Using $P_\nu{}^\mu(z)=P_{-\nu-1}{}^\mu(z)$ and the connection of the Legendre functions with the Gegenbauer polynomials [Sec. 3.15, Eqs. (1), (4), and (10) of Ref. 9] Eq. (B1) becomes

$$\begin{aligned}\lim(l-\rho+m)&T_{i(\rho+1)}{}^l(x)T_{i(\rho+1)}{}^l(y)\\ &=(2\rho-2m+1)\frac{\sin\pi(\rho+1)}{\rho+1}\frac{2^{-2\rho-2}\Gamma(2\rho-m+2)}{[\Gamma(\rho+\frac{3}{2})]^2\Gamma(m+1)}(-1)^m\\ &\quad\times(x^2-1)^{-(\rho-m+1)/2}(y^2-1)^{-(\rho-m+1)/2}\\ &\quad\times(d/dx)^m(x^2-1)^{\rho+1/2}(d/dy)^m(y^2-1)^{\rho+1/2}. \qquad \text{(B2)}\end{aligned}$$

From Eq. (2), it is seen that for physical s,

$$\cosh\beta_t=\frac{1}{m_1m_3}p_1p_3=\frac{\epsilon_1}{m_1}\frac{\epsilon_3}{m_3}-\frac{q_i}{m_1}\frac{q_f}{m_3}\cos\theta,$$

$$\cosh\beta_u=\frac{1}{m_1m_4}p_1p_4=\frac{\epsilon_1}{m_1}\frac{\epsilon_4}{m_4}+\frac{q_i}{m_1}\frac{q_f}{m_4}\cos\theta,$$

so that $x=\epsilon_1/m_1$ and $y=\epsilon_3/m_3$ or ϵ_4/m_4.

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Dispersion Relations for Three-Particle Scattering Amplitudes. III.*

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We continue our discussion of the scattering of three nonrelativistic, spinless particles interacting via two-body Yukawa potentials. We study the on-energy-shell amplitudes for the breakup of a two-particle bound state, for elastic scattering off a bound state, and for rearrangement as a function of the total center-of-mass energy E for fixed physical values of the angle variables. We show that each of these amplitudes is given by a "Fredholm series" which is uniformly convergent with respect to E for all values of E on the physical sheet including the real axis. We also show that each of these amplitudes satisfies a dispersion relation in E for fixed physical values of the angle variables. Except for simple poles which arise from the lowest-order diagrams, all of the singularities lie on the real E axis.

I. INTRODUCTION

In two previous papers we have studied the on-energy-shell scattering amplitude for three free, nonrelativistic particles interacting via two-body Yukawa potentials.[1] We have seen that this amplitude exists for all values of the center-of-mass energy E on the physical sheet, including the real axis, provided the vectors

$$\mathbf{y}_i=(2m_iE)^{-1/2}\mathbf{k}_i,$$
$$\mathbf{y}_i'=(2m_iE)^{-1/2}\mathbf{k}_i',\quad i=1,2,3 \qquad (1.1)$$

are kept in the physical region. Here, m_i are the masses of the particles and $\mathbf{k}_i$ and $\mathbf{k}_i'$ the initial and final center-of-mass momenta. In addition we have shown that the amplitude for the scattering of three free particles satisfies a dispersion relation in E for fixed, physical values of the vectors $\mathbf{y}_i$ and $\mathbf{y}_i'$. All of the singularities lie on the real E axis.

In the present paper we shall extend our results to the amplitude for the elastic scattering of a particle off a two-body bound state and to the amplitudes for bound-state breakup and rearrangement collisions. In order to simplify the kinematics we shall present our proofs for the case of three particles of equal mass m. However, the extension of our results to the case of arbitrary masses is straightforward.

As in I and II we shall parametrize the states of three free particles in terms of the vectors $\mathbf{y}_i$ defined in Eq. (1.1) and the center-of-mass energy E which is measured from the three-particle threshold. States containing a free particle and a two-particle bound state of binding energy B will be expressed in terms of E and the unit vector

$$\hat{n}=[\tfrac{2}{3}(E+B)]^{-1/2}\mathbf{k}_{B'}, \qquad (1.2)$$

where $\mathbf{k}_B$ is the center-of-mass momentum of the free

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[1] M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. **146**, 1130 (1966) (hereafter referred to as I); **159**, 1348 (1967) (hereafter referred to as II).

particle. Note that $k_B^2=\frac{2}{3}(E+B)$. We are using units in which $\hbar=2m=1$. We note that the vectors $\mathbf{y}_i$ and $\hat{n}$ satisfy the relations

$$\sum_{i=1}^{3}\mathbf{y}_i=0\,,\quad \sum_{i=1}^{3}y_i^2=1\,,\qquad \hat{n}^2=1\,,\quad y_i^2\leq\tfrac{2}{3}\,,\quad i=1,2,3. \tag{1.3}$$

We shall study the on-energy-shell amplitudes for elastic scattering, rearrangement and breakup as a function of E for fixed, physical values of the vectors $\hat{n}$ and $\mathbf{y}_i$. Our procedure will be the same as in I and II. In Sec. II we define the amplitudes and write them formally as the ratio of two Fredholm series. The Fredholm denominator is the same as for the scattering of three free particles, and we have already seen that it exists and satisfies a dispersion relation in E, the only singularities being the unitarity cut.[1] On the other hand, the numerator series is just a rearrangement of the perturbation series for the Faddeev equation. (Throughout this paper we shall refer to the iteration solution of the Faddeev equation as the perturbation series.) Thus, in order to find the analyticity properties of a general term in the numerator series it is sufficient to study a general term in the perturbation series. In Sec. III we show that all terms of fourth order or higher in the perturbation series for each of our amplitudes satisfies a dispersion relation in E. All of the singularities are on the *real E axis*. In Sec. IV we show that the same result holds for the second- and third-order terms in the perturbation series. The first-order terms have simple poles in E which can become complex for certain values of the binding energies and the vectors $\hat{n}$ and $\mathbf{y}_i$. These poles become complex only because of our choice of variables. What is important for dynamical calculations based on unitarity and analyticity is that the residue at the poles and the discontinuities across all of the cuts except those associated with the potentials can be evaluated explicitly in terms of on-energy-shell quantities.

In Sec. V we use the method of dual diagrams to find the position of all branch points except those associated with the potentials.

In Sec. VI we show that the numerator series for each of our amplitudes is uniformly convergent for all values of E on the physical sheet. Thus all of the amplitudes exist and satisfy the dispersion relations previously obtained for the individual terms in the Fredholm series.

II. DEFINITION OF AMPLITUDES

In this paper we shall consider the problem of three spinless particles of equal mass interacting via two-body Yukawa potentials. In order to simplify the presentation we shall not work with a general superposition of Yukawa potentials; however, all of our results would remain valid in that case. We write the potential in the form

$$V=V_1+V_2+V_3\,, \tag{2.1}$$

where the subscript indicates the particle which is *not* interacting. In momentum space we have, for example,

$$\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|V_1|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle=\delta^3(\mathbf{q}_1-\mathbf{q}_1')\times\delta^3(\mathbf{q}_2+\mathbf{q}_3-\mathbf{q}_2'-\mathbf{q}_3')[(\mathbf{q}_2-\mathbf{q}_2')^2+\mu^2]^{-1}. \tag{2.2}$$

We are using units in which $\hbar=2m=1$, and we shall consistently omit numerical factors such as coupling constants and 2π's.

It is customary to write the three-body off-energy-shell scattering amplitude in the form

$$T=T_1+T_2+T_3\,, \tag{2.3}$$

where, for example, T_1 is that part of the amplitude in which particles 2 and 3 interact last. In terms of these amplitudes the Faddeev equations are

$$T_\alpha=\hat{t}_\alpha+\hat{t}_\alpha G_0(T_\beta+T_\gamma)\,,\quad \alpha,\beta,\gamma \text{ distinct.} \tag{2.4}$$

G_0 is the three-particle free Green's function,

$$\begin{aligned}\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|G_0(E)|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle&=\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|[E-H_0]^{-1}|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle\\&=\delta^3(\mathbf{q}_1-\mathbf{q}_1')\delta^3(\mathbf{q}_2-\mathbf{q}_2')\delta^3(\mathbf{q}_3-\mathbf{q}_3')\\&\quad\times[E-\mathbf{q}_1^2-\mathbf{q}_2^2-\mathbf{q}_3^2]^{-1}.\end{aligned} \tag{2.5}$$

The three-body operator $\hat{t}_1$ is related to the usual off-energy-shell two-body scattering amplitude $\langle\mathbf{p}|t_1(E)|\mathbf{p}'\rangle$ by

$$\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|\hat{t}_1(E)|\mathbf{q}_1'\mathbf{q}_2'\mathbf{q}_3'\rangle=\delta^3(\mathbf{q}_1-\mathbf{q}_1')\delta^3(\mathbf{q}_2+\mathbf{q}_3-\mathbf{q}_2'-\mathbf{q}_3')\times\langle\tfrac{1}{2}(\mathbf{q}_2-\mathbf{q}_3)|t_1(E-\tfrac{3}{2}\mathbf{q}_1^2)|\tfrac{1}{2}(\mathbf{q}_2'-\mathbf{q}_3')\rangle. \tag{2.6}$$

Similar expressions can be obtained for $\hat{t}_2$ and $\hat{t}_3$ by cyclic permutation of the indices of Eq. (2.6).

In our case it will be convenient to work with amplitudes which specify which particles interact first as well as last. We thus write[2]

$$T_\alpha=T_{\alpha1}+T_{\alpha2}+T_{\alpha3}\,,\quad \alpha=1,2,3. \tag{2.7}$$

The Faddeev equations then become

$$T_{\alpha\delta}=\hat{t}_\alpha\delta_{\alpha\delta}+\hat{t}_\alpha G_0(T_{\beta\delta}+T_{\gamma\delta})\,,\quad \alpha,\beta,\gamma \text{ distinct} \tag{2.8}$$

or, in matrix notation,

$$T=\hat{t}+KT\,, \tag{2.9}$$

where

$$T=\begin{pmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{pmatrix}\,,\quad \hat{t}=\begin{pmatrix}\hat{t}_1&0&0\\0&\hat{t}_2&0\\0&0&\hat{t}_3\end{pmatrix}\,,$$

$$K=\begin{pmatrix}0&\hat{t}_1G_0&\hat{t}_1G_0\\\hat{t}_2G_0&0&\hat{t}_2G_0\\\hat{t}_3G_0&\hat{t}_2G_0&0\end{pmatrix}\,. \tag{2.10}$$

[2] It should be noted that we are using a different notation from that in I and II. In I and II the potential had a pair of subscripts which indicated which particles *were* interacting. Similarly, the T matrix had a pair of subscripts to indicate which particles interacted last. In the present paper the two subscripts on the T matrix will always indicate the particles which do *not* interact in the initial and final states.

We note that the kernel K is identical to the kernel of Eq. (2.4) which was studied in I and II.

All of the on-energy-shell amplitudes which we shall be interested in can be obtained by taking appropriate matrix elements of $T_{\alpha\gamma}$. The amplitude for the breakup of a bound state of particles 1 and 2, $\langle\mathbf{y}|M(E)|B_3\hat{n}\rangle$, is given by[3]

$$\langle\mathbf{y}|M(E)|B_3\hat{n}\rangle=\sum_{\alpha=1}^{3}\langle\mathbf{y}|T_{\alpha1}(E)+T_{\alpha2}(E)|B_3\hat{n}\rangle, \quad (2.11)$$

where $|\mathbf{y}\rangle$ is a plane-wave state for three free particles with center-of-mass energy E and momentum vectors $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$; and $|B_3\hat{n}\rangle$ is a state with center-of-mass energy E in which particles 1 and 2 are bound with binding energy B_3. $\hat{n}$ is a unit vector in the direction of the center-of-mass momentum of the free particle. In momentum space we have

$$\begin{aligned}\langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|\mathbf{y}\rangle &=\delta^3(\mathbf{q}_1-k\mathbf{y}_1)\delta^3(\mathbf{q}_2-k\mathbf{y}_2)\delta^3(\mathbf{q}_3-k\mathbf{y}_3)\,,\\ \langle\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|B_3\hat{n}\rangle &=\delta^3(\mathbf{q}_3-[\tfrac{2}{3}(k^2+B_3)]^{1/2}\hat{n})\psi_{B_3}[\tfrac{1}{2}(\mathbf{q}_1-\mathbf{q}_2)]\,, \end{aligned}\quad (2.12)$$

where $k^2=E$ and ψ_{B_3} is the usual two-body bound-state wave function.

Similarly, typical amplitudes for elastic scattering off a bound state and for rearrangement collisions can be written in the form

$$\begin{aligned}\langle B_3\hat{n}'|M(E)|B_3\hat{n}\rangle &=\langle B_3'\hat{n}'|[T_{11}(E)+T_{12}(E)+T_{21}(E)+T_{22}(E)]|B_3\hat{n}\rangle\,,\\ \langle B_2'\hat{n}'|M(E)|B_3\hat{n}\rangle &=\langle B_2'\hat{n}'|[V_2+T_{11}+T_{12}+T_{31}+T_{32}]|B_3\hat{n}\rangle\\ &=\langle B_2'\hat{n}'|[V_3+T_{11}+T_{12}+T_{31}+T_{32}]|B_3\hat{n}\rangle.\end{aligned}\quad (2.13)$$

The next step is to obtain a "Fredholm series" for the on-energy-shell amplitudes. Our starting point is the off-energy-shell T matrix. Iterating Eq. (2.9) once we have

$$T=(1+K)\hat{t}+K^2T. \quad (2.14)$$

We have studied the kernel K^2 in I and II in connection with the on-energy-shell scattering amplitude for three free particles. We have seen that K^2 is a square-integrable operator in the upper-half $k=E^{1/2}$ plane. As a result, the resolvent

$$R=K^2+K^2R=N/D \quad (2.15)$$

has a Fredholm solution in this region given by[4]

$$D=\sum_{n=0}^{\infty}D_n\,,\quad N=\sum_{n=0}^{\infty}N_n\,, \quad (2.16)$$

where

$$D_0=1\,,$$

$$D_n=\frac{(-)^n}{n!}\begin{vmatrix}0 & n-1 & 0 & \cdots & 0 & 0 & 0\\ \Sigma_2 & 0 & n-2 & \cdots & 0 & 0 & 0\\ \Sigma_3 & \Sigma_2 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ \Sigma_{n-1} & \Sigma_{n-2} & \Sigma_{n-3} & \cdots & \Sigma_2 & 0 & 1\\ \Sigma_n & \Sigma_{n-1} & \Sigma_{n-2} & \cdots & \Sigma_3 & \Sigma_2 & 0\end{vmatrix};$$

$$N_0=K^2\,,$$

$$N_n=\frac{(-)^n}{n!}\begin{vmatrix}K^2 & n & 0 & \cdots & 0 & 0 & 0\\ K^4 & 0 & n-1 & \cdots & 0 & 0 & 0\\ K^6 & \Sigma_2 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ K^{2n} & \Sigma_{n-1} & \Sigma_{n-2} & \cdots & \Sigma_2 & 0 & 1\\ K^{2n+2} & \Sigma_n & \Sigma_{n-1} & \cdots & \Sigma_3 & \Sigma_2 & 0\end{vmatrix};$$

and $\Sigma_i=\mathrm{tr}(K^{2i})$. We have seen in I that the Σ_i are analytic functions of E. Their only singularity is the unitarity cut which runs along the real E axis from $-B_m$ to ∞. B_m is the largest two-particle binding energy in the problem. In II we saw that the series for D is uniformly convergent for all values of E on the physical sheet, so D has the same analyticity properties as the Σ_i.

We can write the solution to Eq. (2.14) in the form

$$\begin{aligned}T&=(1+K)\hat{t}+R(1+K)\hat{t}\\ &=\sum_{n=0}^{7}K^n\hat{t}+K^4R(K^2+K^3)\hat{t}\\ &=\sum_{n=0}^{7}K^n\hat{t}+D^{-1}\sum_{n=0}^{\infty}K^4N_n(K^2+K^3)\hat{t}.\end{aligned}\quad (2.17)$$

We now substitute Eq. (2.17) into Eqs. (2.11) and (2.13) to obtain a formal series for each of the on-energy-shell amplitudes. We have, for example,

$$\begin{aligned}\langle\mathbf{y}|M(E)|B_3\hat{n}\rangle=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{2}\langle\mathbf{y}|[&\sum_{n=0}^{7}(K^n\hat{t})_{\alpha\beta}\\ &+D^{-1}\sum_{n=0}^{\infty}(K^4N_n(K^2+K^3)\hat{t})_{\alpha\beta}]|B_3\hat{n}\rangle.\end{aligned}\quad (2.18)$$

We have iterated the right-hand side of Eq. (2.17) enough times so that none of the external momenta appear in N.

We shall start by studying the analyticity properties of the individual terms in Eq. (2.18) and in the corre-

[3] The amplitude M is related to the S matrix by

$$\langle f|S|i\rangle=\delta_{fi}-2\pi i\delta(E_f-E_i)\delta^3(\mathbf{p}_f-\mathbf{p}_i)\langle f|M|i\rangle.$$

For a derivation of Eqs. (2.11) and (2.13) see, for example, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

[4] F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).

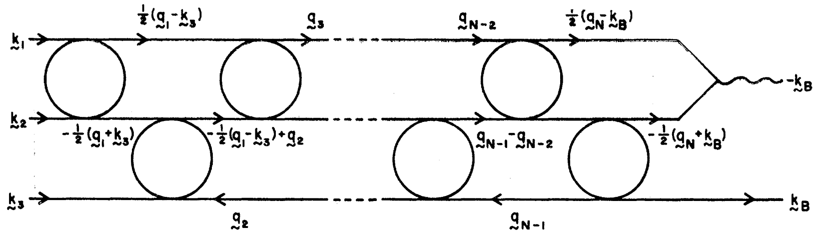

FIG. 1. A typical Nth-order graph for the breakup amplitude.

sponding series for the elastic scattering and rearrangement amplitudes. The proof of the convergence of these series will be given in Sec. VI.

In addition to the "Fredholm series" given in Eq. (2.18), it is convenient to introduce a "perturbation series" for each of the on-energy-shell amplitudes. The perturbation series is obtained by substituting the expression

$$T=\sum_{i=0}^{\infty} K^i\hat{t} \tag{2.19}$$

into Eqs. (2.11) and (2.13). From Eq. (2.16) we see that $\langle\mathbf{y}|[K^4N_n(K^2+K^3)\hat{t}]_{\alpha\beta}|B_3\hat{n}\rangle$ is just a sum of the first $2n+9$ terms in the perturbation series for the breakup amplitude, each term in the series being multiplied by an appropriate product of Σ_i's. Since the analyticity properties of the Σ_i's and of D^{-1} are known, in order to obtain the analyticity properties of a general term in the Freedholm series for the breakup amplitude, it is sufficient to study a general term in the perturbation series. Clearly the same result holds for the elastic scattering and rearrangement amplitudes.

III. PERTURBATION THEORY: $N\geq 4$

We now turn to the problem of studying a general term in the perturbation series for each of the on-energy-shell amplitudes. We shall study the analyticity properties in the upper-half $k=E^{1/2}$ plane for fixed, physical values of the vectors $\mathbf{y}$ and $\hat{n}$. In this section we shall treat terms of order $N\geq 4$.

We start with a typical Nth-order term for the breakup amplitude

$$\begin{aligned}f(k)&=\langle\mathbf{y}|(K^{N-1}\hat{t})_{31}|B_3\hat{n}\rangle\\&=\langle\mathbf{y}|t_3G_0t_1G_0\cdots G_0t_1|B_3\hat{n}\rangle\\&=\int d^3q_1\,d^3q_2\cdots d^3q_n\\&\qquad\times\frac{F(\mathbf{q}_1,\mathbf{q}_2,\cdots\mathbf{q}_n;k,\mathbf{y},\hat{n})\psi_{B_3}(\frac{1}{2}\mathbf{q}_N)}{S(\mathbf{q}_1,\mathbf{q}_2,\cdots\mathbf{q}_N;k,\mathbf{y},\hat{n})},\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}S=&[\tfrac{1}{2}\mathbf{q}_1^2-\tfrac{1}{2}\mathbf{k}_{12}][\tfrac{1}{4}(\mathbf{q}_1-\mathbf{k}_3)^2+\mathbf{q}_2^2+(\mathbf{q}_2-\tfrac{1}{2}\mathbf{q}_1+\tfrac{1}{2}\mathbf{k}_3)^2-k^2]\\&\times[\mathbf{q}_2^2+\mathbf{q}_3^2+(\mathbf{q}_2-\mathbf{q}_3)^2-k^2]\cdots\\&\times[\mathbf{q}_{N-1}^2+\mathbf{q}_{N-2}^2+(\mathbf{q}_{N-1}-\mathbf{q}_{N-2})^2-k^2]\\&\quad\times[\mathbf{q}_{N-1}^2+\tfrac{1}{4}(\mathbf{q}_N-\mathbf{k}_B)^2+(\mathbf{q}_{N-1}-\tfrac{1}{2}\mathbf{q}_N+\tfrac{1}{2}\mathbf{k}_B)^2-k^2],\end{aligned}$$

and

$$\begin{aligned}\mathbf{k}_{12}&=k\mathbf{y}_{12}=\mathbf{k}_1-\mathbf{k}_2,\\\mathbf{k}_B&=\hat{n}[\tfrac{2}{3}(k^2+B_3)]^{1/2},\\&0\leq y_{12}^2\leq 2.\end{aligned}$$

F is a product of two-particle t matrices. Our choice of variables is shown in Fig. 1. The two-particle t matrices are represented by circles, the free Green's functions by three horizontal lines, and the bound-state wave function by a wavy line. We have performed the trivial δ-function integrations arising from momentum conservation.

We first consider the two-particle t matrices which appear in Eq. (3.1). We recall that for a superposition of Yukawa potentials the off-energy-shell two-particle t matrix has a Fredholm solution of the form[1,5]

$$\begin{aligned}\langle\mathbf{p}|t(E)|\mathbf{p}'\rangle&=\langle\mathbf{p}|[V+VG_0t]|\mathbf{p}'\rangle\\&=\langle\mathbf{p}|V|\mathbf{p}'\rangle+d^{-1}(E)\int d^3p''\\&\qquad\times\langle\mathbf{p}|n(E)|\mathbf{p}''\rangle\langle\mathbf{p}''|V|\mathbf{p}'\rangle\\&=\langle\mathbf{p}|V|\mathbf{p}'\rangle+d^{-1}(E)\sum_{i=0}^{\infty}\int d^3p''\\&\qquad\times\langle\mathbf{p}|n_i(E)|\mathbf{p}''\rangle\langle\mathbf{p}''|V|\mathbf{p}'\rangle.\end{aligned}\tag{3.2}$$

Here, n and d are given by the infinite series of Eq. (2.16) if one replaces K^2 by the kernel VG_0 and Σ_i by $\sigma_i=tr(VG_0)^i$. σ_i and d^{-1} have the integral representations[1]

$$\begin{aligned}\sigma_i(E)&=\int_0^\infty\frac{\mathrm{Im}\sigma_i(E')}{E'-E}dE',\\d^{-1}(E)&=1+\sum_i\frac{C_i}{E+B_i}+\int_0^\infty\frac{\mathrm{Im}d^{-1}(E')}{E'-E}dE'.\end{aligned}\tag{3.3}$$

The B_i are the two-particle binding energies and the C_i are constants. We again note that the series for nV is just a rearrangement of the perturbation series for t [see Eq. (2.16)].

We now use the last line of Eq. (3.2) for each of the t matrices in Eq. (3.1). Since the series for nV is relatively uniformly convergent,[4] we can interchange the order of summation and integration if there exists a region of the k plane in which the resulting integrals are uniformly convergent. In such a region we can study

[5] R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).

the analyticity properties of $f(k)$ by studying those of a general term in which each of the t matrices has been expanded in a perturbation series, provided, of course, we take into account the σ_i and d^{-1}. A typical term is shown in Fig. 2. The wavy vertical lines represent two-particle potentials.

We start our discussion of the integral represented by Fig. 2 by seeking a region of the k plane in which the integrand has no singularities, i.e., a region in which the denominators of the Green's functions, the potentials, the bound-state wave functions, the σ_i and the d^{-1} do not vanish. In such a region the integral is an analytic function of k and is uniformly convergent, so $f(k)$ is an analytic function of k.

We first recall that for a superposition of Yukawa potentials the wave function of a bound state of angular momentum l and binding energy B has the integral representation[6,7]

$$\psi_{lB}(\mathbf{p})=\frac{p^l Y_{lm}(\hat{p})}{2p^2+B}\int_{\mu^2}^{\infty}\frac{d\mu'^2\rho_{l,B}(\mu'^2)}{2p^2+\mu'^2}. \qquad (3.4)$$

Y_{lm} is the usual spherical harmonic. The factor of p^l in Eq. (3.4) will not create any difficulties about the convergence of integrals; it merely means that the dispersion integral will fall off rapidly for large values of p^2. In fact the results of II imply that the quantity $(2p^2+B)\psi_{l,B}(\mathbf{p})$ is a square-integrable function of $\mathbf{p}$. In Eq. (3.1) we have chosen the integration variables so that the argument of the bound-state wave function is independent of the external momenta. As a result, neither of its denominators can vanish.

We next consider the denominators of the potentials. Those denominators which occur in the middle of the diagram do not depend on the external momenta, and, therefore, never vanish. Those potential denominators which do depend on the external momenta have the general form

$$(\mathbf{q}-k\mathbf{a})^2+\mu^2 \qquad (3.5)$$

or

$$(\mathbf{q}'-\alpha\mathbf{k}_B)^2+\mu^2,$$

where $\mathbf{a}$ is a fixed vector which depends on the $\mathbf{y}$'s and α is a constant. The first term in Eq. (3.5) cannot vanish in the strip

$$|\mathrm{Im}k|<\mu/|\mathbf{a}|, \qquad (3.6)$$

and the second term cannot vanish in the strip

$$|\mathrm{Im}k|<(\sqrt{\tfrac{2}{3}})\mu/\alpha. \qquad (3.7)$$

By making use of Eq. (1.3) we see that none of the potential denominators in Fig. 2 can vanish in the strip

$$|\mathrm{Im}k|<(\sqrt{\tfrac{3}{2}})\mu. \qquad (3.8)$$

We now consider the denominator S of Eq. (3.1). It is the product of the denominators of the Green's functions which occur between potentials that act on

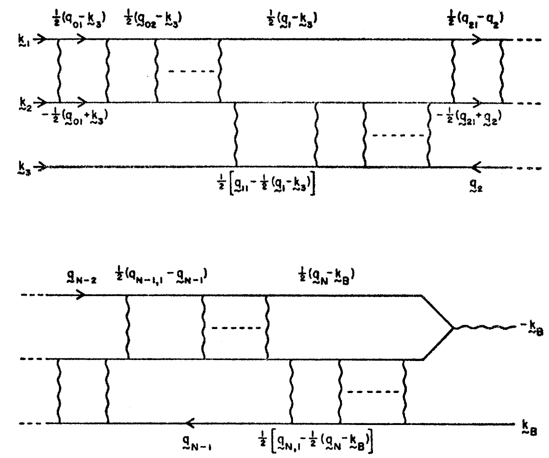

FIG. 2. A typical Nth-order graph for the breakup amplitude in which each two-particle t matrix has been expanded in its perturbation series.

different pairs of particles. We see that these denominators take four distinct forms:

$$\begin{aligned}
&\tfrac{1}{2}\mathbf{q}_1^2-\tfrac{1}{2}k^2\mathbf{y}_{12}^2,\\
&\mathbf{q}_i^2+\mathbf{q}_{i+1}^2+(\mathbf{q}_i-\mathbf{q}_{i+1})^2-k^2,\\
&\mathbf{q}_2^2+\tfrac{1}{4}(\mathbf{q}_1-\mathbf{k}_3)^2+(\mathbf{q}_2-\tfrac{1}{2}\mathbf{q}_1+\tfrac{1}{2}\mathbf{k}_3)^2-k^2, \qquad (3.9)\\
&\mathbf{q}_{N-1}^2+\tfrac{1}{4}(\mathbf{q}_N-\mathbf{k}_B)^2+(\mathbf{q}_{N-1}-\tfrac{1}{2}\mathbf{q}_N+\tfrac{1}{2}\mathbf{k}_B)^2-\mathbf{k}^2.
\end{aligned}$$

The first two denominators in Eq. (3.9) cannot vanish for complex values of k since $\mathbf{q}_i$ and $\mathbf{y}_{12}$ are real vectors. To see that the third term in Eq. (3.9) cannot vanish for complex k we introduce the six-dimensional vectors

$$\mathbf{Q}=\begin{pmatrix}\sqrt{\tfrac{1}{2}}(\mathbf{q}_1-\mathbf{q}_2)\\ (\sqrt{\tfrac{3}{2}})\mathbf{q}_2\end{pmatrix},\quad \mathbf{K}=k\begin{pmatrix}\sqrt{2}\mathbf{y}_3\\ 0\end{pmatrix}, \qquad (3.10)$$

and write

$$\begin{aligned}
&\mathbf{q}_2^2+\tfrac{1}{4}(\mathbf{q}_1-\mathbf{k}_3)^2+(\mathbf{q}_2-\tfrac{1}{2}\mathbf{q}_1+\tfrac{1}{2}\mathbf{k}_3)^2-k^2\\
&=\tfrac{3}{2}\mathbf{q}_2^2+\tfrac{1}{2}(\mathbf{q}_1-\mathbf{q}_2)^2-(\mathbf{q}_1-\mathbf{q}_2)\cdot\mathbf{k}_3-k^2(1-\tfrac{1}{2}y_3^2)\\
&=\mathbf{Q}^2-\mathbf{K}\cdot\mathbf{Q}-k^2(1-\tfrac{1}{2}y_3^2)=Q^2-Qk\sqrt{2}y_3z\\
&\qquad -k^2(1-\tfrac{1}{2}y_3^2), \qquad (3.11)
\end{aligned}$$

where

$$-1\le z=\hat{Q}\cdot\hat{K}\le 1,$$
$$Q=|\mathbf{Q}|.$$

This denominator can vanish for

$$Q/k=y_3z/\sqrt{2}\pm[1-\tfrac{1}{2}y_3^2(1-z^2)]^{1/2}. \qquad (3.12)$$

Equation (3.12) cannot be satisfied for k complex since $y_3^2\le\tfrac{2}{3}$, and Q is real. It should be noted that for $\mathrm{Im}k>0$ the limit $\mathrm{Im}k\to 0$ is equivalent to the usual $\epsilon\to 0$ prescription.

To study the last denominator in Eq. (3.9) we introduce the six-dimensional vectors

$$\mathbf{Q}'=\begin{pmatrix}\sqrt{\tfrac{1}{2}}(\mathbf{q}_N-\mathbf{q}_{N-1})\\ (\sqrt{\tfrac{3}{2}})\mathbf{q}_{N-1}\end{pmatrix},\quad \mathbf{K}'=k_B\begin{pmatrix}\sqrt{2}\hat{n}\\ 0\end{pmatrix}, \qquad (3.13)$$

[6] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).
[7] J. B. Hartle and R. Sugar (to be published).

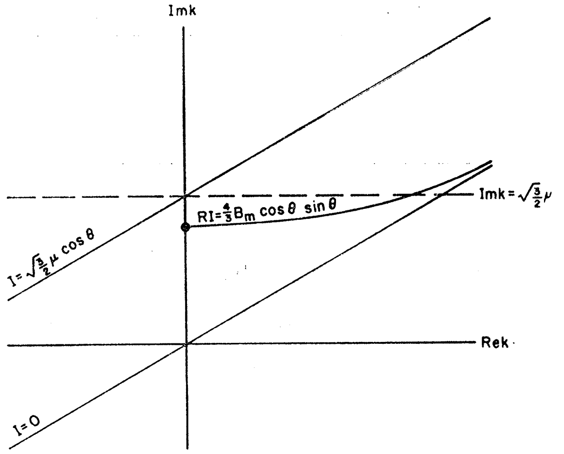

FIG. 3. The curves defined in Eqs. (3.21) and (3.29).

so that

$$\begin{aligned}\mathbf{q}_{N-1}^2+\tfrac{1}{4}(\mathbf{q}_N-\mathbf{k}_B)^2+(\mathbf{q}_{N-1}-\tfrac{1}{2}\mathbf{q}_N+\tfrac{1}{2}\mathbf{k}_B)^2-k^2\\ =\mathbf{Q}'^2-\mathbf{Q}'\cdot K'-k^2+\tfrac{1}{2}k_B{}^2\\ =Q'^2-Q'k_B\sqrt{2}z-k^2+\tfrac{1}{2}k_B{}^2.\end{aligned}\tag{3.14}$$

This denominator vanishes for

$$Q'=(z/\sqrt{3})[k^2+B_3]^{1/2}+[k^2-\tfrac{1}{3}(1-z^2)(k^2+B_3)]^{1/2}.\tag{3.15}$$

Since

$$\begin{aligned}&\operatorname{Im}[k^2+B_3]^{1/2}\le \operatorname{Im}k,\\ &[1-\tfrac{1}{3}(1-z^2)]^{1/2}\operatorname{Im}k\\ &\qquad\le \operatorname{Im}[k^2-\tfrac{1}{3}(1-z^2)(k^2+B_3)]^{1/2},\end{aligned}\tag{3.16}$$

Eq. (3.15) cannot hold for k complex. As a result, none of the denominators of Eq. (3.9) can vanish if $\operatorname{Im}k\neq 0$.

We next consider the denominators of the Green's functions that occur between potentials which act on the *same* pair of particles. From Eq. (2.5) and Fig. 2 we see that these denominators also fall into four distinct classes:

$$\begin{aligned}&\tfrac{3}{2}\mathbf{q}_i{}^2+\tfrac{1}{2}\mathbf{q}_{ij}{}^2-k^2,\\ &\tfrac{1}{2}\mathbf{q}_{0i}{}^2-\tfrac{1}{2}k^2\mathbf{y}_{12}{}^2,\\ &\tfrac{3}{8}(\mathbf{q}_1-\mathbf{k}_3)^2+\tfrac{1}{4}\mathbf{q}_{1i}{}^2-k^2,\\ &\tfrac{3}{8}(\mathbf{q}_N-\mathbf{k}_B)^2+\tfrac{1}{4}\mathbf{q}_{Ni}{}^2-k^2.\end{aligned}\tag{3.17}$$

By using the techniques employed to study the denominators of Eq. (3.9) we see that none of the denominators of Eq. (3.17) can vanish for $\operatorname{Im}k\neq 0$.

Finally, we must examine the denominators of the σ_i and the d^{-1}. From Fig. 1 and Eq. (2.6) we see that the denominators in the dispersion integrals of Eq. (3.3) are of the same form as the denominators of Eq. (3.17). As a result, they can not vanish for $\operatorname{Im}k>0$. The denominators of the bound-state poles of the d^{-1} fall into four classes:

$$\begin{aligned}&B_i+\tfrac{1}{2}k^2y_{12}{}^2,\\ &B_i+k^2-\tfrac{3}{2}\mathbf{q}_j{}^2,\\ &B_i+k^2-\tfrac{3}{8}(\mathbf{q}_1-\mathbf{k}_3)^2,\\ &B_i+k^2-\tfrac{3}{8}(\mathbf{q}_N-\mathbf{k}_B)^2.\end{aligned}\tag{3.18}$$

The first type of denominator in Eq. (3.18) gives rise to simple poles in the breakup amplitude. Of course these poles do not occur in the amplitudes for rearrangement or elastic scattering off a bound state. The second denominator can only vanish on the real k axis or on the imaginary axis for $B_i{}^{1/2}\ge|\operatorname{Im}k|$. However, the last two denominators can vanish for complex values of k. We note that these denominators occur only in the first and last closed loops of Fig. 1.

Let us consider the Fredholm denominator $d^{-1}[k^2-\tfrac{3}{8}(\mathbf{q}_1-\mathbf{k}_3)^2]$ more carefully. When we substitute the integral representation [Eq. (3.3)] for $d^{-1}[k^2-\tfrac{3}{8}(\mathbf{q}_1-\mathbf{k}_3)^2]$ into Eq. (3.1) we obtain a sum of terms. We have seen that the terms containing the dispersion integral and the constant do not give rise to any difficulties. However, in general there will be several additional terms each containing a bound-state pole with a different binding energy. In each such term we make use of the Feynman identity to combine the bound-state pole denominator with the Green's-function denominator $[\tfrac{1}{2}q_1{}^2-\tfrac{1}{2}k_{12}{}^2]^{-1}$. We have

$$\begin{aligned}&[q_1{}^2-k_{12}{}^2]^{-1}[(\mathbf{q}_1-\mathbf{k}_3)^2-(8/3)(k^2+B_i)]^{-1}\\ &\quad=\int_0^1 dx[(\mathbf{q}_1-x\mathbf{k}_3)^2-k^2[2+\tfrac{2}{3}x-y_3{}^2(1-x)\\ &\qquad\qquad\times(3+x)]-(8/3)xB_i]^{-2}.\end{aligned}\tag{3.19}$$

We now make the change of variables $\mathbf{q}_1\to\mathbf{q}_1+x\mathbf{k}_3$. Then the Feynmanized denominator of Eq. (3.19) can only vanish on the real k axis or on the imaginary k axis for $(\tfrac{4}{3}B_i)^{1/2}\ge|\operatorname{Im}k|$. From Eqs. (3.5), (3.9), and (3.17) we see that this change of variables does not alter our conclusions about the vanishing of the other denominators.

Those terms which contain bound-state poles of the form $B_i+k^2-\tfrac{3}{8}(\mathbf{q}_N-\mathbf{k}_B)^2$ can be treated in a similar manner. Since we are making use of the integral representation of Eq. (3.4) for the bound-state wave function, it is simplest to combine the bound-state pole denominators. We make use of the Feynman identity in the form

$$\begin{aligned}&[q_N{}^2+2B_3]^{-1}[q_N{}^2+2\mu^2]^{-1}[(\mathbf{q}_N-\mathbf{k}_B)^2-(8/3)(k^2+B_i)]^{-1}\\ &\quad=2\int dx_1dx_2dx_3\delta(1-x_1-x_2-x_3)[(\mathbf{q}_N-x_3\mathbf{k}_B)^2\\ &\qquad-k^2(2x_3+\tfrac{2}{3}x_3{}^2)-(8/3)x_3B_i\\ &\qquad+B_3[2x_1+\tfrac{2}{3}x_3(1-x_3)]+2x_2\mu^2]^{-3}.\end{aligned}\tag{3.20}$$

After we make the change of variables $\mathbf{q}_N\to\mathbf{q}_N+x_3\mathbf{k}_B$, the Feynmanized denominator can only vanish on the real k axis or on the imaginary k axis for $(\frac{4}{3}B_i)^{1/2} \geq |\mathrm{Im}k|$. We again note that the change of variables does not alter our conclusions about the vanishing of the other denominators. However, it does introduce the external momenta into the spherical harmonics in the wave functions. [see Eq. (3.4)]. Now the quantity

$$|\mathbf{q}_N + x_3\mathbf{k}_B|^l Y_{lm}\left(\frac{\mathbf{q}_N + x_3\mathbf{k}_B}{|\mathbf{q}_N + x_3\mathbf{k}_B|}\right)$$

is merely a polynomial in the components of the vector $\mathbf{q}+x_3\mathbf{k}_B$, so it can only give rise to kinematic singularities at the points $k=i(B_3)^{1/2}$ and $k=0$.[7] These singularities can be eliminated in the usual way by writing the full amplitude in terms of the invariant amplitudes.[7] We shall ignore them in the remainder of this paper.

We have now seen that in the strip

$$(\sqrt{\tfrac{3}{2}})\mu > \mathrm{Im}k > 0 \tag{3.21}$$

the denominators in our problem can only vanish along the imaginary k axis. The first type of denominator in Eq. (3.18) gives rise to simple poles in the breakup amplitude, and the other denominators in Eq. (3.18) give rise to cuts which are restricted to that part of the imaginary axis for which

$$(\tfrac{4}{3}B_m)^{1/2} \geq \mathrm{Im}k \geq 0. \tag{3.22}$$

Except for these singularities $f(k)$ is an analytic function of k in the strip given by Eq. (3.21). In I and II we assumed that

$$2B_m \leq \mu^2. \tag{3.23}$$

We shall retain this assumption throughout this paper. As a result, it will always be possible to continue $f(k)$ from the first to the second quadrant of the k plane (for $N \geq 4$).

We note that in its domain of analyticity $f(k)$ is a real analytic function in the sense that

$$f(k) = f^*(-k^*). \tag{3.24}$$

We can show that $f(k)$ is analytic in the entire first quadrant of the k plane by making use of the rotation of contours argument introduced in I and II. We again consider the integral represented by Fig. 2. We start by multiplying each of the $\mathbf{q}$ variables by $e^{i\theta}$ and seeking a region of the k plane in which the integrand of this new integral is free of singularities.

The first denominator in Eq. (3.5) becomes $(qe^{i\theta}-k\mathbf{a})^2 + \mu^2$. It cannot vanish in the strip

$$|I| < (\mu \cos\theta)/|\mathbf{a}|. \tag{3.25}$$

We have written $ke^{-i\theta} = R+iI$. The second denominator in Eq. (3.5) becomes $(\mathbf{q}'e^{i\theta} - \alpha\mathbf{k}_B)^2 + \mu^2$. If we restrict ourselves to that part of the first quadrant of the k plane which lies above the curve $RI = B_3 \sin\theta \cos\theta$, then

$$\mathrm{Im}(k_B e^{-i\theta}) \leq I. \tag{3.26}$$

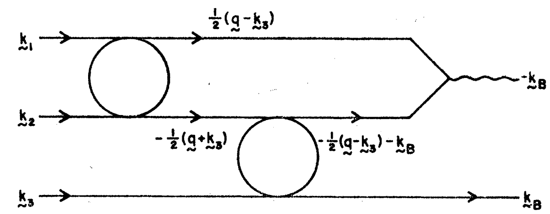

FIG. 4. A typical second-order diagram for the breakup amplitude.

Thus neither of the denominators of Eq. (3.5) can vanish in the region bounded by the curves

$$\begin{aligned} I &= (\sqrt{\tfrac{3}{2}})\mu \cos\theta, \\ \mathrm{Re}k &= 0, \\ RI &= B_3 \cos\theta \sin\theta. \end{aligned} \tag{3.27}$$

The first three denominators of Eqs. (3.9) and (3.17) can only vanish along the line $I=0$. By the same reasoning which led to Eq. (3.16) we see that the fourth denominator of Eqs. (3.9) and (3.17) can not vanish in the first quadrant of the k plane provided we again restrict ourselves to the region above the hyperbola $RI = B_3 \sin\theta \cos\theta$.

The denominators of the dispersion integrals for the σ_i and the d^{-1} are of the form

$$\begin{aligned} &E' + \tfrac{3}{2}\mathbf{q}_i^2 e^{2i\theta} - k^2, \\ &E' - \tfrac{1}{2}k^2 \mathbf{y}_{12}^2, \\ &E' + \tfrac{3}{8}(\mathbf{q}_1 e^{i\theta} - \mathbf{k}_3)^2 - k^2, \\ &E' = \tfrac{3}{8}(\mathbf{q}_N e^{i\theta} - \mathbf{k}_B)^2 - k^2. \end{aligned} \tag{3.28}$$

None of these denominators can vanish in the first quadrant of the k plane above the curve $RI = B_3 \sin\theta \times \cos\theta$. In addition, in this region of the k plane the arguments of $d^{-1}(k^2 - \frac{3}{2}\mathbf{q}_1^2 e^{2i\theta})$, $d^{-1}[k^2 - \frac{3}{8}(\mathbf{q}_1 e^{i\theta} - \mathbf{k}_2)^2]$, and $d^{-1}[k^2 - \frac{3}{8}(\mathbf{q}_N e^{i\theta} - \mathbf{k}_B)^2]$ remain on the physical sheet so we never encounter the resonance poles.

Finally we consider the bound-state pole denominators of Eq. (3.18). The second denominator can only vanish for k along the hyperbola $RI = B_i \sin\theta \cos\theta$. In order to study the last two denominators of Eq. (3.18) we make use of the Feynman identities of Eqs. (3.19) and (3.20). We see that none of the Feynmanized denominators can vanish above the hyperbola $RI = \frac{4}{3}B_m \times \sin\theta \cos\theta$.

Our net result is that if all of the q variables are multiplied by a phase $e^{i\theta}$, $\pi/2 \geq \theta \geq 0$, none of the resulting denominators will vanish in the region of the k plane bounded by the curves

$$\begin{aligned} I &= (\sqrt{\tfrac{3}{2}})\mu \cos\theta, \\ \mathrm{Re}k &= 0, \\ RI &= \tfrac{4}{3}B_m \cos\theta \sin\theta, \end{aligned} \tag{3.29}$$

where $ke^{-i\theta} = R+iI$. From Fig. 3 we see that there is always an overlap between this region and the strip $\sqrt{\frac{3}{2}}\mu \geq \mathrm{Im}k \geq 0$. This means that if we start with the original integral ($\theta=0$) and fix k in the region of overlap,

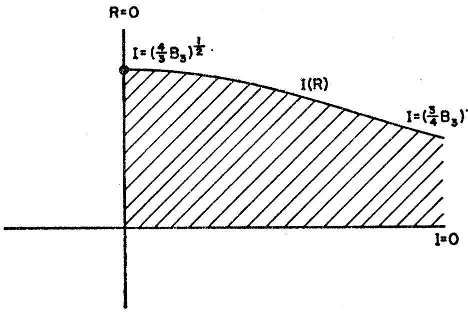

FIG. 5. The curve $I(R)$ defined in Eq. (4.5).

we can simultaneously rotate all contours of integration through an angle θ, $\pi/2>\theta>0$, without crossing a singularity of the integrand. The integral on the rotated contour will therefore be the analytic continuation of the original integral and, by our previous argument, it will define an analytic function in the region given by Eq. (3.29). Since this region sweeps out the entire first quadrant of the k plane as θ varies from 0 to $\pi/2$, it follows that $f(k)$ is analytic in the entire first quadrant. From Eq. (3.24) it follows that $f(k)$ is also an analytic function in the entire second quadrant. The only possible singularities of $f(k)$ are cuts lying along the imaginary axis above the point $k=i(\sqrt{\frac{3}{2}})\mu$ or below the point $k=i(\frac{4}{3}B_m)^{1/2}$ and poles at the points $k=i(2B_i)^{1/2}/y_{12}$. Therefore, for any term of order $N\geq 4$ in the perturbation series for the breakup amplitude, we can write a dispersion relation in the form

$$g(k^2)=f(k)=\int_{-\infty}^{-3\mu^2/2}\frac{\operatorname{Im}g(k'^2)dk'^2}{k'^2-k^2}+\int_{-4B_m/3}^{\infty}\frac{\operatorname{Im}g(k'^2)dk'^2}{k'^2-k^2}+\text{pole terms.} \tag{3.30}$$

So far we have restricted ourselves to the breakup amplitude. The only difference between this amplitude and the ones for elastic scattering and rearrangement is that the latter will have bound-state wave functions on both ends of the diagram.[8] Clearly our proof of analyticity will go through unchanged for these amplitudes, the only difference being that there will be no pole terms.

IV. PERTURBATION THEORY: $N\leq 3$

We now consider the low-order diagrams, which present a slightly more difficult problem. Let us start with the second-order diagram for the breakup amplitude shown in Fig. 4. We have

$$f(k)=\int d^3q\, t_{12}[\tfrac{1}{2}\mathbf{k}_{12},\tfrac{1}{2}\mathbf{q};\tfrac{1}{2}k_{12}^2]\times t_{23}[\tfrac{3}{4}\mathbf{k}_3+\tfrac{1}{4}\mathbf{q},\mathbf{k}_B-\tfrac{1}{4}\mathbf{q};k^2-\tfrac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2]\times\psi_{B_3}(\tfrac{1}{2}\mathbf{q}+\tfrac{1}{2}\mathbf{k}_B-\tfrac{1}{2}\mathbf{k}_3)[\tfrac{1}{2}\mathbf{q}^2-\tfrac{1}{2}\mathbf{k}_{12}^2]^{-1}. \tag{4.1}$$

If we now use the last line of Eq. (3.2) for t_{12} and t_{23}, we obtain a sum of integrals with the same denominators that were studied in Sec. III. The only difference is that the external momenta now appear in the argument of ψ_{B_3}, so we must find a region of the k plane in which the denominators of ψ_{B_3} do not vanish.

The wave-function pole now takes the form [see Eq. (3.4)]

$$\tfrac{1}{2}(\mathbf{q}-\mathbf{k}_3+\mathbf{k}_B)^2+B_3. \tag{4.2}$$

Clearly this denominator cannot vanish if

$$\operatorname{Im}k_3+\operatorname{Im}k_B<(2B_3)^{1/2}. \tag{4.3}$$

Writing $k=R+iI$ we see that

$$\operatorname{Im}k_B=\tfrac{1}{3}[[(R^2-I^2+B_3)^2+4R^2I^2]^{1/2}-(R^2-I^2+B_3)]^{1/2} \tag{4.4}$$

is an increasing function of I. Since $y_3\leq\sqrt{\frac{2}{3}}$, the denominator of Eq. (4.2) can not vanish in the region of the k plane bounded by the real axis, the imaginary axis, and the curve (see Fig. 5)

$$I+\tfrac{1}{2}[[(R^2-I^2+B_3)^2+4R^2I^2]^{1/2}-(R^2-I^2+B_3)]^{1/2}=(3B_3)^{1/2}. \tag{4.5}$$

We denote the curve defined by Eq. (4.5) by $I(R)$. We note that $I(R)$ is a decreasing function of R and that

$$I(0)=(\tfrac{4}{3}B_3)^{1/2},\quad I(\infty)=(\tfrac{3}{4}B_3)^{1/2}. \tag{4.6}$$

The denominator of the dispersion integral for the wave function [see Eq. (3.4)] can not vanish in the region defined by Eq. (4.5) since $\mu^2>B_3$. In fact the dispersion denominator can never vanish in a region in which the pole denominator does not vanish, so we can ignore this denominator in the rest of our work.

The remaining denominators in the integral for $f(k)$ were discussed in Sec. III. It follows from the results of that section that $f(k)$ is analytic in the first quadrant of the k plane in the region bounded by the real axis, the imaginary axis, and the curve $I(R)$.

In order to show that $f(k)$ is analytic in the entire first quadrant of the k plane we shall use the rotation of contours argument employed in Sec. III.

As in Sec. III we shall make use of the integral representation of Eq. (3.3) for $d_{23}^{-1}[k^2-\frac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2]$. Let us first consider the terms in $f(k)$ which contain the constant and the dispersion integral of d_{23}^{-1}. After rotating the contours of integration through an angle θ, $\frac{1}{2}\pi\geq\theta\geq 0$, the denominator of the wave-function pole becomes

$$\tfrac{1}{2}(\mathbf{q}e^{i\theta}-\mathbf{k}_3+\mathbf{k}_B)^2+B_3. \tag{4.7}$$

[8] We define the Nth-order amplitudes for rearrangement and elastic scattering to have one less power of the kernel than the Nth-order breakup amplitude, so that the total number of Green's functions plus wave functions will be the same for all three Nth-order amplitudes.

It can not vanish if

$$\mathrm{Im}(k_3e^{-i\theta})+\mathrm{Im}(k_Be^{-i\theta})<(2B_3)^{1/2}\cos\theta. \tag{4.8}$$

Writing $ke^{-i\theta}=R+iI$ we see that

$$\mathrm{Im}k_B=\tfrac{1}{3}[[(R^2-I^2+B_3\cos2\theta)^2+(2RI-B_3\sin2\theta)^2]^{1/2} -(R^2-I^2+B_3\cos2\theta)]^{1/2} \tag{4.9}$$

is an increasing function of I in that part of the first quadrant of the k plane lying above the hyperbola $RI=B_3\sin\theta\cos\theta$. As a result, the denominator of Eq. (4.7) can not vanish in the first quadrant in a region bounded by the imaginary k axis, the hyperbola $RI=B_3\sin\theta\cos\theta$, and the curve $I_\theta(R)$ defined by [see Fig. 6(a)]

$$I+\tfrac{1}{2}[[(R^2-I^2+B_3\cos2\theta)^2+(2RI-B_3\sin2\theta)^2]^{1/2} -(R^2-I^2+B_3\cos2\theta)]^{1/2} =(3B_3)^{1/2}\cos\theta. \tag{4.10}$$

We note that $I_\theta(R)$ is a decreasing function of R and that

$$I_\theta((\tfrac{4}{3}B_3)^{1/2}\sin\theta)=(\tfrac{4}{3}B_3)^{1/2}\cos\theta,$$
$$I_\theta(\infty)=(\tfrac{3}{4}B_3)^{1/2}\cos\theta. \tag{4.11}$$

From the results of Sec. III we see that none of the other denominators can vanish in the region bounded by the imaginary k axis, the curve $I_\theta(R)$, and the hyperbola $RI=B_3\sin\theta\cos\theta$. As we vary θ from 0 to $\frac{1}{2}\pi$, this region sweeps out the entire first quadrant of the k plane, so it follows that those terms in $f(k)$ which contain the constant or the dispersion integral of d_{23}^{-1} are analytic in the entire first quadrant.

For those terms in $f(k)$ which do contain the bound-state poles of d_{23}^{-1} we make use of the Feynman identity to combine the pole denominator with the Green's-function denominator $(q^2-k_{12}^2)$, and all of the Green's-function denominators of t_{23}. Our choice of variables is shown in Fig. 7. The combined denominator is

$$S=\sum_{i=1}^{N}x_i[(\mathbf{q}-\mathbf{k}_3)^2+\tfrac{4}{3}\mathbf{q}_{0i}^2-(8/3)k^2] +x_{N+1}[(\mathbf{q}-\mathbf{k}_3)^2-(8/3)(k^2+B_i)] +x_{N+2}[\mathbf{q}^2-\mathbf{k}_{12}^2]. \tag{4.12}$$

After making the change of variables

$$\mathbf{q}\rightarrow\mathbf{q}+(1-x_{N+2})\mathbf{k}_3, \tag{4.13}$$

we have

$$S=q^2+\tfrac{4}{3}\sum_{i=1}^{N}x_iq_{0i}^2-ak^2-bB_i, \tag{4.14}$$

where

$$a=(8/3)(1-x_{N+2})-y_3^2x_{N+2}(1-x_{N+2})+x_{N+2}y_{12}^2,$$
$$b=(8/3)x_{N+1}. \tag{4.15}$$

We note that $a\geq0$ and $b/a\leq\frac{4}{3}$. Under this change of variables the wave-function pole denominator becomes

$$\tfrac{1}{2}(\mathbf{q}-x_{N+2}\mathbf{k}_3+\mathbf{k}_B)^2+B_3, \tag{4.16}$$

so it is still true that none of the denominators can vanish in the region bounded by $I(R)$, the real k axis, and the imaginary k axis.

If we now rotate the contours of integration through an angle θ, the denominator S can only vanish on the hyperbola $RI=(b/a)B_i\sin\theta\cos\theta$. From our previous discussion it is clear that if $(b/a)B_i\leq\frac{4}{3}B_3$, none of the denominators in the problem can vanish in the region

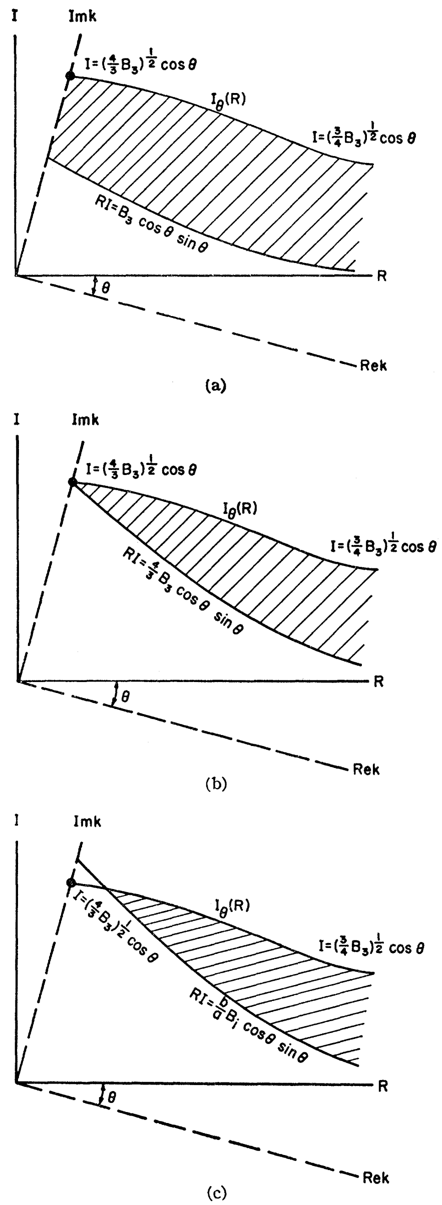

FIG. 6. (a) The region bounded by the curves $\mathrm{Im}k=0$, $RI=B_3\cos\theta\sin\theta$, and $I_\theta(R)$. (b) The region bounded by the curves $RI=\frac{4}{3}B_3\cos\theta\sin\theta$ and $I_\theta(R)$. (c) The region bounded by the curves $RI=(b/a)B_i\cos\theta\sin\theta$ and $I_\theta(R)$ when $(b/a)B_i>\frac{4}{3}B_3$.

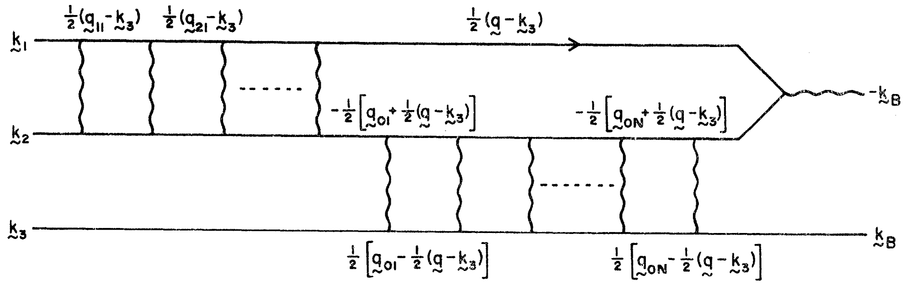

FIG. 7. A typical second-order diagram for the breakup amplitude in which each two-particle t matrix has been expanded in its perturbation series.

bounded by $I_\theta(R)$ and the hyperbola $RI=\frac{4}{3}B_3\sin\theta\cos\theta$. From Fig. 6(b) we see that this region sweeps out the entire first quadrant of the k plane as θ varies from 0 to $\pi/2$.

However, B_i can be greater than B_3, so in general there will be a range of x_i's such that $(b/a)B_i>\frac{4}{3}B_3$. For such values of the x_i's the region in which none of the denominators can vanish is bounded by $I_\theta(R)$ and $RI=(b/a)B_i\sin\theta\cos\theta$. From Fig. 6(c) we see that the region between these curves does not sweep out the entire first quadrant of the k plane. As a result, it is convenient to divide the x integration into two terms: one in which $(b/a)B_i\leq\frac{4}{3}B_3$ and one in which $(b/a)B_i\geq\frac{4}{3}B_3$. The first term is clearly analytic in the first quadrant of the k plane, so we need only study the second term.

For those values of the x_i for which $(b/a)B_i\geq\frac{4}{3}B_3$ we would like to make a charge of variables which would prevent the denominator S from vanishing above the hyperbola $RI=\frac{4}{3}B_3\sin\theta\cos\theta$. To this end we make use of the identity[9]

$$J=\int d^3q/\prod_i[(\mathbf{q}-\mathbf{A}_i)^2+B_i][q^2-c]$$

$$=\int d^3q/[1+\alpha^2]\prod_i[(\mathbf{q}-\mathbf{A}_i)^2+\mathbf{q}^2\alpha^2+B_i]\times[q^2(1+\alpha^2)-c]. \quad (4.17)$$

Here, α is an arbitrary real constant, the $\mathbf{A}_i$ are fixed vectors, and B_i and C are constants. Before rotating contours and before allowing k to become complex we apply this identity to the $\mathbf{q}$ integration and to each of the $\mathbf{q}_{0i}$ integrations. In each case we choose

$$\alpha=\left[\frac{(b/a)B_i-\frac{4}{3}B_3}{k^2+\frac{4}{3}B_3}\right]^{1/2}. \quad (4.18)$$

[9] This identity follows trivially from the Feynman identity

$$J=\int\frac{d^3q\,\Pi_i\,dx_i\delta(1-\sum_i x_i)}{[\mathbf{q}^2-2\mathbf{q}\cdot(\sum_i x_i\mathbf{A}_i)+\sum_i x_i(\mathbf{A}_i^2+B_i)]^N[\mathbf{q}^2-c]}.$$

We now make the change of variables $\mathbf{q}\to\mathbf{q}+\alpha\mathbf{q}\times\hat{n}/\sin\theta$, where $\hat{n}$ is a unit vector in the direction of $\sum_i x_i\mathbf{A}_i$, θ is the angle between $\mathbf{q}$ and $\hat{n}$, and α is an arbitrary real constant. The Jacobian of this transformation is $[1+\alpha^2]^{-1}$, so

$$J=\int\frac{d^3q\,\Pi_i\,dx_i\delta(1-\sum_i x_i)}{[1+\alpha^2][\mathbf{q}^2(1+\alpha^2)-2\mathbf{q}\cdot(\sum_i x_i\mathbf{A}_i)+\sum_i x_i(\mathbf{A}_i^2+B_i)]^N[\mathbf{q}^2(1+\alpha^2)-c]}$$

$$=\int\frac{d^3q}{[1+\alpha^2]\Pi_i[(\mathbf{q}-\mathbf{A}_i)^2+\mathbf{q}^2\alpha^2+B_i][\mathbf{q}^2(1+\alpha^2)-c]}.$$

Since we are only considering values of the x_i for which $(b/a)B_3\geq\frac{4}{3}B_3$, α is real.

After making this transformation the denominator S becomes

$$S=[q^2+\tfrac{4}{3}\sum_i x_iq_{0i}^2]\left[\frac{k^2+(b/a)B_i}{k^2+\frac{4}{3}B_3}\right]-ak^2-bB_i$$

$$=[q^2+\tfrac{4}{3}\sum_i x_iq_{0i}^2-a(k^2+\tfrac{4}{3}B^3)]\left[\frac{k^2+(b/a)B_i}{k^2+\frac{4}{3}B_3}\right]. \quad (4.19)$$

If we now rotate contours of integration through an angle θ, S can only vanish on the hyperbola $RI=\frac{4}{3}B_3\times\sin\theta\cos\theta$. Clearly the term $[k^2+(b/a)B_i]/(k^2+\frac{4}{3}B_3)$ gives no trouble for k^2 complex.

With the present choice of variables the denominator of the wave-function pole becomes

$$\tfrac{1}{2}(\mathbf{q}-x_{N+2}\mathbf{k}_3+\mathbf{k}_B)^2+\tfrac{1}{2}q^2\frac{(b/a)B_i-\frac{4}{3}B_3}{k^2+\frac{4}{3}B_3}+B_3. \quad (4.20)$$

In the Appendix we show that this denominator and the potential denominators which depend on $\mathbf{q}$ or $\mathbf{q}_{0i}$ can not vanish in the region bounded by $I(R)$, the real k axis, and the imaginary k axis. After rotating contours of integration these denominators can not vanish in the region bounded by $I_\theta(R)$ and $RI=\frac{4}{3}B_3\sin\theta\cos\theta$.

As a result, the second-order term for the breakup amplitude is analytic in the first quadrant of the k plane. It is also analytic in the second quadrant since we define it there by

$$f(k)=f^*(-k^*). \quad (4.21)$$

The third-order term for the breakup amplitude can be treated in exactly the same way and we find that it is also analytic in both the first and second quadrants.

Except for terms of the form $\langle B_2'\hat{n}|V_2|B_3\hat{n}\rangle$, the second- and third-order terms for the bound-state scattering and rearrangement amplitudes can be treated in the same way as the corresponding terms in the breakup amplitude.[8] They are also analytic in both the first and second quadrant of the k plane.

We next consider a typical first-order diagram for the breakup amplitude which is shown in Fig. 8(a).

The only new singularities arise from the wave function. The denominator of the wave-function pole is

$$2(\mathbf{k}_1+\tfrac{1}{2}\mathbf{k}_B)^2+B_3. \quad (4.22)$$

It vanishes at

$$k^2=-\tfrac{4}{3}B_3[(\tfrac{1}{3}+2y_1^2-y_1^2z^2)+zy_1(2-4y_1^2+y_1^2z^2)^{1/2}]\\ \times[(\tfrac{1}{3}+2y_1^2)^2-(8/3)y_1^2z^2]^{-1}. \quad (4.23)$$

This pole can become complex for $\frac{1}{2}\leq y_1^2\leq\frac{2}{3}$. However, what is important for dynamical calculations based on analyticity and unitarity is that the residue of this pole is just the product of the on-energy-shell two-body t matrix and the coupling constant g_3 defined by

$$g_3=\lim_{2p^2\to -B_3}(2p^2+B_3)\psi_{B_3}(\mathbf{p}). \quad (4.24)$$

The diagram shown in Fig. 8(a) also has a cut arising from the dispersion integral for the wave function. Making use of our requirement that $\mu^2>2B_m$, it is easy to see that this cut always lies on the imaginary k axis.

Finally we turn to the terms in the rearrangement amplitude of the form $\langle B_2'\hat{n}'|V_2|B_3\hat{n}\rangle$. A typical term is shown in Fig. 8(b). There is a simple pole at

$$k^2=-[(20+8z^2)(B_3+B_3')+4z[4(z^2-1)\\ \times(B_3-B_2')^2 9B_3B_2']^{1/2}][25-16z^2]^{-1}. \quad (4.25)$$

The position of the pole becomes complex if $B_3>4B_2'$ or $B_2'>4B_3$, but the residue of the pole is given by the product of two coupling constants.

We have thus seen that except for the simple poles given by Eq. (4.23) and (4.25) all terms in the perturbation series for each of our amplitudes are analytic in the entire first quadrant of the k plane. Since they are defined in the second quadrant by Eq. (4.21) they are analytic there also.

V. POSITION OF SINGULARITIES

We now turn to the problem of locating the various branch points of our amplitudes.

The vanishing of the free Green's-function denominators gives rise to the three-particle unitarity cut whose branch point is at $k=0$. The vanishing of the bound-state pole denominators of the two-particle t matrices gives rise to the unitarity cuts for the scattering of a particle off the various bound states. The branch points are located at $k=iB_i^{1/2}$.

All of the other branch points arise from the simultaneous vanishing of several denominators. We can obtain the position of these singularities by making use of the Landau-Bjorken (LB) method.[10] We know that all of these singularities lie on the imaginary k axis and that we can continue to this axis by rotating the contours of integration of all of the $\mathbf{q}$ variables through an angle of $\frac{1}{2}\pi$. It is important to notice that we do not need to deform the integration path of any of the Feynman parameters. This means that we need only consider real positive values of the Feynman parameters in the LB equations. As a result, after writing $k=iK$, $\mathbf{q}_i=i\mathbf{p}_i$, where K and $\mathbf{p}_i$ are real, we can realize the LB conditions in terms of dual diagrams in real Euclidean $\mathbf{p}$ space. By studying these dual diagrams it is possible to obtain the position of all of the branch points that lie on the imaginary k axis. For the present we shall confine ourselves to the study of those singularities which arise from pinches among free Green's functions, bound-state poles, and wave-function poles.

[10] See, for example, J. D. Bjorken and G. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965).

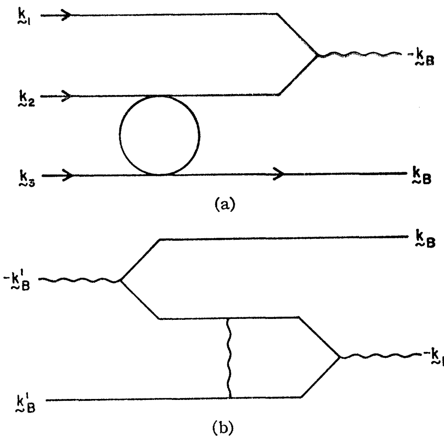

FIG. 8. (a) A first-order diagram for the breakup amplitude. (b) A first-order diagram for the rearrangement amplitude.

Let us start with the Nth-order breakup diagram of Fig. 1. After writing $k=iK$, $\mathbf{q}_i=i\mathbf{p}_i$, we apply the Feynman identity to the Green's functions, the bound-state poles, and the wave-function pole. The combined denominator is

$$S=x_1[\mathbf{p}_1^2+\mathbf{K}_3^2+(\mathbf{p}_1-\mathbf{K}_3)^2-K^2]+x_2[\mathbf{p}_1^2+\mathbf{p}_2^2\\ +(\mathbf{p}_1-\mathbf{p}_2)^2-K^2]+\cdots+x_N[\mathbf{p}_{N-1}^2+\mathbf{p}_N^2\\ +(\mathbf{p}_{N-1}-\mathbf{p}_N)^2-K^2]+x_{n+1}[(\mathbf{p}_N-\tfrac{1}{2}\mathbf{K}_B)^2-2B_3]\\ +x_{N+2}[\mathbf{p}_{l1}^2-\tfrac{2}{3}(K^2-B_{l1})]\\ +x_{N+3}[\mathbf{p}_{l2}^2-\tfrac{2}{3}(K^2-B_{l2})]+\cdots\\ +x_{N+M}[p_{lM}^2-\tfrac{2}{3}(K^2-B_{lm})], \quad (5.1)$$

where

$$N\geq M;\quad l_m>l_{m-1}>\cdots>l1\geq 1,\\ \mathbf{K}_3=\mathbf{y}_3K,\\ \mathbf{K}_B=\hat{n}[\tfrac{2}{3}(K^2-B_3)]^{1/2}.$$

We are using the same variables as in Fig. 1 except that we have let $\frac{1}{2}(\mathbf{p}_1-\mathbf{K}_3)\to\mathbf{p}_1$ and $\frac{1}{2}(\mathbf{p}_N-\mathbf{K}_B)\to\mathbf{p}_N$ in order to simplify the geometry of the dual diagrams. The dual diagram corresponding to the denominator of Eq. (5.1) is shown in Fig. 9. The LB equations require that all vectors lie in a plane. The positivity of the Feynman parameters requires that the polygonal line $CA_1A_2A_3\cdots A_ND$ is convex and lies inside the triangle OCD. The LB equations further require that

$$|A_1C|=|\mathbf{p}_1-\tfrac{1}{2}\mathbf{K}_3|=\tfrac{1}{2}Ky_{12}\\ |OA_i|^2+|A_iA_{i+1}|^2+|OA_{i+1}|^2\\ =\mathbf{p}_i^2+(\mathbf{p}_i-\mathbf{p}_{i+1})^2+\mathbf{p}_{i+1}^2=K^2,\\ i=1,2,\cdots N-1 \quad (5.2)\\ |A_ND|=|\mathbf{p}_N-\tfrac{1}{2}\mathbf{K}_B|^2=\sqrt{\tfrac{1}{2}}B_3.$$

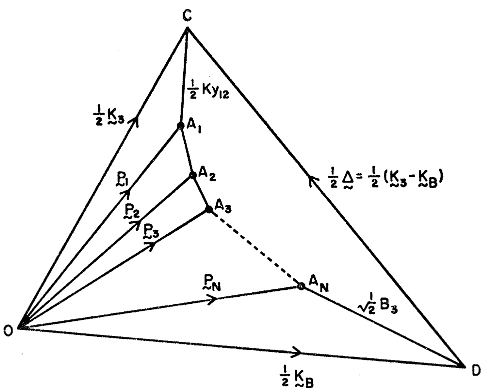

FIG. 9. The dual diagram corresponding to Fig. 1.

For each bound-state pole denominator in the problem we have the added condition

$$|OA_{li}|=\mathbf{p}_{li}^2=\tfrac{2}{3}(K^2-B_{li}). \qquad (5.3)$$

Reduced diagrams corresponding to, for example, $x_i=0$ may also be possible. In such a case the polygonal line breaks in two. The points $A_1, A_2, \cdots, A_i$ must now lie on the line OC and the points $A_{i+1}, \cdots, A_N$ must lie on the line OD.

Given all these geometric constraints it is easy to see that only four types of dual diagrams are possible. They are shown in Fig. 10.

For terms of order $N\geq 3$ the only possible dual diagrams are Figs. 10(a) and 10(b). Figure 10(a) corresponds to a "pinch" between the bound-state pole and the free Green's function in the first closed loop. The pinching condition is

$$|OA_1|+|A_1C|=|OC|, \qquad (5.4)$$

which tells us that the branch point is at

$$K^2=A_{l1}=\frac{B_{l1}}{1-\frac{3}{8}(y_3-y_{12})^2}, \quad y_3^2\geq\tfrac{1}{2}. \qquad (5.5)$$

This type of branch point was discussed in I, where we referred to it as an anomalous threshold of type B. It is not on the physical sheet for $y_3^2<\frac{1}{2}$. At $y_3^2=\frac{1}{2}$ it enters the physical sheet through the normal threshold at $K^2=B_{li}$. As y_3^2 is increased, the branch point moves up the imaginary k axis, reaching the point $K^2=\frac{4}{3}B_{l1}$ when $y_3^2=\frac{2}{3}$.

The dual diagram of Fig. 10(b) corresponds to a pinch between the bound-state pole and the wave-function pole in the last closed loop. The branch point is at

$$K^2=A_{lN}'=\tfrac{4}{3}[B_3+B_{lN}-(B_3B_{lN})^{1/2}], \quad B_{lN}\geq 4B_3. \quad (5.6)$$

We shall call this type of branch point an anomalous threshold of type B'. It is not on the physical sheet for $B_{lN}<4B_3$. If we imagine varying the binding energy B_3, the branch point enters the physical sheet through the normal threshold at $K^2=B_{lN}$ when $B_{lN}=4B_3$. As B_3 decreases, the branch point moves up the imaginary k axis approaching the point $K^2=\frac{4}{3}B_{lN}$ as $B_{lN}/B_3\to\infty$.

If we consider the amplitudes for rearrangement or elastic scattering we find that each term of order $N\geq 3$ can have two anomalous thresholds of type B', since there can be a pinch between a wave-function pole and a bound-state pole at each end of the diagram.

The second-order term in the breakup amplitude will have one anomalous threshold of type B and one of type B'. It will also have singularities corresponding to the dual diagrams of Figs. 10(c) and 10(d). Figure 10(c) corresponds to a pinch between the free Green's function and the wave-function pole. The position of the branch point is given by

$$(2B_3)^{1/2}+K_{12}=[K_3^2+K_B^2-2zK_3K_B]^{1/2}, \qquad (5.7)$$

where z is the cosine of the angle between K_3 and K_B. Let us denote the solution of Eq. (5.7) by $K=C(z)$. For $z=-1$ we find

$$K=C(-1)=B_3^{1/2}\frac{\sqrt{2}\alpha-[(16/4)-\frac{2}{3}\alpha^2]^{1/2}}{\alpha^2-\frac{2}{3}}, \qquad y_3^2\geq\tfrac{1}{6} \qquad (5.8)$$

where $\alpha=y_3-y_{12}$. This branch point is not on the physical sheet for $y_3^2<\frac{1}{6}$. It enters the physical sheet at $K=\infty$ and travels down the imaginary k axis as y_3^2 increases, reaching the point $K^2=\frac{4}{3}B_3$ when $y_3^2=\frac{2}{3}$. It is easy to see that for $C(z)$ on the physical sheet

$$C(z)\geq C(-1), \quad 1\geq z\geq -1. \qquad (5.9)$$

Clearly the branch point at $K=C(z)$ gives rise to a cut which runs from $K=C(z)$ to $K=\infty$. From Eqs. (5.5), (5.6), and (5.8) we see that unless B_3 is the largest binding energy in the problem, there will always be a range of values of y_3 such that this cut overlaps the unitarity cut. As a result, it will not always be possible to continue the second-order terms from the first to the second quadrant of the k plane. This will not prevent us from making N/D-type calculations. The second-

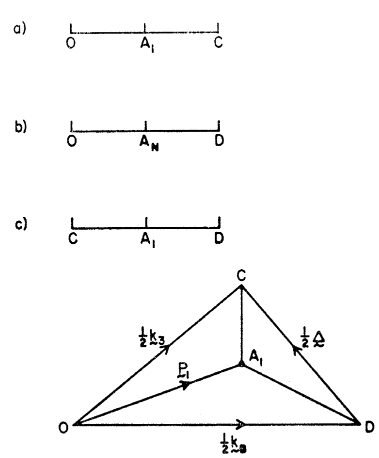

FIG. 10. The four possible dual diagrams satisfying the Landau-Bjorken conditions.

order terms are defined in the second quadrant of the k plane by Eq. (4.21). The discontinuity across the cuts associated with the anomalous thresholds and the cut associated with $C(z)$ can be calculated explicitly in terms of on-energy-shell two-particle quantities. The details of obtaining these discontinuities are given in I.

We recall from Sec. III that all terms of order $N \geq 4$ do have a gap between the "left- and right-hand cuts," so we can continue them from the first to the second quadrant. Presumably the same result holds for the third-order terms since they have no singularities of the type $C(z)$.

We next consider the dual diagram of Fig. 10(d). It corresponds to a pinch between the free Green's function, the wave-function pole, and a bound-state pole. The algebraic expression for the position of the branch point can be read off from the dual diagram, and we will not write it out here. We merely note that if we denote the position of the branch point by $K=D(z)$, then

$$C(z) \geq D(z) \geq A_1^{1/2}, \quad A_1'^{1/2}, \qquad (5.10)$$

where $A_1^{1/2}$ and $A_1'^{1/2}$ are the positions of the anomalous thresholds of type B and B'. We see from Fig. 10(d) that as we vary y_3 or the binding energies $D(z)$ can leave the physical sheet through either of the anomalous thresholds or through the branch point at $C(z)$.

Finally we must consider the second-order terms in the amplitudes for rearrangement or elastic scattering. The dual diagram of Fig. 10(c) now corresponds to a pinch between two wave-function poles. If the initial state contains a bound state with binding energy B_3, and the final state one with binding energy B_3', then the condition for the pinch is

$$(2B_3)^{1/2}+(2B_3')^{1/2}=[K_B^2+K_B'^2-2zK_BK_B']^{1/2}, \qquad (5.11)$$

where z is the cosine of the angle between $\mathbf{K}_B$ and $\mathbf{K}_B'$. If we denote the position of the branch point by $K=C'(z)$ we find

$$C'(z) \geq C'(-1)=\tfrac{4}{3}[B_3+B_3'+(B_3B_3')^{1/2}]. \qquad (5.12)$$

Again there will be cases in which the left- and right-hand cuts overlap, but the discontinuities across all of the cuts can still be obtained in terms of on-energy-shell two-body quantities.

The dual diagram of Fig. 10(d) now corresponds to a pinch between two wave-function poles and a bound-state pole. The branch point again lies between $C'(z)$ and the anomalous thresholds, and it can leave the physical sheet through any of these branch points.

VI. CONVERGENCE OF THE FREDHOLM SERIES

We have seen that all terms in the perturbation series for our amplitudes are analytic in the upper-half-k plane. The lowest-order terms can have complex poles, but all other singularities lie on the imaginary k axis. It follows from our discussion in Sec. II that the individual terms in the Fredholm series have the same analyticity properties. We shall now show that the Fredholm series are uniformly convergent with respect to k in the upper-half plane. As a result, the full amplitudes can only have the singularities of the individual terms in the perturbation series and poles at the zeroes of D.

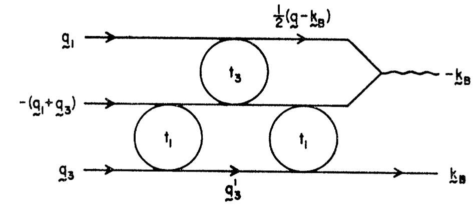

FIG. 11. The diagram corresponding to the function $g(\mathbf{q}_1,\mathbf{q}_3)$ of Eq. (4.1).

In II we gave a detailed proof of the convergence of the Fredholm series for the amplitude for the scattering of three free particles. The proof for the amplitudes which we have been studying is identical, so we will only outline it briefly. The reader is referred to II for the technical details.

Our starting point is the Faddeev equation for the off-energy-shell three-particle T matrix, Eq. (2.14). In II we showed that the kernel K^2 is square-integrable (L^2) in the entire upper-half k plane. As a result, the resolvent exists and is given by a relatively, uniformly convergent series.[4] The on-energy-shell amplitude for the scattering of three free particles is obtained by taking the matrix element of the resolvent between the vectors $(K^2+K^3)\hat{t}|\mathbf{y}\rangle$ and $K^{+4}|\mathbf{y}'\rangle$. In II we showed that these vectors are L^2, so the series obtained by taking the matrix element of the resolvent between them is uniformly convergent with respect to k.

From Eqs. (2.11), (2.13), and (2.18) we see that the Fredholm series for the amplitudes for breakup, rearrangement, and elastic scattering are obtained by taking the matrix elements of the resolvent between vectors of the form $K^{+4}|\mathbf{y}\rangle$, $(K^2+K^3)\hat{t}|B_3\hat{n}\rangle$, and $K^{+4}|B_3\hat{n}\rangle$. In order to complete our proof of the convergence of the Fredholm series, it is only necessary to show that the last two vectors are L^2.

It will suffice to consider a typical term

$$g(\mathbf{q}_1,\mathbf{q}_3)=\langle \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|t_1G_0t_3G_0t_1|B_3\hat{n}\rangle. \qquad (6.1)$$

Our choice of variables is shown in Fig. 11. We wish to show that g is an L^2 function of $\mathbf{q}_1$ and $\mathbf{q}_3$. In II we saw that the off-energy-shell two-particle t matrices are L^2 functions of either of their momenta, the norm being bounded by a constant. Since g depends on $\mathbf{q}_3$ only through the first t_1 we can do the $\mathbf{q}_3$ integration at once by making use of the Schwartz inequality. If we exclude values of k on the real axis, the Green's functions which depend on $\mathbf{q}_1$ can be bounded by a constant. Then the only $\mathbf{q}_1$ dependence will be in t_3 so we can do the $\mathbf{q}_1$ integration. The norm of t_3 can be bounded by a constant for $6\mu > \mathrm{Im}k$ (see II). At this point the $\mathbf{q}_3'$ dependence is in the remaining Green's function and the remaining t_1. Both are L^2 functions of $\mathbf{q}_3'$for complex values of k, so the $\mathbf{q}_3'$integration can be done by making use of the Schwartz inequality. The norm of t_1 is bounded by a constant for $(\frac{3}{2})^{1/2}\mu > \mathrm{Im}k$. We are now left with the bound

$$\int d^3q_1\, d^3q_3 |g(\mathbf{q}_1,\mathbf{q}_3)|^2 \leq C\left[\int d^3q \psi_{B_3}(\tfrac{1}{2}\mathbf{q})\right]^2. \quad (6.2)$$

We can write the wave function in the form

$$\psi_{B_3}(\tfrac{1}{2}\mathbf{q}) = \frac{f(\frac{1}{2}\mathbf{q})}{\frac{1}{2}q^2 + B_3}. \quad (6.3)$$

The results of II imply that f is an L^2 function of $\mathbf{q}$, so a final application of the Schwartz inequality gives

$$\int d^3q_1 d^3q_3 |g(\mathbf{q}_1,\mathbf{q}_3)|^2 \leq C' \quad (6.4)$$

for

$$\mathrm{Re}k \geq \epsilon > 0, \quad (\sqrt{\tfrac{3}{2}})\mu - \epsilon \geq \mathrm{Im}k \geq \epsilon > 0. \quad (6.5)$$

We have ignored the possibility of the last t matrix having bound-state poles. If it does, we make use of the Feynman identity to combine each of them with the wave-function pole. The proof then goes through unchanged.

The proof for the other vectors goes through in a similar fashion. It then follows that the Fredholm series are uniformly convergent with respect to k in the strip defined by Eq. (6.5). In order to show that they converge in the entire upper-half k plane we use the rotation of contours argument introduced in II. There is no added difficulty in showing that g is L^2 on the rotated contours.

Our final result is that the Fredholm series for all of the on-energy-shell amplitudes are uniformly convergent with respect to k in the entire upper half-plane including the real and imaginary axis. The only possible except is the imaginary axis above the point $k = i(\sqrt{\frac{3}{2}})\mu$, where our proof breaks down. This is the location of the "left-hand cut."

VII. SUMMARY OF RESULTS

We have completed our study of the scattering of three nonrelativistic particles interacting via two-body Yukawa potentials. We now summarize our results.

(1) We have proven the existence of the on-energy-shell amplitudes describing the scattering of three free particles, the scattering of one particle off a bound state of two particles, the breakup of a bound state, and rearrangement collisions among three particles.

(2) Each of these amplitudes can be written as the ratio of two Fredholm series which are uniformly convergent with respect to the total center-of-mass energy E for all values of E on the physical sheet.

(3) Each of the amplitudes satisfies of dispersion relation in E for fixed physical values of the vectors $\mathbf{y}_i$ and $\hat{n}$. Except for the simple poles discussed in Sec. IV all of the singularities lie on the real energy axis.

(4) The amplitudes have been defined so that they are real analytic functions of E, i.e., $T(E) = T^*(E^*)$.

(5) The analytic properties proved for the amplitudes with the vectors $\mathbf{y}_i$ and $\hat{n}$ fixed also hold for the partial-wave amplitude. As mentioned in I, in order to project out a partial-wave amplitude one merely multiplies the amplitude by a polynomial in the components of the vectors $\mathbf{y}_i$ and $\hat{n}$ and integrates over a compact domain. Such a procedure can introduce no new singularities in E.

(6) We have encountered three types of singularities in our discussion:

(i) *Unitarity cuts.* These are associated with the various types of intermediate assymptotic states. There will be one cut running from $E=0$ to $+\infty$ corresponding to intermediate states of three free particles. There will also be a cut from $E = -B_{ij}$ to $+\infty$ for each bound state of the (i,j) pair with binding energy B_{ij}.

(ii) *Anomalous singularities.* The various types of anomalous singularities are enumerated in Sec. VI of I and in Sec. V of this paper. The important feature of these singularities is that the discontinuities across the cuts and the residues of the poles can be evaluated in terms of on-energy-shell two-body scattering amplitudes (suitably continued to unphysical energies). It is necessary to take these singularities into account in any realistic calculation of the three-body amplitudes since they are the "nearest singularities" to the physical region.

(iii) *"Left-hand cuts."* In the language of perturbation theory these singularities arise from pinches among the denominators of the potentials and(or) the bound-state wave functions. In a fully relativistic theory one might hope to obtain the discontinuities across these cuts from the unitarity relations in the crossed channels. However, in a practical calculation the discontinuity across the left-hand cuts would have to be taken as input information just as it is in the two-body problem.

(7) It is possible to carry out a calculation of all of the three-body amplitudes solely in terms of on-energy-shell quantities. N/D equations were written down in I. The necessary input information is the discontinuity across the left-hand cut and the on-energy-shell two-body scattering amplitude.

It should be noted that no matter how simple an approximation one makes for the left-hand cut, an N/D calculation of the three-body amplitudes will be quite difficult. Even after the N/D equations have been solved one will be faced with the problem of inverting D, which is an operator in the space of the $\mathbf{y}$ variables.

Finally we should mention that the simple techniques employed in this series of papers can be readily extended to the general case of N-particle scattering. It should be simple to show that the N-particle amplitudes exist and satisfy dispersion relations in the total energy. More challenging questions are whether the same methods can be used to study analyticity in the complex angular momentum plane for the nonrelativistic three-body problem, or to study the amplitudes for the scattering of three relativistic particles.

APPENDIX

In this Appendix we shall study the wave-function denominator of Eq. (4.20) and the corresponding potential denominators.

We start with the wave-function denominator. After rotating contours of integration through an angle θ, we have

$$Se^{2i\theta}=(\mathbf{q}e^{i\theta}-x_{N+2}\mathbf{k}_3+\mathbf{k}_B)^2 +\mathbf{q}^2e^{2i\theta}\left(\frac{(b/a)B_i-\frac{4}{3}B_3}{k^2+\frac{4}{3}B_3}\right)+2B_3. \quad \text{(A1)}$$

We now write

$$\begin{aligned} ke^{-i\theta}&=R+iI\,,\\ \mathbf{\Delta}&=\mathbf{k}_B-x_{N+2}\mathbf{k}_3=\mathbf{\Delta}_R+i\mathbf{\Delta}_I\,,\\ C&=R^2-I^2+\tfrac{4}{3}B_3\cos2\theta\,,\\ D&=2RI-\tfrac{4}{3}B_3\sin2\theta\,,\\ N&=q^2[(b/a)B_i-\tfrac{4}{3}B_3)(C^2+D^2)^{-1}\,,\\ \mathbf{p}&=\mathbf{q}+\mathbf{\Delta}_R\,. \end{aligned} \quad \text{(A2)}$$

Since we are only considering values of the x_i's such that $(b/a)B_i\geq\frac{4}{3}B_3$, N is a positive quantity which can take on any value from zero to infinity.

With the present notation, the real and imaginary parts of S become

$$\mathrm{Im}S=2\Delta_I pz-(NC+2B_3)\sin2\theta-ND\cos2\theta\,, \quad \text{(A3)}$$

$$\mathrm{Re}S=p^2-\Delta_I^2+(NC+2B_3)\cos2\theta-ND\sin2\theta\,. \quad \text{(A4)}$$

z is the cosine of the angle between $\mathbf{p}$ and $\mathbf{\Delta}_I$. Setting $\mathrm{Im}S=0$ and substituting into Eq. (A4) gives

$$\mathrm{Re}S\geq\frac{1}{4\Delta_I^2}[(NC+2B_3)\sin2\theta+ND\cos2\theta]^2 -\Delta_I^2+(NC+2B_3)\cos2\theta-ND\sin2\theta\,. \quad \text{(A5)}$$

The right-hand side of Eq. (A5) vanishes for

$$\Delta_I^2=\tfrac{1}{2}[(NC+2B_3)^2+N^2D^2]^{1/2} +\tfrac{1}{2}(NC+2B_3)\cos2\theta-\tfrac{1}{2}ND\sin2\theta\,. \quad \text{(A6)}$$

Since the right-hand side of Eq. (A5) is a decreasing function of Δ_I^2, S can not vanish if Δ_I^2 is less than the right-hand side of Eq. (A6).

We want to show that S can not vanish between the curves $I_\theta(R)$ and $RI=\frac{4}{3}B_3\sin\theta\cos\theta$. The region above the curve $RI=\frac{4}{3}B_3\sin\theta\cos\theta$ is filled by hyperbolas of the form $RI=\frac{4}{3}B_3x\sin\theta\cos\theta$, where $x\geq1$. Since the curve $I_\theta(R)$ lies below the line $I=(\frac{4}{3}B_3)^{1/2}\cos\theta$, we need only consider that part of the hyperbola for which $I\geq(\frac{4}{3}B_3)^{1/2}\cos\theta$ or $R\leq(\frac{4}{3}B_3)^{1/2}x\sin\theta$. From these considerations we see that both C and D are positive between the curves $I_\theta(R)$ and $RI=\frac{4}{3}B_3\sin\theta\cos\theta$. In addition, after a little algebra we find that in this region

$$\tfrac{1}{2}[(NC+2B_3)^2+N^2D^2]^{1/2}+\tfrac{1}{2}(NC+2B_3)\cos2\theta -\tfrac{1}{2}ND\sin2\theta\geq2B_3\cos^2\theta \quad \text{(A7)}$$

for all values of N.

The maximum value of Δ_I^2 is $[(\sqrt{\tfrac{2}{3}})I+\mathrm{Im}(k_Be^{-i\theta})]^2$. The curve $I_\theta(R)$ was defined by setting this quantity equal to $2B_3\cos^2\theta$ [see Eqs. (4.8) and (4.10)]. Since the maximum value of Δ_I^2 is an increasing function of I above the hyperbola $RI=B_3\sin\theta\cos\theta$,

$$\Delta_I^2\leq2B_3\cos^2\theta \quad \text{(A8)}$$

between the curves $I_\theta(R)$ and $RI=\frac{4}{3}B_3\sin\theta\cos\theta$. As a result, the denominator of Eq. (A1) can not vanish between these curves.

The transformation defined by Eqs. (4.17) and (4.18) also introduces terms of the form

$$q^2\left(\frac{(b/a)B_i-\frac{4}{3}B_3}{k^2+\frac{4}{3}B_3}\right) \quad \text{or} \quad q_{0i}^2\left(\frac{(b/a)B_i-\frac{4}{3}B_3}{k^2+\frac{4}{3}B_3}\right)$$

into each of the potentials of t_{23} and into the first potential of t_{12}. These potential denominators can be studied in exactly the same way as the wave-function denominator. The only difference is that in Eqs. (A6), (A7), and (A8), B_3 is replaced everywhere by μ^2. Since we have assumed that $2B_m<\mu^2$, it follows that none of the potential denominators vanish between the curves $I_\theta(R)$ and $RI=\frac{4}{3}B_3\sin\theta\cos\theta$.