}$$

Introducing a complete set of states, one can bring Eq. (4) to the form

$$\textstyle\sum_n |\langle\rho^+|Q_A{}^{(+)}|n\rangle|^2 - \sum_{n'} |\langle\rho^+|Q_A{}^{(-)}|n'\rangle|^2 = 2. \tag{5}$$

We now assume that only *known*, *single*-particle and resonance intermediate states contribute appreciably to this sum rule, and neglect all other states. Since on doubly charged mesons are known, this approximation immediately allows us to drop the second sum in Eq. (5), and we can write

$$\textstyle\sum_n |\langle\rho^+|Q_A{}^{(+)}|n\rangle|^2 \cong 2, \tag{6}$$

where the summation extends over all known single particle and resonance states. For nonzero $\mathbf{p}_\rho$, the states that contribute are the π^0, ω, φ, $A_1{}^0$, and $A_2{}^0$ states.[14] One can show from the known widths of the φ and A_2 decays into $\rho\pi$ that their contribution to sum rule (6) is negligible. We can therefore write that

$$\sum_{\mathbf{p}_\pi} |\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)|Q_A{}^{(+)}|\pi^0;\, \mathbf{p}_\pi\rangle|^2 + \sum_{\mathbf{p}_\omega}\sum_{M'=1}^{3} |\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)|Q_A{}^{(+)}|\omega;\, \mathbf{p}_\omega,\xi^{(M')}(\omega)\rangle|^2 + \sum_{\mathbf{p}_{A1}}\sum_{M'=1}^{3} |\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)| \times Q_A{}^{(+)}|A_1{}^0;\, \mathbf{p}_{A_1},\xi^{(M')}(A_1)\rangle|^2 \cong 2, \tag{7}$$

where p_π, p_ω, and p_{A_1} are the π, ω, and A_1 momenta, respectively, and $\xi^{(M')}(\omega)$ and $\xi^{(M')}(A_1)$, the polarization vectors of ω and A_1.

For $M=1, 2, 3$, Eq. (7) gives three sum rules, of which only two are independent.[8] The sum rules obtained for $M=1$, and $M=2$ are identical.

We can write

$$2\Omega(E_\rho E_\pi)^{1/2}\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)|A_\mu{}^{(+)}(0)|\pi^0;\, \mathbf{p}_\pi\rangle = i[\xi_\mu{}^{(M)*}F_A{}^{(1)}[\pi^0\to\rho^+;\, (p_\rho-p_\pi)^2] + (\xi^{(M)*}\cdot p_\pi) \times\{(p_\rho-p_\pi)_\mu F_A{}^{(2)}[\pi^0\to\rho^+;\, (p_\rho-p_\pi)^2] + (p_\rho+p_\pi)_\mu F_A{}^{(3)}[\pi^0\to\rho^+;\, (p_\rho-p_\pi)^2]\}], \tag{8a}$$

$$\begin{aligned}2\Omega(E_\rho E_\omega)^{1/2}&\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)|A_\mu{}^{(+)}(0)|\omega;\, \mathbf{p}_\omega,\xi^{(M')}(\omega)\rangle \\ &= i[\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha{}^{(M)*}(\rho)\xi_\beta{}^{(M')}(\omega)\{(p_\rho-p_\omega)_\gamma F_A{}^{(1)}[\omega\to\rho^+;\, (p_\rho-p_\omega)^2] + (p_\rho+p_\omega)_\gamma \\ &\quad\times F_A{}^{(2)}[\omega\to\rho^+;\, (p_\rho-p_\omega)^2]\} + \epsilon_{\alpha\beta\gamma\delta}\xi_\alpha{}^{(M)*}(\rho)\xi_\beta{}^{(M')}(\omega)(p_\rho)_\gamma(p_\omega)_\delta \\ &\quad\times\{(p_\rho-p_\omega)_\mu F_A{}^{(3)}[\omega\to\rho^+;\, (p_\rho-p_\omega)^2]\times(p_\rho+p_\omega)_\mu F_A{}^{(4)}[\omega\to\rho^+;\, (p_\rho-p_\omega)^2]\}]\end{aligned} \tag{8b}$$

and

$$\begin{aligned}2\Omega(E_\rho E_{A_1})^{1/2}&\langle\rho^+;\, \mathbf{p}_\rho,\xi^{(M)}(\rho)|A_\mu{}^{(+)}(0)|A_1{}^0;\, \mathbf{p}_{A_1},\xi^{(M')}(A_1)\rangle \\ &= i[(\xi^{(M)*}(\rho)\cdot\xi^{(M')}(A_1))\{(p_\rho-p_{A_1})_\mu F_A{}^{(1)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2] + (p_\rho+p_{A_1})_\mu F_A{}^{(2)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2]\} \\ &\quad+(\xi^{(M)*}(\rho)\cdot p_{A_1})\xi_\mu{}^{(M')}(A_1)F_A{}^{(3)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2] + (\xi^{(M')}(A_1)\cdot p_\rho)\xi_\mu{}^{(M)*}(\rho) \\ &\quad\times F_A{}^{(4)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2] + (\xi^{(M)*}(\rho)\cdot p_{A_1})(\xi^{(M')}(A_1)\cdot p_\rho) \\ &\quad\times\{(p_\rho-p_{A_1})_\mu F_A{}^{(5)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2] + (p_\rho+p_{A_1})_\mu F_A{}^{(6)}[A_1{}^0\to\rho^+;\, (p_\rho-p_{A_1})^2]\}].\end{aligned} \tag{8c}$$

In the above, Ω is the normalization volume of our wave functions, and E_ρ, E_π, E_ω, and E_{A_1}, the ρ, π, ω, and A_1 energies, respectively.[15] These are the most general expressions one can write down consistent with the parity and Lorentz transformation properties of the currents $V_\mu{}^{(i)}(x)$ and $A_\mu{}^{(i)}(x)$.

[11] B. Renner, Phys. Letters **21**, 453 (1966).

[12] D. A. Geffen, Ann. Phys. (N. Y.) 42, 1 (1967).

[13] q_μ is a fictitious 4-vector introduced in covariant calculations. It satisfies $q^2=0$.

[14] In determining the G parity of the intermediate states we have assumed that the currents $V_\mu(x)$ and $A_\mu(x)$ are first-class currents, i.e., $GV_\mu(x)G^{-1} = V_\mu(x)$ and $GA_\mu(x)G^{-1} = -A_\mu(x)$. See S. Weinberg, Phys. Rev. **112**, 1375 (1958).

We have also defined

$$\xi_\mu{}^{(M)\bigstar}=\xi_\mu{}^{(M)*} \quad \text{if} \quad \mu=1,2,3\,,$$
$$=-\xi_4{}^{(M)*} \quad \text{if} \quad \mu=4\,, \tag{9}$$

where * indicates complex conjugation. With our choice of representation for the $\xi_\mu{}^{(M)}$,[6] it is easy to see that $\xi_\mu{}^{(M)\bigstar}=\xi_\mu{}^{(M)}$.

Using the Gell-Mann–Lévy version[16] of PCAC (partially conserved axial-vector current) hypothesis, i.e.,

$$\partial_\mu A_\mu{}^{(+)}(x)=m_\pi{}^3 a_\pi \varphi_\pi{}^{(+)}(x)\,, \tag{10}$$

where a_π is the dimensionless π-meson-decay coupling constant, and $\varphi_\pi{}^{(+)}(x)$, the renormalized π-meson field ($\varphi_\pi{}^{(+)}$ creates a positively charged pion), one can show that

$$\begin{aligned} &F_A{}^{(1)}[\pi^0\to\rho^+;(p_\rho-p_\pi)^2=0] \\ &\quad+(m_\rho{}^2-m_\pi{}^2)F_A{}^{(3)}[\pi^0\to\rho^+;(p_\rho-p_\pi)^2=0] \\ &\qquad=m_\pi a_\pi g_{\rho\pi\pi}[(p_\rho-p_\pi)^2=0]\,, \end{aligned} \tag{11a}$$

$$\begin{aligned} &2F_A{}^{(2)}[\omega\to\rho^+;(p_\rho-p_\omega)^2=0] \\ &\quad+(m_\omega{}^2-m_\rho{}^2)F_A{}^{(4)}[\omega\to\rho^+;(p_\rho-p_\omega)^2=0] \\ &\qquad=m_\pi a_\pi g_{\omega\rho\pi}[(p_\rho-p_\pi)^2=0]\,, \end{aligned} \tag{11b}$$

$$\begin{aligned} &(m_{A_1}{}^2-m_\rho{}^2)F_A{}^{(2)}[A_1{}^0\to\rho^+;(p_\rho-p_{A_1})^2=0] \\ &\qquad=m_\pi a_\pi g_{A_1\rho\pi}{}^{(1)}[(p_\rho-p_{A_1})^2=0]\,, \end{aligned} \tag{11c}$$

$$\begin{aligned} &F_A{}^{(3)}[A_1{}^0\to\rho^+;(p_\rho-p_{A_1})^2=0] \\ &\quad-F_A{}^{(4)}[A_1{}^0\to\rho^+;(p_\rho-p_{A_1})^2=0] \\ &\quad+(m_{A_1}{}^2-m_\rho{}^2)F_A{}^{(6)}[A_1{}^0\to\rho^+;(p_\rho-p_{A_1})^2=0] \\ &\qquad=m_\pi a_\pi g_{A_1\rho\pi}{}^{(2)}[(p_\rho-p_{A_1})^2=0]\,. \end{aligned} \tag{11d}$$

The form factors $g_{\rho\pi\pi}$, $g_{\omega\rho\pi}$, $g_{A_1\rho\pi}{}^{(1)}$, and $g_{A_1\rho\pi}{}^{(2)}$ are defined by

$$\begin{aligned} \langle\rho^+;\mathbf{p}_\rho,\xi^{(M)}(\rho)|j_\pi{}^{(+)}(0)|\pi^0;\mathbf{p}_\pi\rangle &=\frac{-1}{2\Omega(E_\rho E_\pi)^{1/2}} \\ &\quad\times(\xi^{(M)\bigstar}(\rho)\cdot p_\pi)g_{\rho\pi\pi}[(p_\rho-p_\pi)^2]\,, \end{aligned} \tag{12a}$$

$$\begin{aligned} &\langle\rho^+;\mathbf{p}_\rho,\xi^{(M)}(\rho)|j_\pi{}^{(+)}(0)|\omega;\mathbf{p}_\omega,\xi^{(M')}(\omega)\rangle \\ &=\frac{1}{2\Omega(E_\rho E_\omega)^{1/2}}\epsilon_{\alpha\beta\gamma\delta}\xi_\alpha{}^{(M)\bigstar}(\rho)\xi_\beta{}^{(M')}(\omega) \\ &\qquad\times(p_\rho)_\gamma(p_\omega)_\delta g_{\omega\rho\pi}[(p_\rho-p_\omega)^2]\,, \end{aligned} \tag{12b}$$

$$\begin{aligned} &\langle\rho^+;\mathbf{p}_\rho,\xi^{(M)}(\rho)|j_\pi{}^{(+)}(0)|A_1{}^0;\mathbf{p}_{A_1},\xi^{(M')}(A_1)\rangle \\ &=\frac{1}{2\Omega(E_\rho E_{A_1})^{1/2}}[\{\xi^{(M)\bigstar}(\rho)\cdot\xi^{(M')}(A_1)\} \\ &\quad\times g_{A_1\rho\pi}{}^{(1)}[(p_\rho-p_{A_1})^2]+\{\xi^{(M)\bigstar}(\rho)\cdot p_{A_1}\} \\ &\qquad\times\{\xi^{(M')}(A_1)\cdot p_\rho\}g_{A_1\rho\pi}{}^{(2)}[(p_\rho-p_{A_1})^2]]\,, \end{aligned} \tag{12c}$$

where the current $j_\pi{}^{(+)}(x)$ is the source of the π-meson field, i.e.,

$$\left[\left(\nabla^2-\frac{\partial^2}{\partial t^2}\right)-m_\pi{}^2\right]\varphi_\pi{}^{(+)}(x)=j_\pi{}^{(+)}(x)\,. \tag{13}$$

Equations (12a), (12b), and (12c) are the most general expressions one can write down consistent with the parity and Lorentz-transformation properties of $j_\pi{}^{(+)}(x)$. One can show from time-reversal invariance that $g_{A_1\rho\pi}{}^{(1)}$ and $g_{A_1\rho\pi}{}^{(2)}$ are real.

From Eqs. (7), (8), and (11) it is now quite easy to derive the following two sum rules:

For M=1 or 2:

$$\begin{aligned} &\{\tfrac{1}{2}m_\pi a_\pi g_{\omega\rho\pi}[(p_\rho-p_\omega)^2=0]\}^2 \\ &\quad+\left\{\frac{m_\pi a_\pi}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}[(p_\rho-p_{A_1})^2=0]\right\}^2\cong 2\,. \end{aligned} \tag{14a}$$

For M=3:

$$\begin{aligned} &\left\{\frac{1}{2}\frac{m_\pi}{m_\rho}a_\pi g_{\rho\pi\pi}[(p_\rho-p_\pi)^2=0]\right\}^2 \\ &\quad+\left\{\frac{m_\pi a_\pi(m_{A_1}{}^2+m_\rho{}^2)}{2m_\rho m_{A_1}(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}[(p_\rho-p_{A_1})^2=0]\right\}^2 \\ &\qquad\times\{1-g_{A_1\rho\pi}'[(p_\rho-p_{A_1})^2=0]\}^2\cong 2\,. \end{aligned} \tag{14b}$$

We have defined

$$\begin{aligned} g_{A_1\rho\pi}(q^2)&=g_{A_1\rho\pi}{}^{(1)}(q^2)\,; \\ g_{A_1\rho\pi}'(q^2)&=\frac{(m_{A_1}{}^2-m_\rho{}^2)^2}{2(m_{A_1}{}^2+m_\rho{}^2)}\frac{g_{A_1\rho\pi}{}^{(2)}(q^2)}{g_{A_1\rho\pi}{}^{(1)}(q^2)}\,. \end{aligned} \tag{15}$$

These sum rules reduce exactly to those derived by Gasiorowicz and Geffen[8] using the covariant method, provided we substitute the specific value which they assigned for the coupling constant $g_{A_1\rho\pi}'$.[9]

B. Application

The sum rules (14a) and (14b) can be solved for the coupling constants $g_{A_1\rho\pi}$ and $g_{A_1\rho\pi}'$, provided all other quantities appearing in them are known. We determined $g_{\omega\rho\pi}$ from the GSW[10] ρ-dominance model for the decay $\omega\to3\pi$. The coupling constant[17] $g_{\rho\pi\pi}$ was calculated from the measured width of the decay $\rho\to2\pi$. Since Eqs. (14a) and (14b) are quadratic in the coupling con-

[15] We use a metric with an imaginary fourth component: $x=(\mathbf{x},it)$ and $p=(\mathbf{p},iE)$. $x^2=\mathbf{x}^2-t^2$; $p^2=\mathbf{p}^2-E^2=-m^2$. p_ρ, p_π, p_ω and p_{A_1} are the 4-momenta of the ρ, π, ω, and A_1, respectively.

[16] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[17] We assume throughout this paper that "physical" coupling constants such as $g_{\rho\pi\pi}(-m_\pi{}^2)$, $g_{A_1\rho\pi}(-m_\pi{}^2)$, etc. do not differ much from the "unphysical" ones $g_{\rho\pi\pi}(0)$, $g_{A_1\rho\pi}(0)$, etc., and shall in fact take them to be approximately equal whenever necessary.

stants, we obtained the following two solutions[18]:

$$|g_{A_1\rho\pi}{}^{\mathrm{I}}(0)|\cong 4.05\times 10^3\ \mathrm{MeV},\quad g_{A_1\rho\pi}{}'^{\mathrm{I}}(0)\cong 2.04;\quad (16\mathrm{I})$$

$$|g_{A_1\rho\pi}{}^{\mathrm{II}}(0)|\cong 4.05\times 10^3\ \mathrm{MeV},\quad g_{A_1\rho\pi}{}'^{\mathrm{II}}(0)\cong -0.04.\quad (16\mathrm{II})$$

The width of the A_1 is given by

$$\Gamma(A_1{}^0\to \text{all }\rho\pi)=\frac{1}{12\pi}\frac{|\mathbf{p}_\rho|}{m_{A_1}{}^2}\{g_{A_1\rho\pi}(-m_\pi{}^2)\}^2 \times\left[2+\frac{1}{m_\rho{}^2}\left\{E_\rho-\frac{2m_{A_1}(m_{A_1}{}^2+m_\rho{}^2)}{(m_{A_1}{}^2-m_\rho{}^2)^2}\times|\mathbf{p}_\rho|^2 g_{A_1\rho\pi}{}'(-m_\pi{}^2)\right\}^2\right].\quad (17)$$

Substituting the values (16I) and (16II) into relation (17), we find that

$$\Gamma^{\mathrm{I}}(A_1{}^0\to \text{all }\rho\pi)\cong 223\pm 60\ \mathrm{MeV},\quad (18\mathrm{I})$$

$$\Gamma^{\mathrm{II}}(A_1{}^0\to \text{all }\rho\pi)\cong 285\pm 75\ \mathrm{MeV}.\quad (18\mathrm{II})$$

These values are to be compared with the experimental width[19]

$$\Gamma_{\mathrm{expt}}(A_1{}^0\to \text{all }\rho\pi)=(130\pm 40)\ \mathrm{MeV}.\quad (19)$$

The quoted theoretical error in Eq. (18) is that introduced by the PCAC assumption, Eq. (10). It is well known that in the nucleon case, the PCAC assumption gives a Goldberger-Treiman relation which is accurate roughly to within 13%.[17] Use of this Goldberger-Treiman relation in deriving the Adler-Weisberger sum rule, for example, is therefore expected to introduce an error of some 26% in the sum rule. If we now further assume that an additional error of, say, some 15% is introduced by the truncation of the sum over intermediate states, then both Γ^{I} and Γ^{II} are further reduced, and Γ^{II} can be brought within the experimental limits as well. We would therefore tend to conclude that both results (16I) and (16II) are not inconsistent with the experimental A_1 width, within the limits of the approximations introduced. To the same accuracy, this justifies the commutation relation (1).

III. SUM RULES FROM CHARGE-CURRENT COMMUTATORS

A. Derivation of Sum Rules

We now consider the matrix element of commutator (2a) between a ρ^0 state and the vacuum state, and of commutator (2b) between an $A_1{}^+$ state and the vacuum state:

$$\langle\rho^0;\mathbf{p}_\rho,\xi^{(M)}(\rho)|[Q_A{}^{(+)},A_\mu{}^{(-)}(0)]|0\rangle =2\langle\rho^0;\mathbf{p}_\rho,\xi^{(M)}(\rho)|V_\mu{}^{(3)}(0)|0\rangle,\quad (20\mathrm{a})$$

$$\langle A_1{}^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|[Q_A{}^{(+)},V_\mu{}^{(3)}(0)]|0\rangle =-\langle A_1{}^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|A_\mu{}^{(+)}(0)|0\rangle.\quad (20\mathrm{b})$$

Introducing a complete set of states in the usual fashion gives

$$\sum_{n_a}\langle\rho^0;\mathbf{p}_\rho,\xi^{(M)}(\rho)|Q_A{}^{(+)}|n_a\rangle\langle n_a|A_\mu{}^{(-)}(0)|0\rangle -\sum_{n_a'}\langle\rho^0;\mathbf{p}_\rho,\xi^{(M)}(\rho)|A_\mu{}^{(-)}(0)|n_a'\rangle\langle n_a'|Q_A{}^{(+)}|0\rangle =2\langle\rho^0;\mathbf{p}_\rho,\xi^{(M)}(\rho)|V_\mu{}^{(3)}(0)|0\rangle\quad (21\mathrm{a})$$

and

$$\sum_{n_b}\langle A_1{}^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|Q_A{}^{(+)}|n_b\rangle\langle n_b|V_\mu{}^{(3)}(0)|0\rangle -\sum_{n_b'}\langle A_1{}^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|V_\mu{}^{(3)}(0)|n_b'\rangle \times\langle n_b'|Q_A{}^{(+)}|0\rangle =-\langle A_1{}^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|A_\mu{}^{(+)}(0)|0\rangle.\quad (21\mathrm{b})$$

In Table I we have summarized the properties of the intermediate states that contribute to these sum rules, and indicated the possible, *known*, *single*-particle and resonance candidates.[14,20] Some formal manipulations similar to those of the previous section reduce Eqs. (21a) and (21b) to the following form

$$A^{(a)}\xi_\mu{}^{(M)*}(\rho)\cong B^{(a)}(p_\rho)_\mu+C^{(a)}\delta_{4\mu},\quad (22\mathrm{a})$$

$$A^{(b)}\xi_\mu{}^{(M)*}(A_1)\cong B^{(b)}(p_{A_1})_\mu+C^{(b)}\delta_{4\mu},\quad (22\mathrm{b})$$

where $A^{(j)}$, $B^{(j)}$, and $C^{(j)}$ ($j=a$ or b) are given in the Appendix. One can easily show that relations (22a) and (22b) imply that

$$A^{(j)}\cong 0,\quad B^{(j)}\cong 0;\quad C^{(j)}\cong 0\quad (j=a\text{ or }b).\quad (23)$$

In the limit when $|\mathbf{p}_\rho|\to\infty$ and $|\mathbf{p}_{A_1}|\to\infty$, the con-

TABLE I. Properties of intermediate states contributing to sum rules (21a) and (21b). The last column lists the possible known single-particle and resonance candidates. Q, S, and B refer to the charge (in units of proton charge), strangeness, and baryon number.

	Q	S	B	$\mathbf{p}$	I^G	J^P	Known single-particle candidates
$\|n_a\rangle$:	−1	0	0	$\mathbf{p}_\rho$	1^-	$0^-,1^+$	π,A_1(1080)
$\|n_a'\rangle$:	+1	0	0	0	1^-	0^-	π
$\|n_b\rangle$:	0	0	0	$\mathbf{p}_{A_1}$	1^+	1^-	ρ
$\|n_b'\rangle$:	1	0	0	0	1^-	0^-	π

[18] All of our input data are taken from the compilation by A. H. Rosenfeld *et al.* (see Ref. 7). The ρ^+ width we used was 132 MeV, which represents the weighted average of published results. The π-decay constant as determined from the observed π-decay rate is $a_\pi=0.95$.

[19] A. H. Rosenfeld *et al.* (see Ref. 7).

[20] Possible contributions of disconnected graphs are also neglected. They do not change the discussions of sum rules in Sec. III B if all the form factors that appear in sum rules satisfy unsubtracted dispersion relations, which we eventually assume in the discussions. If certain form factors are known to satisfy subtracted dispersion relations, the disconnected graphs should be included in evaluating sum rules. We thank Professor B. W. Lee and Professor D. A. Geffen for clarifying this point.

ditions (23) give the following sum rules:

$$2v_\rho+m_\pi a_\pi\left\{\frac{\sqrt{2}a_{A_1}}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}(0)-\tfrac{1}{2}F_A{}^{(1)}[\pi^+\to\rho^0;\,k^2=\infty]\right\}\cong 0\,, \quad (24a)$$

$$2\sqrt{2}\frac{a_{A_1}}{m_{A_1}{}^2}g_{A_1\rho\pi}(0)\left\{1+\frac{(m_{A_1}{}^2+m_\rho{}^2)}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}'(0)\right\}+2m_\pi a_\pi g_{\rho\pi\pi}(0)-k^2\{F_A{}^{(2)}[\pi^+\to\rho^0;\,k^2=\infty]+F_A{}^{(3)}[\pi^+\to\rho^0;\,k^2=\infty]\}\cong 0\,, \quad (24a')$$

$$k^2\{F_A{}^{(2)}[\pi^+\to\rho^0;\,k^2=\infty]-F_A{}^{(3)}[\pi^+\to\rho^0;\,k^2=\infty]\}\cong 0\,, \quad (24a'')$$

$$-\sqrt{2}a_{A_1}+m_\pi a_\pi\left\{\frac{v_\rho}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}(0)-\tfrac{1}{2}F_V{}^{(1)}[\pi^+\to A_1{}^+;\,k^2=\infty]\right\}\cong 0\,, \quad (24b)$$

$$2\frac{v_\rho}{m_\rho{}^2}g_{A_1\rho\pi}(0)\left\{1-\frac{(m_{A_1}{}^2+m_\rho{}^2)}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}'(0)\right\}+k^2\{F_V{}^{(2)}[\pi^+\to A_1{}^+;\,k^2=\infty]+F_V{}^{(3)}[\pi^+\to A_1{}^+;\,k^2=\infty]\}\cong 0\,, \quad (24b')$$

$$k^2\{F_V{}^{(2)}[\pi^+\to A_1{}^+;\,k^2=\infty]-F_V{}^{(3)}[\pi^+\to A_1{}^+;\,k^2=\infty]\}\cong 0\,, \quad (24b'')$$

where $k^2\to\infty$. The constants v_ρ and a_{A_1}, as well as the form factors appearing in the above, are defined in the Appendix. Discussion of sum rules (24) is postponed to the end of the paper.

Consider now the matrix element of relation (2c) between an ω particle state of momentum $\mathbf{p}_\omega$, polarization $\xi^{(M)}(\omega)$, and the vacuum state:

$$\langle\omega;\,\mathbf{p}_\omega,\xi^{(M)}(\omega)|[Q_A{}^{(+)},V_\mu{}^{(-)}(0)]|0\rangle=2\langle\omega;\,\mathbf{p}_\omega,\xi^{(M)}(\omega)|A_\mu{}^{(3)}(0)|0\rangle=0\,. \quad (25)$$

Introducing a complete set of states gives

$$\sum_n\langle\omega|Q_A{}^{(+)}|n\rangle\langle n|V_\mu{}^{(-)}(0)|0\rangle-\sum_{n'}\langle\omega|V_\mu{}^{(-)}(0)|n'\rangle\langle n'|Q_A{}^{(+)}|0\rangle=0\,. \quad (26)$$

In Table II we have summarized the properties of the intermediate states $|n\rangle$ and $|n'\rangle$ contributing to sum rule (26). Assuming our sum rule to be saturated by the known single-particle and resonance states listed in the last column of the table,[20] we find that

$$\frac{\sqrt{2}v_\rho}{2E_\rho}F_A[\rho^-\to\omega;\,(p_\rho-p_\omega)^2]|_{\mathbf{p}_\rho=\mathbf{p}_\omega}\cong\tfrac{1}{2}m_\pi{}^2a_\pi F_V[\pi^+\to\omega;\,(p_\omega-p_\pi)^2]|_{\mathbf{p}_\pi=0}\,, \quad (27)$$

TABLE II. Properties of intermediate states contributing to sum rule (26).

	Q	S	B	$\mathbf{p}$	I^G	J^P	Known single-particle candidates
$\|n\rangle$:	-1	0	0	$\mathbf{p}_\omega$	1^+	1^-	ρ
$\|n'\rangle$:	$+1$	0	0	0	1^-	0^-	π

where we have used relation (A7) together with

$$\langle\omega;\,\mathbf{p}_\omega,\xi^{(M)}(\omega)|Q_A{}^{(+)}|\rho^-;\,\mathbf{p}_\rho,\xi^{(M')}(\rho)\rangle=\frac{-i\delta_{\mathbf{p}_\rho,\mathbf{p}_\omega}}{(2\Omega E_\rho)^{1/2}(2\Omega E_\omega)^{1/2}}\epsilon_{4\alpha\beta\gamma}\xi_\alpha{}^{(M)*}(\omega)\times\xi_\beta{}^{(M')}(\rho)(p_\omega)_\gamma F_A[\rho^-\to\omega;\,(p_\rho-p_\omega)^2] \quad (28)$$

and

$$\langle\omega;\,\mathbf{p}_\omega,\xi^{(M)}(\omega)|V_\mu{}^{(-)}(0)|\pi^+;\,\mathbf{p}_\pi\rangle=\frac{1}{(2\Omega E_\mu)^{1/2}(2\Omega E_\pi)^{1/2}}\epsilon_{\mu\alpha\beta\gamma}\xi_\alpha{}^{(M)*}(\omega)(p_\omega)_\beta(p_\pi)_\gamma\times F_V[\pi^+\to\omega;\,(p_\omega-p_\pi)^2]\,. \quad (29)$$

In the limit when $|\mathbf{p}_\omega|\to\infty$, the sum rule (27) reduces to

$$F_V[\pi^+\to\omega;\,(p_\omega-p_\pi)^2=\infty]\cong 0\,. \quad (30)$$

B. Discussion

If we assume that the form factors $F_A{}^{(1)}[\pi^+\to\rho^0;\,k^2]$ and $F_V{}^{(1)}[\pi^+\to A_1{}^+;\,k^2]$ satisfy unsubtracted dispersion relations, i.e.,

$$F_A{}^{(1)}[\pi^+\to\rho^0;\,k^2=\infty]=0\,,\quad F_V{}^{(1)}[\pi^+\to A_1{}^+;\,k^2=\infty]=0\,, \quad (31)$$

relations (24a) and (24b) reduce to

$$2v_\rho\cong-\frac{\sqrt{2}m_\pi a_\pi a_{A_1}}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}(0) \quad (32a)$$

and

$$\sqrt{2}a_{A_1}\cong\frac{m_\pi a_\pi v_\rho}{(m_{A_1}{}^2-m_\rho{}^2)}g_{A_1\rho\pi}(0)\,. \quad (32b)$$

These sum rules are identical to those obtained by Renner[11] and Geffen[12] using the covariant method. These authors assumed unsubtracted dispersion relations for the form factors $G_1{}^{(j)}(\nu)$ ($j=a$ or b) given by

$$\xi_\mu{}^{(M)}(\rho)G_1{}^{(a)}(\nu_\rho)+\cdots=\int d^4x\,e^{iq\cdot x}\theta(t)\times\langle\rho^0|[\partial_\alpha A_\alpha{}^{(+)}(x),A_\mu{}^{(-)}(0)]|0\rangle\,, \quad (33a)$$

and

$$\xi_\mu{}^{(M)}(A_1)G_1{}^{(b)}(\nu_{A_1})+\cdots=\int d^4x\,e^{iq\cdot x}\theta(t)\times\langle A_1{}^+|[\partial_\alpha A_\alpha{}^{(+)}(x),V_\mu{}^{(3)}(0)]|0\rangle\,, \quad (33b)$$

where $\nu_\rho = p_\rho \cdot q$, $\nu_{A_1} = p_{A_1} \cdot q$, and q_μ is a fictitious 4-vector satisfying the condition $q^2=0$. At the end of the calculation Renner and Geffen set $q_\mu = 0$.

From Eqs. (32a) and (32b) one can show that

$$v_\rho / a_{A_1} \cong \pm 1 \,, \tag{34a}$$

and

$$\{[m_\pi a_\pi / (m_{A_1}{}^2 - m_\rho{}^2)] g_{A_1\rho\pi}(0)\}^2 \cong 2 \,. \tag{34b}$$

Using Eq. (17), relation (34b) implies that $\Gamma(A_1 \to \text{all } \rho\pi) > 445$ MeV. This is more than two and a half times larger than the upper experimental limit [see Eq. (19)]. Furthermore, comparing Eqs. (14a) and (34b), we see that they imply $g_{\omega\rho\pi}=0$, in contradiction to the GSW estimate.[10] Since we tend to accept the validity of sum rules (14a) and (14b),[21] these discrepancies could be accounted for by one or more of the following possibilities:

(a) Either one or both form factors $F_A{}^{(1)}[\pi^+ \to \rho^+; k^2]$ and $F_V{}^{(1)}[\pi^+ \to A_1{}^+; k^2]$ satisfy subtracted dispersion relations.

(b) One or both of the commutators (2a) and (2b) are not valid.

(c) Limiting the intermediate states in (21a) and (21b) to known single-particle and resonance states is a bad approximation.

We now turn our attention to sum rules (24a′), (24a″), (24b′), and (24b″) which were obtained together with sum rules (24a) and (24b). These sum rules were not obtained by Renner[11] and Geffen[12] when they used the covariant approach. We wish to point out that the reason for this stems from the fact that these authors considered directly the limit $q_\mu \to 0$ (Ref. 13) (one of the advantages in taking such a limit is that ambiguities due to possible Schwinger terms are removed). This is easily observed if we remember that the sum rules derived by them were obtained from equations of the form

$$\mathcal{A}^{(a)}(\nu_\rho)\xi_\mu{}^{(M)}(\rho) = \mathcal{B}^{(a)}(\nu_\rho)\{\xi^{(M)}(\rho)\cdot q\}(p_\rho)_\mu + \mathcal{C}^{(a)}(\nu_\rho)\{\xi^{(M)}(\rho)\cdot q\}q_\mu \,, \tag{35a}$$

and

$$\mathcal{A}^{(b)}(\nu_{A_1})\xi_\mu{}^{(M)}(A_1) = \mathcal{B}^{(b)}(\nu_{A_1})\{\xi^{(M)}(A_1)\cdot q\}(p_{A_1})_\mu + \mathcal{C}^{(b)}(\nu_{A_1})\{\xi^{(M)}(A_1)\cdot q\}q_\mu \,. \tag{35b}$$

[Compare these with Eqs. (23a) and (23b).] If we now simply set $q_\mu = 0$, we obtain just two sum rules:

$$\mathcal{A}^{(j)}(0) = 0; \quad (j = a \text{ or } b). \tag{36}$$

These are identical to sum rules (32a) and (32b). If we do not set $q_\mu = 0$, it is easy to show that Eqs. (35a) and (35b) imply

$$\mathcal{A}^{(j)}(\nu) = 0 \,, \tag{37a}$$

$$\mathcal{B}^{(j)}(\nu) = 0, \quad (j = a \text{ or } b) \tag{37b}$$

$$\mathcal{C}^{(j)}(\nu) = 0 \,. \tag{37c}$$

It is therefore clear that both covariant and noncovariant methods give exactly the same number of sum rules.[22]

We will now attempt to discuss the implications of the new sum rules Eqs. (24a′), (24a″), (24b′), and (24b″). Using the PCAC hypothesis, one can show that[23]

$$\lim_{k^2\to\infty} k^2 F_A{}^{(2)}[\pi^+ \to \rho^0; k^2] = 0 \,. \tag{38}$$

Substituting Eq. (38) into Eq. (24a″) gives

$$\lim_{k^2\to\infty} k^2 F_A{}^{(3)}[\pi^+ \to \rho^0; k^2] = 0 \,. \tag{39}$$

In Ref. 23, it was shown that

$$F_A{}^{(3)}[\pi^+ \to \rho^0; k^2] \cong \frac{\sqrt{2} a_{A_1}\{-\frac{1}{2} g_{A_1\rho\pi}{}^{(2)}(-m_\pi{}^2)\}}{m_{A_1}{}^2 + k^2} \,. \tag{40}$$

This implies[17]

$$F_A{}^{(3)}[\pi^+ \to \rho^0; k^2=0] = -\frac{\sqrt{2}}{2m_{A_1}{}^2} a_{A_1} g_{A_1\rho\pi}{}^{(2)}(0) = 0 \,. \tag{41}$$

Furthermore, substituting Eqs. (38), (39), and (41) into Eq. (24a′), gives the following relation:

$$\sqrt{2}\frac{a_{A_1}}{m_{A_1}{}^2} g_{A_1\rho\pi}(0) + m_\pi a_\pi g_{\rho\pi\pi}(0) \cong 0 \,. \tag{42}$$

It is interesting to note that Eq. (42) follows directly from the PCAC hypothesis alone,[23] when, as implied by Eq. (41), $g_{A_1\rho\pi}{}^{(2)}(0) = 0$ [a_{A_1} cannot vanish since otherwise Eq. (42) would imply that $g_{\rho\pi\pi} = 0$, in violent contradiction with experiment].

Combining Eqs. (32a) and (42), it is easy to show that

$$v_\rho g_{\rho\pi\pi} \cong 2m_\rho{}^2 \left[\frac{m_\pi}{\sqrt{2} m_\rho} a_\pi g_{\rho\pi\pi}\right]^2 \cong 2m_\rho{}^2 \times 2 \,. \tag{43}$$

This result is not in agreement with the relation

$$v_\rho g_{\rho\pi\pi} = 2m_\rho{}^2 \tag{44}$$

derived by Sakurai[24] on the basis of ρ dominance and the CVC (conserved vector current) hypothesis.[25] This discrepancy may serve perhaps as further evidence for conclusions (a), (b), and (c) given following Eq. (34b). Consider now Eqs. (24b′) and (24b″). The CVC

---

[21] This is based on the fact that (a) sum rules (14a) and (14b) are not inconsistent with experiment, within the limits of the approximations introduced, and (b) they were derived from the commutator (1) which is at the basis of the Adler-Weisberger sum rule.

[22] We are now investigating sum rules (37b) and (37c) and comparing them with Eqs. (24a′), (24″), (24b′), and (24b″). The results of this investigation will be the subject of another paper.

[23] C. W. Kim and Michael Ram, following paper, Phys. Rev. **162**, 1584 (1967). In this paper, the authors use the so-called pole dominance version of PCAC and assume all form factors of interest to satisfy unsubtracted dispersion relations. The same results are equally obtainable from the Gell-Mann–Lévy version of PCAC, Eq. (10), provided one also assumes all form factors are unsubtracted and that $\lim_{k^2\to\infty} g_{\rho\pi\pi}(k^2)/k^2 = 0$.

[24] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[25] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

hypothesis

$$\partial_\mu V_\mu^{(i)}=0, \quad (i=1, 2, 3) \tag{45}$$

together with the assumption

$$\lim_{k^2\to\infty} F_V^{(1)}[\pi^+\to A_1^+; k^2]=0$$

and

$$\lim_{k^2\to\infty} F_V^{(3)}[\pi^+\to A_1^+; k^2]=0, \tag{46}$$

implies

$$\lim_{k^2\to\infty} k^2F_V^{(2)}[\pi^+\to A_1^+; k^2]=0. \tag{47}$$

Substituting Eq. (47) into Eq. (24b″), we find

$$\lim_{k^2\to\infty} k^2F_V^{(3)}[\pi^+\to A_1^+; k^2]=0. \tag{48}$$

In analogy to Eq. (40), one can show that

$$F_V^{(3)}[\pi^+\to A_1^+; k^2]\cong\frac{\sqrt{2}v_\rho\{-\frac{1}{2}g_{A_1\rho\pi}^{(2)}(-m_\pi^2)\}}{m_\rho^2+k^2}. \tag{49}$$

From Eq. (49) we obtain[17]

$$F_V^{(3)}[\pi^+\to A_1^+; k^2=0]\cong-\frac{\sqrt{2}}{2m_\rho^2}v_\rho g_{A_1\rho\pi}^{(2)}(0)=0. \tag{50}$$

Using Eqs. (24b′), (47), (48), and (50) one can now easily derive the relation

$$v_\rho g_{A_1\rho\pi}(0)=0. \tag{51}$$

Equations (50) and (51) have the following two solutions:

$$\text{(a)}\quad v_\rho=0, \qquad \text{(b)}\quad g_{A_1\rho\pi}=g_{A_1\rho\pi}'=0. \tag{52}$$

Both solutions are in disagreement with experiment as (a) rules out the decay of the ρ meson into lepton pairs, which has been observed,[26] while (b) forbids the decay $A_1\to\rho+\pi$ (this is the dominant decay mode of the A_1[19]).

Let us now return to sum rule (30). If the form factor $F_V[\pi^+\to\omega; k^2]$ is known to satisfy an unsubtracted dispersion relation, the sum rule reduces to the trivial identity $0\equiv0$. The covariant method gives directly the result $0\equiv0$ (simply from covariance) without the necessity of imposing any restrictions on the form factors.

ACKNOWLEDGMENTS

We would like to thank Professor Gordon Feldman and Professor Thomas Fulton for many helpful discussions.

APPENDIX

$$A^{(a)}=2v_\rho-\sqrt{2}a_{A_1}\left\{\frac{(E_\rho-E_{A_1})}{2E_{A_1}}F_A^{(1)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2] +\frac{(E_\rho+E_{A_1})}{2E_{A_1}}F_A^{(2)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]\right\}\Bigg|_{\mathbf{p}_{A1}=\mathbf{p}_\rho} -\tfrac{1}{2}a_\pi m_\pi F_A^{(1)}[\pi^+\to\rho^0; (p_\rho-p_\pi)^2]|_{\mathbf{p}_\pi=0}, \tag{A1}$$

$$B^{(a)}=\frac{i\sqrt{2}a_{A_1}}{2E_{A_1}}\xi_4^{(M)*}(\rho)\Bigg[\frac{(E_{A_1}-E_\rho)}{m_{A_1}^2}\{(E_\rho-E_{A_1})F_A^{(1)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]$$
$$+(E_\rho+E_{A_1})F_A^{(2)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]\}+\frac{E_{A_1}}{m_{A_1}^2}(E_{A_1}-E_\rho)F_A^{(3)}[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]$$
$$-\left\{1+\frac{(p_{A_1}\cdot p_\rho)}{m_{A_1}^2}\right\}F_A^{(4)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]+\left\{1+\frac{(p_{A_1}\cdot p_\rho)}{m_{A_1}^2}\right\}(E_{A_1}-E_\rho)$$
$$\times\{(E_\rho-E_{A_1})F_A^{(5)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]+(E_\rho+E_{A_1})F_A^{(6)}[A_1^-\to\rho^0; (p_\rho-p_{A_1})^2]\}\Bigg]\Bigg|_{\mathbf{p}_{A1}=\mathbf{p}_\rho}+\frac{im_\pi a_\pi}{2E_\pi}$$
$$\times\xi_4^{(M)*}(\rho)[F_A^{(1)}[\pi^-\to\rho^0; (p_\rho-p_\pi)^2]+(E_\rho-E_\pi)\{(E_\rho-E_\pi)F_A^{(2)}[\pi^-\to\rho^0; (p_\rho-p_\pi)^2]$$
$$+(E_\rho+E_\pi)F_A^{(3)}[\pi^-\to\rho^0; (p_\rho-p_\pi)^2]\}]|_{\mathbf{p}_\pi=\mathbf{p}_\rho}+\tfrac{1}{2}im_\pi^2a_\pi\xi_4^{(M)*}(\rho)$$
$$\times[F_A^{(2)}[\pi^+\to\rho^0; (p_\rho-\rho_\pi)^2]+F_A^{(3)}[\pi^+\to\rho^0; (p_\rho-p_\pi)^2]]|_{\mathbf{p}_\pi=0}, \tag{A2}$$

[26] S. S. Hertzbach *et al.*, Phys. Rev. **155**, 1461 (1967).

$$C^{(a)}=\frac{\sqrt{2}a_{A_1}}{2E_{A_1}}\xi_4^{(M)*}(\rho)\Bigg[-\frac{(E_{A_1}-E_\rho)^2}{m_{A_1}^2}\{(E_\rho-E_{A_1})F_A^{(1)}[A_1^-\to\rho^0;(p_A-p_{A_1})^2]+(E_\rho+E_{A_1})F_A^{(2)}$$
$$\times[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]\}+\left\{1-\frac{E_{A_1}(E_{A_1}-E_\rho)}{m_{A_1}^2}\right\}(E_{A_1}-E_\rho)F_A^{(3)}[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]+\frac{(p_{A_1}\cdot p_\rho)}{m_{A_1}^2}$$
$$\times(E_{A_1}-E_\rho)F_A^{(4)}[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]-\frac{(p_{A_1}\cdot p_\rho)}{m_{A_1}^2}(E_{A_1}-E_\rho)^2\{(E_\rho-E_{A_1})F_A^{(5)}[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]$$
$$+(E_\rho+E_{A_1})F_A^{(6)}[A_1^-\to\rho^0;(p_\rho-p_{A_1})^2]\}\Bigg]\Bigg|_{\mathbf{p}_{A1}=\mathbf{p}_\rho}-\frac{m_\pi a_\pi}{2E_\pi}(E_\pi-E_\rho)\xi_4^{(M)*}(\rho)$$
$$\times[F_A^{(1)}[\pi^-\to\rho^0;(p_\rho-p_\pi)^2]+(E_\rho-E_\pi)\{(E_\rho-E_\pi)F_A^{(2)}[\pi^-\to\rho^0;(p_\pi-p_\rho)^2]$$
$$+(E_\rho+E_\pi)F_A^{(3)}[\pi^-\to\rho^0;(p_\rho-p_\pi)^2]\}]|_{\mathbf{p}_\rho=\mathbf{p}_\pi}+\tfrac{1}{2}m_\pi^3a_\pi\xi_4^{(M)*}(\rho)$$
$$\times[F_A^{(2)}[\pi^+\to\rho^0;(p_\rho-p_\pi)^2]-F_A^{(3)}[\pi^+\to\rho^0;(p_\rho-p_\pi)^2]]|_{\mathbf{p}_\pi=0},\quad\text{(A3)}$$

$$A^{(b)}=-\sqrt{2}a_{A_1}-v_\rho\left\{\frac{(E_{A_1}-E_\rho)}{2E_\rho}F_A^{(1)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\right.$$
$$\left.+\frac{(E_{A_1}+E_\rho)}{2E_\rho}F_A^{(2)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\right\}\Bigg|_{\rho=\mathbf{p}_{A1}}-\tfrac{1}{2}m_\pi a_\pi F_V^{(1)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]|_{\mathbf{p}_\pi=0},\quad\text{(A4)}$$

$$B^{(b)}=\frac{iv_\rho}{2E_\rho}\xi_4^{(M)*}(A_1)\Bigg[\frac{(E_\rho-E_{A_1})}{m_\rho^2}\{(E_{A_1}-E_\rho)F_A^{(1)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]$$
$$+(E_{A_1}+E_\rho)F_A^{(2)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\}+\frac{E_\rho}{m_\rho^2}(E_\rho-E_{A_1})F_A^{(3)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]$$
$$-\left\{1+\frac{(p_{A_1}\cdot p_\rho)}{m_\rho^2}\right\}F_A^{(4)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]+\left\{1+\frac{(p_{A_1}\cdot p_\rho)}{m_\rho^2}\right\}(E_\rho-E_{A_1})$$
$$\times\{(E_{A_1}-E_\rho)F_A^{(5)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]+(E_{A_1}+E_\rho)F_A^{(6)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\}\Bigg]\Bigg|_{\mathbf{p}_\rho=\mathbf{p}_{A1}}$$
$$+\tfrac{1}{2}im_\pi^2a_\pi\xi_4^{(M)*}(A_1)\{F_V^{(2)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]+F_V^{(3)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]\}|_{\mathbf{p}_\pi=0},\quad\text{(A5)}$$

$$C^{(b)}=\frac{v_\rho}{2E_\rho}\xi_4^{(M)*}(A_1)\Bigg[-\frac{(E_\rho-E_{A_1})^2}{m_\rho^2}\{(E_{A_1}-E_\rho)F_A^{(1)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]$$
$$+(E_{A_1}+E_\rho)F_A^{(2)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\}+(E_\rho-E_{A_1})\left\{1-\frac{E_\rho}{m_\rho^2}(E_\rho-E_{A_1})\right\}F_A^{(3)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]$$
$$+\frac{(p_{A_1}\cdot p_\rho)}{m_\rho^2}(E_\rho-E_{A_1})F_A^{(4)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]-\frac{(p_{A_1}\cdot p_\rho)}{m_\rho^2}(E_\rho-E_{A_1})^2$$
$$\times\{(E_{A_1}-E_\rho)F_A^{(5)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]+(E_{A_1}+E_\rho)F_A^{(6)}[\rho^0\to A_1^+;(p_{A_1}-p_\rho)^2]\}\Bigg]\Bigg|_{\mathbf{p}_\rho=\mathbf{p}_{A1}}$$
$$+\tfrac{1}{2}m_\pi^3a_\pi\xi_4^{(M)*}(A_1)\{F_V^{(2)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]-F_V^{(3)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]\}|_{\mathbf{p}_\pi=0}.\quad\text{(A6)}$$

In the above we have defined v_ρ, a_{A_1} and the form factor $F_V^{(i)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]$, $i=1,2,3$, through the relations

$$\langle\rho^0|V_\mu^{(3)}(0)|0\rangle=\frac{i}{(2\Omega E_\rho)^{1/2}}v_\rho\xi_\mu^{(M)*}(\rho)\,, \tag{A7}$$

$$\langle A_1^+|A_\mu^{(+)}(0)|0\rangle=\frac{i}{(2\Omega E_{A_1})^{1/2}}\sqrt{2}a_{A_1}\xi_\mu^{(M)*}(A_1)\,, \tag{A8}$$

and

$$\langle A_1^+;\mathbf{p}_{A_1},\xi^{(M)}(A_1)|V_\mu^{(3)}(0)|\pi^+;\mathbf{p}_\pi\rangle=\frac{1}{(2\Omega E_{A_1})^{1/2}(2\Omega E_\pi)^{1/2}}[\xi_\mu^{(M)*}(A_1)F_V^{(1)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]$$
$$+\{\xi^{(M)*}(A_1)\cdot p_\pi\}\{(p_{A_1}-p_\pi)_\mu F_V^{(2)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]+(p_{A_1}+p_\pi)_\mu F_V^{(3)}[\pi^+\to A_1^+;(p_{A_1}-p_\pi)^2]\}]. \tag{A9}$$

The remaining form factors are defined by Eqs. (8a) and (8c), together with the isotopic-spin transformation properties of $A_\mu^{(i)}$.

PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Remarks on the Pole Dominance Version of the Hypothesis of Partially Conserved Axial-Vector Current*

C. W. KIM AND MICHAEL RAM†
Department of Physics, The Johns Hopkins University, Baltimore, Maryland
(Received 15 June 1967)

We review and discuss the dispersion-theory version of the PCAC hypothesis in the case of nucleon leptonic weak decays. The discussion is extended to the case of meson weak decays, and the feasibility of a direct test of the Goldberger-Treiman relation for the meson case is considered.

I. INTRODUCTION

THERE have recently been many applications of the PCAC (partially conserved axial-vector current) hypothesis, particularly in conjunction with the derivation of sum rules from current algebra. The PCAC hypothesis is essential in these applications, as it relates weak-interaction form factors (which are in most cases difficult, if not impossible, to measure at present) appearing in the sum rules, to strong-coupling constants which can be determined from decay widths or scattering experiments. In general, to test the PCAC hypothesis directly, one would have to measure independently both the weak form factors and strong-coupling constants. The best and only known case that has been tested directly to date is that involving nucleon $n\to p$ weak form factor (the famous Goldberger-Treiman relation).[1]

In this paper we would like to review the possibility of directly testing the PCAC hypothesis in the case of meson decays. As we shall see, no direct test (as in the nucleon case) is feasible. We shall only discuss the dispersion-theory (pole-dominance) version[2] of the PCAC hypothesis. The more commonly used version due to Gell-Mann and Lévy[3] which relates the divergence of the axial-vector current to the pion field will not be considered.

In Sec. II, we review the application of PCAC to nucleon leptonic weak interactions. In Sec. III, the results of Sec. II are extended to the case of meson leptonic weak interactions. Throughout this paper we use natural units ($\hbar=c=1$).

II. APPLICATION OF PCAC TO NUCLEON LEPTONIC WEAK INTERACTIONS

Let us first review the application of PCAC to nucleon leptonic weak interactions. Consider the matrix element $\langle 0|A_\mu^{(+)}(0)|\bar{p},n;\text{in}\rangle$, where $|\bar{p},n;\text{in}\rangle$ represents an antiproton and neutron "in" state with antiproton and neutron 4-momenta $\bar{p}=(\bar{\mathbf{p}},iE_{\bar{p}})$ and $n=(\mathbf{n},iE_n)$, respectively. $A_\mu^{(+)}(x)$, with $\mu=1,\cdots 4$, is the strangeness-conserving axial-vector weak hadron current operative in β decay and muon capture. This matrix element is easily related to the matrix element $\langle p|A_\mu^{(+)}(0)|n\rangle$ involved in β decay and muon capture,

* Work supported in part by the National Science Foundation.
† Present address: Physics Department, State University of New York at Buffalo, Buffalo, New York.
[1] Even for the nucleon case, the PCAC hypothesis has been tested at only two momentum transfers, corresponding to those occurring in β decay ($q^2\approx 0$) and muon capture ($q^2\approx m_\mu^2$).

[2] Y. Nambu, Phys. Rev. Letters 4, 380 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960).
[3] M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).