driving forces this cancellation must be handled with adjustable parameters. The n-pole approximation provides a convenient parametrization[16] in that the threshold conditions lead to constraints on the residues of the poles. It is also known[17] that this cancellation has an important effect on partial-wave calculations and cannot be ignored as has been the practice in the past.

Perhaps the most interesting application of the n-pole solution will be in self-consistent S-matrix calculations. Because the partial-wave driving forces are determined by the physical amplitudes in the crossed channels, and these in turn can be found by the N/D method for the crossed-channel partial-wave amplitudes,[18] it is advantageous to have a closed-form solution of the N/D equations. Self-consistent solutions involving more than a few partial-wave amplitudes would be extremely difficult to obtain if the integral equation had to be solved numerically at each step. But with the n-pole solution much of the problem reduces to a complicated but manageable algebraic problem. The n-pole solution, both for the unsubtracted and the once-subtracted case, also has the very important property that as the number of poles is increased the solution converges to the solution of the integral equation.

[16] J. Dilley, Nuovo Cimento (to be published).
[17] R. W. Childers and A. W. Martin (unpublished).
[18] See Ref. 14 for a formulation of the self-consistency problem in terms of convergent sums over partial-wave amplitudes alone.

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Unitarity and the Mandelstam Representation. I. Moment Conditions for the Double Spectral Functions*

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It was shown by Mandelstam that the double spectral function in the elastic strip is determined by double integrals over crossed-channel absorptive parts. The content of this consequence of elastic unitarity can be expressed in a number of different ways, some of which significantly simplify the problem of determining the spectral function for given absorptive parts. Among the results presented are rigorous moment conditions, "open-ended" integrals, and the Laplace transform of the double spectral function. An approximation procedure for the determination of the spectral function in terms of simple, one-dimensional integrals over the absorptive parts is developed.

I. INTRODUCTION

THE Mandelstam representation[1] for two-body scattering amplitudes has provided a powerful tool in the description of properties of scattering processes. As examples one may cite the Froissart bounds[2] for the high-energy behavior of the scattering amplitude and the numerous more recent results[3] of a similar nature. It is also evident that the requirements of crossing symmetry in the S-matrix approach to the strong interactions are most easily satisfied in a framework such as the Mandelstam representation.

But the Mandelstam representation has not been a useful tool in the actual calculation of scattering amplitudes, and the reason is a simple one. It is well known that the requirements of unitarity are of profound importance in the description of the strong interactions and the mathematical problem of imposing unitarity upon the Mandelstam representation is formidable. Not only is it a problem in two variables but also the complications of inelastic unitarity must be overcome.

In the elastic domain, Mandelstam derived[1] rigorous expressions for the double spectral functions in terms of integrals over combinations of crossed-channel absorptive parts. Insofar as these double spectral functions determine the imaginary part of the s-channel scattering amplitude in the elastic region, and hence the partial-wave phase shifts up to a sign, there is a good deal of physical information to be obtained from further investigation of this consequence of elastic unitarity. One practical difficulty with the expressions deduced by Mandelstam is the complicated structure of the double integrals which must be evaluated and their sensitivity to regions of integration in which the integrand becomes singular.

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[3] See, for example, T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. **135**, B1464 (1964); Y. S. Jin and A. Martin, *ibid.* **135**, B1375 (1964).

The purpose of this work is to derive further rigorous consequences of elastic unitarity relating the double spectral functions to integrals over the absorptive parts. The primary usefulness of these results will be in determining the spectral functions in the elastic "strips" in terms of the gross properties of the absorptive parts. The more fundamental problem of determining the spectral functions self-consistently by means of unitarity and crossing symmetry requires the introduction of inelastic unitarity and is outside the immediate domain of these considerations. It is possible, however, that the techniques employed here will be of use in the self-consistency problem.

The central results to be presented are "moment" integrals (more accurately, inverse moments) over the double spectral functions. From these moment conditions additional identities can be deduced. In particular, some rigorous expressions for finite integrals over the spectral functions are obtained in which the integrand singularities of Mandelstam's basic result are removed. The moment conditions are derived in two ways. First, by means of the generating function for Legendre functions of the second kind, and second, by consideration of the Laplace transform of the double spectral function. After the derivation of these results it is shown how the moment conditions can be used to determine the spectral functions in the elastic strip.

For the sake of simplicity in the mathematical development the complications of spin and internal symmetry will be suppressed. It will also be assumed, for simplicity, that the scattering amplitude satisfies an unsubtracted Mandelstam representation. This is by no means a crucial assumption as it will be evident how the development is extended to the n-times subtracted representation. Finally, to simplify the kinematics the scattering of identical neutral pseudoscalar mesons will be considered.

In Sec. II the basic formulas, including Mandelstam's expression for the double spectral function in the elastic strip, are reviewed. Section III concerns the derivation of the moment conditions for the spectral function by means of the generating function for the Legendre functions of the second kind. It is shown that the application of known sums for Legendre functions leads to further identities for the spectral functions. A simple integration procedure is applied in Sec. IV to obtain "open-ended" expressions for the double spectral function. These are integral expressions with arbitrary end points of integration and possess the property, in contrast to Mandelstam's basic expression, that the integrands do not become singular in the range of integration.

The integration procedure is used in Sec. V to derive the Laplace transform of the double spectral function. Application of further transforms yields additional identities including the generalization of the moment conditions to complex values of the parameter. In Sec. VI the moment conditions are used to develop a simple and accurate approximation scheme for the determination of the double spectral function in the elastic strip. Section VII is devoted to conclusions.

II. BASIC FORMULAS

The invariant amplitude for the scattering of neutral pseudoscalar mesons is related to the S-matrix element by

$$S=1+\frac{(2\pi)^4 i\delta^4(P_f-P_i)}{(16\omega_1\omega_2\omega_3\omega_4)^{1/2}}A(s,t,u),$$

where $\omega_1\cdots\omega_4$ are the energies of the incoming and outgoing mesons and s, t, and u are the familiar Mandelstam variables satisfying[4] $s+t+u=1$. It is assumed that there are no bound states in this amplitude and that a G parity forbids three meson intermediate states. In this case the elastic region is $1<s<4$ $(4\mu^2<s<16\mu^2)$ and in terms of $z=\cos\theta$, where θ is the center-of-mass scattering angle, the elastic unitarity condition reads

$$\operatorname{Im}A(s,z)=[2(8\pi)^2]^{-1}[(s-1)/s]^{1/2}\times\int d\Omega_n\, A^*(s,z_{fn})A(s,z_{ni}),\quad (1)$$

where the additional factor of $\frac{1}{2}$ accounts for the identical nature of the particles.

In this simplest of possible cases crossing symmetry implies that $A(s,t,u)$ is totally symmetric in the three Mandelstam variables. The unsubtracted Mandelstam representation can then be written in terms of one double spectral function

$$A(s,t,u)=\frac{1}{\pi^2}\int_1^\infty\int\frac{ds'dt'\,\rho(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_1^\infty\int\frac{ds'du'\,\rho(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int_1^\infty\int\frac{dt'du'\,\rho(t',u')}{(t'-t)(u'-u)},$$

and the double spectral function is itself symmetric:

$$\rho(s,t)=\rho(t,s).$$

Mandelstam's derivation of the elastic unitarity expression for $\rho(s,t)$ utilizes the fixed-s dispersion relation for the invariant amplitude

$$A(s,t,u)=\frac{1}{\pi}\int_1^\infty\frac{dx\,A_t(s,x)}{x-t}+\frac{1}{\pi}\int_1^\infty\frac{dx\,A_u(s,x)}{x-u},\quad (2)$$

where the absorptive parts in the crossed channels in this highly symmetric problem are

$$A_t(s,x)=A_u(s,x)=\frac{1}{\pi}\int_1^\infty\frac{dy\,\rho(x,y)}{y-s}+\frac{1}{\pi}\int_1^\infty\frac{dy\,\rho(x,y)}{y+x+s-1}.$$

[4] The units $\hbar=c=2\mu=1$ are used.

Inserting the fixed-s relation (2) into the elastic unitarity condition (1) and carrying out the angular integration leads[1] to the expression[5]

$$\int_1^\infty \frac{dx\,\rho(s,x)}{x-t}=\frac{1}{[s(s-1)]^{1/2}16\pi^2}\times\int_1^\infty\int\frac{dxdy\,A_t^*(s,x)A_t(s,y)}{K(t,x,y)}\times\ln\left[\frac{f(t,x,y)+K(t,x,y)}{f(t,x,y)-K(t,x,y)}\right], \quad (3)$$

where

$$f(t,x,y)=x+y+2xy/(s-1)-t, \quad (4)$$

and $K(t,x,y)$, the Mandelstam kernel, is given by

$$K(t,x,y)=[t^2+x^2+y^2-2tx-2ty-2xy-4txy/(s-1)]^{1/2}. \quad (5)$$

The expression (3) is valid only for $1\leqslant s<4$ since that is the range of validity of the elastic unitarity condition (1). The next step in Mandelstam's procedure is to identify the discontinuity for positive t of the right-hand side of (3). This involves tracing the singularity structure of the logarithm and leads to the basic result

$$\rho(s,t)=\frac{1}{[s(s-1)]^{1/2}8\pi^2}\times\int_1^\theta\int\frac{dxdy\,A_t^*(s,x)A_t(s,y)\theta(K^2)}{K(t,x,y)}, \quad (6)$$

where θ at the upper limits of integration indicates that the range of integration is finite, for finite t, and is determined by the step function in the integrand.

One well-known consequence of (6) is that the double spectral function has a curved boundary, coming from the step function, outside of which the spectral function vanishes. This boundary is not symmetric in s and t indicating that purely elastic unitarity and crossing symmetry are not compatible. When crossing symmetry is taken into account it is seen that the spectral function can be written

$$\rho(s,t)=\theta(st-4s-t)\rho_1(s,t)+\theta(st-s-4t)\rho_1(t,s), \quad (7)$$

where $\rho_1(s,t)$ has no particular symmetry properties in its two variables. The first term on the right-hand side of (7) can be called the "first wing" of the spectral function and the second term, the "second wing." The second wing vanishes for s in the elastic region and (6) applies only to $\rho_1(s,t)$. The subscript will be dropped for simplicity in what follows but it should be borne in mind that all results hold only for $\rho_1(s,t)$ in the elastic strip.

[5] In the general case in which the three double-spectral functions are not identical, the elastic unitarity condition leads to expressions of the form of Eq. (3) for both $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$. Furthermore, as is well known, the combination of absorptive parts on the right-hand side of (3) is replaced by $A_t^*(s,x)A_t(s,y)\to\frac{1}{2}[A_t^*(s,x)\times A_t(s,y)+A_u^*(s,x)A_u(s,y)]$ for ρ_{st} and by a similar replacement for ρ_{su}.

Equation (6) determines the double spectral function in terms of the absorptive parts in the crossed channels. It is not an easy integral to evaluate, however, even for approximate forms for the absorptive parts. It is readily seen that the integral is quite sensitive to the values of the integrand in the vicinity of the upper limit since the Mandelstam kernel vanishes there. The equations derived in the following sections are implicitly contained in (6). Their usefulness stems from the fact that they express the content of (6) in different ways, some of which promise to make the determination of the double spectral function a simpler problem, if not a more direct one.

III. GENERATING FUNCTION APPROACH

With the expression for the Legendre function of the second kind $Q_0(z)$ in terms of a logarithm it is possible to write (3) in the form

$$\int_1^\infty\frac{dx\,\rho(s,x)}{x-t}=\frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int\frac{dxdy\,A_t^*(s,x)A_t(s,y)}{K(t,x,y)}\times Q_0\left[\frac{f(t,x,y)}{K(t,x,y)}\right] \quad (8)$$

and to recognize that the integrand involves a generating function for the Legendre functions[6]

$$\sum_{n=0}^\infty h^nQ_n(z)=(1-2hz+h^2)^{-1/2}Q_0[(z-h)(1-2hz+h^2)^{-1/2}]. \quad (9)$$

The sum in (9) converges for $|h|<|z+(z^2-1)^{1/2}|$ [>1 for z not in the interval $(-1,1)$]. With the choice

$$h=t(x-y)^{-1},\quad z=[x+y+2xy(s-1)^{-1}](x-y)^{-1} \quad (10)$$

the right-hand side of (9) reduces to the function in the integrand of (8), and the domain of convergence of the infinite sum in (9) is readily seen to coincide with the onset of the singularity of the integral.

It follows that for $|t|<4s/(s-1)$ the left-hand side of (8) can be expanded binomially:

$$\int_1^\infty\frac{dx\,\rho(s,x)}{x-t}=\sum_{n=0}^\infty t^n\int_1^\infty\frac{dx\,\rho(s,x)}{x^{n+1}},$$

[6] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 154.

while the right-hand side can be expanded by means of the generating function with the choice (10). Coefficients of the two power series in t can be equated with the result

$$\int_1^\infty \frac{dx\,\rho(s,x)}{x^{n+1}} = \frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int \frac{dxdy\,A_t^*(s,x)A_t(s,y)}{(x-y)^{n+1}} \times Q_n\left[\frac{x+y+2xy/(s-1)}{(x-y)}\right]. \quad (11)$$

This is the moment condition referred to in the Introduction. It holds for s in the elastic domain and for $n=0, 1, 2, \cdots$ (with the assumption that the unsubtracted Mandelstam representation holds). It will be shown later that (11) holds for complex n. This later derivation will also indicate that in case the Mandelstam representation requires M subtractions then (11) holds for $n\geqslant M$.

It should be noted that the right-hand side of (11) is well behaved at the points $x=y$ because of the asymptotic behavior of the Legendre functions for large argument. It is similarly clear that the integrand is symmetric in x and y (apart from the absorptive parts) as is necessary if the double spectral function is to be real valued. Finally, note that

$$\int_1^\infty \frac{dx\,\rho(s,x)}{x^{n+1}} \propto (s-1)^{n+1/2} \quad \text{as} \quad s\to 1,$$

which is the necessary condition such that the imaginary parts of the partial-wave amplitudes possess the correct threshold behavior. This is not a surprising result. The Mandelstam representation leads to the correct threshold behavior of the real parts of the partial-wave amplitudes and so the threshold behavior of the imaginary parts will follow from the elastic unitarity condition which is satisfied by (11).

With the rigorous moment conditions it is possible to find other identities for the double spectral function by employing various sums over Legendre functions of the second kind. For example, the sum

$$\sum_{n=0}^\infty \frac{(n+k)!}{n!k!}h^n Q_{n+k}(z) = (1-2hz+h^2)^{-(k+1)/2}Q_k[(z-h)(1-2hz+h^2)^{-1/2}]$$

leads to the generalization of (8) and (11)

$$\int_1^\infty \frac{dx\,\rho(s,x)}{(x-t^{n+1})} = \frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int \frac{dxdy\,A_t^*(s,x)A_t(s,y)}{K^{n+1}(t,x,y)} \times Q_n\left[\frac{f(t,x,y)}{K(t,x,y)}\right], \quad (12)$$

where $f(t,x,y)$ and $K(t,x,y)$ are given by (4) and (5), respectively. This result holds, by analytic continuation, for all t. It is worth while to note that (12) can be derived by successive differentiation of (8), and this procedure forms an alternative derivation of the moment conditions.

The sum which follows from integrating the generating function,

$$2h\sum_{n=0}^\infty (n+1)^{-1}h^nQ_n(z) = \ln[z-h+(1-2hz+h^2)^{1/2}] \times \ln[z-h-(1-2hz+h^2)^{1/2}] - \ln(z+1)\ln(z-1),$$

leads to the identity

$$\int_1^\infty dx\,\rho(s,x)\ln(1-t/x) = [s(s-1)]^{-1/2}(16\pi^2)^{-1} \times\left\{\left|\int_1^\infty dx\,A_t(s,x)\ln[1+(s-1)/x]\right|^2 - \int_1^\infty\int dxdy \times A_t^*(s,x)A_t(s,y)\ln[1+(s-1)(x+y-t+K)/(2xy)] \times \ln[1+(s-1)(x+y-t-K)/(2xy)]\right\}. \quad (13)$$

The identities presented here are based on a few of the sums derivable from the generating function. As a consequence they are also derivable from the basic form [Eq. (8)]. One useful feature of the moment conditions is that they suggest identities, through the use of known sums over Legendre functions, that are not suggested by the basic form.

It seems likely that consideration of other sums, such as generalizations of Heine's formula, will lead to rigorous identities for the double spectral function that will help in the determination of unitary and crossing symmetric scattering amplitudes. One such approach, based upon the moment conditions themselves, will be developed in Sec. VI.

IV. OPEN-ENDED IDENTITIES

The purpose of this section is to develop some rigorous identities involving integrals over the spectral function with a finite range of integration. These identities can be obtained with the techniques of the preceding section but there is a simpler way. Consider Mandelstam's expression [Eq. (6)] for the spectral function. It can be written in the form

$$\theta(st-4s-t)\rho(s,t) = \frac{1}{[s(s-1)]^{1/2}8\pi^2} \times\int_1^\infty\int \frac{dxdy\,A_t^*(s,x)A_t(s,y)\theta(t-t_+)}{(t-t_+)^{1/2}(t-t_-)^{1/2}}, \quad (14)$$

where

$$t_\pm=\{[x(y+s-1)]^{1/2}\pm[y(x+s-1)]^{1/2}\}^2/(s-1), \quad (15)$$

and the step function on the left in (14) serves as a reminder of the boundary of the spectral function while the step function on the right cuts off the integral for sufficiently large x or y.

In the following development t will be a positive number satisfying the inequality

$$t>t_0=4s/(s-1).$$

Consider (14) with t replaced by t' and the operation of integrating this expression with $\int_0^t dt' f(t')$. The result is easily seen to be

$$\int_{t_0}^{t} dx\,\rho(s,x)f(x) =\frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int dxdy\,A_t^*(s,x)A_t(s,y)\theta(t-t_+) \times\int_{t_+}^{t}\frac{dt'\,f(t')}{(t'-t_+)^{1/2}(t'-t_-)^{1/2}}. \quad (16)$$

Choosing $f(t')=1$ leads to the "open-ended" identity

$$\int_{t_0}^{t} dx\,\rho(s,x)=\frac{1}{[s(s-1)]^{1/2}4\pi^2}\int_1^\infty\int dxdy\,A_t^*(s,x) \times A_t(s,y)\theta(t-t_+)Q_0\left[\frac{(t-t_-)^{1/2}}{(t-t_+)^{1/2}}\right]. \quad (17)$$

This equation, which can also be deduced from (13) by consideration of the singularity structure, gives the integral of the double-spectral function over any finite (or infinite) range in terms of integrals over the absorptive parts in which the integrand is not singular. In fact, whereas Mandelstam's equation for the spectral function [Eq. (6)] involves a kernel that becomes large near the upper limits of integration, the integrand in (17) vanishes there. This is nothing more than the observation that the integrable square root singularity has been integrated out.

A more general open-ended identity follows from the choice $f(t')=(t'-\lambda)^{-1}$ in which case (16) becomes

$$\int_{t_0}^{t}\frac{dx\,\rho(s,x)}{(x-\lambda)} =\frac{1}{[s(s-1)]^{1/2}4\pi^2}\int_1^\infty\int\frac{dxdy\,A_t^*(s,x)A_t(s,y)\theta(t-t_+)}{[(t_+-\lambda)(t_--\lambda)]^{1/2}} \times Q_0\left[\frac{(t-t_-)^{1/2}(t_+-\lambda)^{1/2}}{(t-t_+)^{1/2}(t_--\lambda)^{1/2}}\right]. \quad (18)$$

This expression holds for $t\geqslant t_0$ and for all values of λ. It is a simple matter to verify that in the limit $t\to\infty$ (18) reduces to (8). Both (17) and (18) are also easily checked by differentiation with respect to t. Finally, note that successive differentiation of (18) with respect to λ leads to open-ended moment conditions for the double-spectral function which generalize (11) and (12).

The use of (18) and similar open-ended identities should simplify the Mandelstam iteration program.[1] In the following section it will be shown that this integration procedure [Eq. (16)] also leads to further identities by means of Laplace transform techniques.

V. LAPLACE TRANSFORM APPROACH

Because the theory and application of Laplace transforms has been developed to a high degree, it will be useful to express the restrictions of elastic unitarity on the double-spectral function in terms of such transforms. This is readily accomplished with the techniques of the previous section. Choose $f(t')=\exp(-\lambda t')$ and $t=\infty$ in which case (16) becomes, after a trivial change of integration variable,

$$\int_{t_0}^{\infty} dx\,\rho(s,x)e^{-\lambda x}=\frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int dxdy\,A_t^*(s,x) \times A_t(s,y)e^{-\lambda t_+}\int_0^\infty\frac{dz\,e^{-\lambda z}}{[z(z+t_+-t_-)]^{1/2}}.$$

Bateman[7] evaluates the last transform with the result that the Laplace transform of the double spectral function in the elastic strip is

$$\int_{t_0}^{\infty} dx\,\rho(s,x)e^{-\lambda x}=\frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int dxdy\,A_t^*(s,x) \times A_t(s,y)e^{-\frac{1}{2}\lambda(t_++t_-)}K_0[\tfrac{1}{2}\lambda(t_+-t_-)], \quad (19)$$

where K_0 is the modified Bessel function of the third kind.

It follows from (19) that any Laplace transforms involving the modified Bessel function will yield further identities for the spectral function. In particular, operating on (19) with $\int_0^\infty d\lambda\,\lambda^\mu e^{\lambda\beta}$ leads[8] to the generalized moment condition

$$\int_{t_0}^{\infty}\frac{dx\,\rho(s,x)}{(x-\beta)^{\mu+1}} =\frac{1}{[s(s-1)]^{1/2}8\pi^2}\int_1^\infty\int\frac{dxdy\,A_t^*(s,x)A_t(s,y)}{[(t_+-\beta)(t_--\beta)]^{(\mu+1)/2}} \times Q_\mu\left[\frac{t_++t_--2\beta}{2(t_+-\beta)^{1/2}(t_--\beta)^{1/2}}\right], \quad (20)$$

[7] *Tables of Integral Transforms*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. I, p. 138.
[8] Reference 7, Vol. I, pp. 137 and 198.

which holds for complex μ with $\mathrm{Re}\mu > -1$. This result is the generalization to complex n of the moment conditions (11) and (12). Furthermore, because (20) was derived from Mandelstam's expression for the double spectral function [Eq. (14)] rather than from an integrated form such as (8), it is clear that the moment conditions hold for the N-times subtracted Mandelstam representation. In this case the necessary number of subtractions for fixed s in the elastic domain is determined by the minimum value of (real) μ for which the right-hand side of (20) converges.

Application of other Laplace transforms leads to the result

$$\int_{t_0}^{\infty} dx\, \rho(s,x) Q_\nu[1+2x/(s-1)] = [s(s-1)]^{-1/2}(8\pi^2)^{-1} \times \left| \int_1^{\infty} dx\, A_t(s,x) Q_\nu[1+2x/(s-1)] \right|^2, \quad (21)$$

which is recognized to be the continuation of the elastic unitarity condition for the partial-wave amplitudes,

$$\mathrm{Im}A_l(s+i\epsilon) = [(s-1)/s]^{1/2} |A_l(s)|^2,$$

to arbitrary real $l=\nu > -1$. Again, the range of ν for which (21) is valid is determined by the convergence of the integrals. As is well known, this result can be continued to complex ν by appropriate modification of the right-hand side.

In addition to further, and for the most part rather complicated identities resulting from Laplace transforms of (19), it will be noted that (20) and (21), being functions of a continuous parameter, allow the development of additional identities. As the simplest example set $\beta=0$ in (20), use the results [from (15)]

$$(t_+t_-)^{1/2} = |x-y|, \quad \tfrac{1}{2}(t_+ + t_-) = x+y+2xy/(s-1),$$

and employ changes of variables in the left-hand side of (20) to obtain the Laplace transform and the equivalent Mellin transform

$$\int_0^{\infty} dz\, e^{-\mu z} \rho(s,e^z) = \int_0^{\infty} dt\, t^{\mu-1} \rho(s,t^{-1}) = \frac{1}{[s(s-1)]^{1/2} 8\pi^2} \int_1^{\infty}\int \frac{dx dy\, A_t^*(s,x) A_t(s,y)}{|x-y|^{\mu+1}} \times Q_\mu\left[\frac{x+y+2xy/(s-1)}{|x-y|}\right]. \quad (22)$$

Laplace transforms involving Legendre functions will then yield further results.

It has been shown that the content of Mandelstam's expression for the double-spectral function in terms of integrals over the absorptive parts can be expressed in a variety of ways. To this point the derivation has been formal with the purpose of suggesting alternative approaches to the problem of computing the double-spectral functions in the elastic strip. One approach, based directly upon the moment conditions, is developed in the following section.

VI. DETERMINATION OF THE SPECTRAL FUNCTIONS

The assumption of S-matrix theory that the constraints of analyticity, crossing symmetry, and unitarity, together with the possible specification of a few constants, lead to unique scattering amplitudes requires the statement of unitarity for all physical energies. In other words, elastic unitarity is insufficient for the self-consistent determination of the double-spectral functions. In the present case, because a rigorous expression for the spectral function derived from inelastic unitarity does not exist, the absorptive parts must be considered as given and the problem reduces to finding the corresponding double-spectral function.

This problem is not so far removed from reality as it may appear. The behavior of $A_t(s,x)$ for $x \approx 1$ is determined by the low-energy behavior of the partial-wave amplitudes in the crossed channels (the Legendre expansion of the imaginary part of the crossed-channel amplitudes converges in a usable domain determined by the boundaries of the spectral functions). The behavior of $A_t(s,x)$ for large x can be inferred to some extent from the phenomenological analysis of high-energy scattering data, or from the Regge hypothesis. In any case, reasonable estimates for the absorptive parts can be made which incorporate inelastic unitarity. It follows that the determination of the spectral function in the elastic strip, and hence the elastic phase shifts up to a sign, in terms of these estimates can be useful.

In the present work specific and more or less realistic models for the absorptive parts will not be studied. The results for such models will be reported at a later time. Instead the purpose of this section is to develop some tools for the analysis. The starting point is the moment conditions of Eq. (11). It will be assumed, for convenience only, that the moment conditions hold for $n \geqslant 0$. The disadvantage of the exact moment conditions, as in Mandelstam's direct expression for the spectral function, is the complicated double integral which must be evaluated.

It would clearly be advantageous to work only with one-dimensional integrals as in the generalized partial-wave unitary condition of Eq. (21). To see that the moment conditions can be well approximated by such forms consider one of the standard hypergeometric expansions of the Legendre functions[9]

$$Q_n(z)=\frac{1}{w^{n+1}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{1}{2})\Gamma(k+n+1)}{k!\Gamma(k+n+\frac{3}{2})w^{2k}}, \tag{23}$$

where

$$w=z+(z^2-1)^{1/2}.$$

It follows easily[10] from (23) that for $z>1$ the Legendre function satisfies the bounds

$$\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})w^{n+1}}<Q_n(z)<\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})w^{n+1}}(1-w^{-2})^{-1/2}. \tag{24}$$

The lower bound in (24) is just the first term in the expansion (23) (all terms of which are positive for $z>1$) while the upper bound is an overestimate of the infinite sum.

For the Legendre function appearing in the moment conditions the upper and lower bounds are very close. That is, for

$$z=[x+y+2xy(s-1)^{-1}](x-y)^{-1}, \tag{25}$$

one finds

$$w=\frac{t_+}{(x-y)}=\frac{[x^{1/2}(y+s-1)^{1/2}+y^{1/2}(x+s-1)^{1/2}]^2}{(s-1)(x-y)}, \tag{26}$$

and this function is bounded for fixed s and $1\leqslant x$, $y\leqslant\infty$ by

$$(s^{1/2}+1)^2/(s-1)\leqslant w\leqslant\infty.$$

It follows that

$$\max[(1-w^{-2})^{-1/2}]=(s^{1/2}+1)/(2s^{1/4}),$$

which for $1\leqslant s<4$ is very close to unity. With z and w given by (25) and (26), respectively, the bounds on the Legendre function can be written

$$\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})w^{n+1}}<Q_n(z)<\frac{(s^{1/2}+1)}{2s^{1/4}}\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})w^{n+1}}. \tag{27}$$

These estimates indicate that the expansion (23) converges extremely rapidly for the Legendre functions in the moment conditions, and that it provides an excellent approximation procedure for those functions. But the goal of expressing the moment conditions in terms of single integrals has not as yet been accomplished. This is done by noting the bounds

$$(s-1)(x-y)/(4sxy)\leqslant w^{-1}\leqslant(s-1)(x-y)/(4xy)$$

with which (27) can be written

$$\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})(xy)^{n+1}}\left(\frac{s-1}{4s}\right)^{n+1}<\frac{1}{(x-y)^{n+1}}Q_n(z)<\frac{(s^{1/2}+1)}{2s^{1/4}}\frac{\pi^{1/2}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})(xy)^{n+1}}\left(\frac{s-1}{4}\right)^{n+1}. \tag{28}$$

[9] E. W. Hobson, *Theory of Spherical and Ellipsoidal Harmonics* (Chelsea Publishing Company, New York, 1955), p. 61.

[10] Reference 9, p. 61. A trivial improvement of the upper bound derived in this reference is used in Eq. (24).

In the estimates to follow the lower bound will be used. This approximation can be systematically improved by keeping more terms in the expansion (23) and by including corrections to the estimate of w^{-1}.

The bounds given in (28) have the desired property that the x and y dependences factor. As a consequence the moment conditions [Eq. (11)] are well approximated by

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x^{n+1}}=\frac{\pi^{1/2}\Gamma(n+1)}{32s\Gamma(n+\frac{3}{2})}\left(\frac{s-1}{s}\right)^{1/2}\left(\frac{s-1}{4s}\right)^n\times\left|\frac{1}{\pi}\int_1^{\infty}\frac{dx\,A_t(s,x)}{x^{n+1}}\right|^2. \tag{29}$$

To see how this form involving relatively simple one-dimensional integrals can be used to obtain first approximations to the double-spectral functions suppose that the integral over the absorptive part has been evaluated with the result

$$\left|\frac{1}{\pi}\int^{\infty}\frac{dx\,A_t(s,x)}{x^{n+1}}\right|^2=f_n(s).$$

Then summing t^n times the moment condition yields

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x-t}=\frac{\pi^{1/2}}{32s}\left(\frac{s-1}{s}\right)^{1/2}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}f_n(s)\times\left[\frac{(s-1)t}{4s}\right]^n. \tag{30}$$

If the integrals in (29) do not converge for $n<N$ a simple modification is needed. One such example is treated below.

The sum on the right-hand side of (30) will generally have a radius of convergence of unity. That is, as a function of t the sum will have a branch point at $(st-4s-t)=0$ which is the proper boundary of the first wing of the double spectral function. So the double spectral function is estimated by evaluating the sum and identifying the discontinuity across the branch cut. As one extreme example consider the (physically unreasonable) case where $f_n(s)$ is approximated by a positive function of s independent of n, $f_n(s)=C(s)$. The sum in (30) is then seen to be a hypergeometric function,

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x-t}=\frac{C(s)}{16s}\left(\frac{s-1}{s}\right)^{1/2}\times F(1,1;\tfrac{3}{2};(s-1)t/(4s)). \tag{31}$$

The discontinuity of a hypergeometric function with one of its first pair of arguments equal to unity is readily evaluated by a change of integration variable in the standard integral representation. That is,

$$F\left(1,b;c;\frac{(s-1)t}{4s}\right)=\frac{\Gamma(c)(4s)^b}{\Gamma(b)\Gamma(c-b)(s-1)^{c-1}}\times\int_{t_0}^{\infty}\frac{dx\,(sx-4s-x)^{c-b-1}}{x^{c-1}(x-t)},\quad(32)$$

where $t_0=4s/(s-1)$ as in Sec. IV. Identifying the discontinuity gives for the spectral function of (31)

$$\rho(s,t)=\frac{C(s)\theta(st-4s-t)}{8(st)^{1/2}(st-4s-t)^{1/2}}.$$

This estimate is clearly a poor one; the double spectral function is singular at the boundary whereas it is known for the present case (as in pion-pion scattering) that the double spectral function must be proportional to $(st-4s-t)^{5/2}$ at the boundary.[11]

As an extreme example in the other direction consider the (also physically unreasonable) case in which $f_n(s)$ is approximated by $f_n(s)=C(s)/\Gamma(n+1)$, that is, the moment integrals of the absorptive part fall off as the inverse square root of $\Gamma(n+1)$. In this case (30) becomes

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x-t}=\frac{\pi^{1/2}C(s)}{32s}\left(\frac{s-1}{s}\right)^{1/2}\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\frac{3}{2})}\left[\frac{(s-1)t}{4s}\right]^n$$

and the sum is recognized as a confluent hypergeometric function,

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x-t}=\frac{C(s)}{16s}\left(\frac{s-1}{s}\right)^{1/2}F\left(1;\tfrac{3}{2};\frac{(s-1)t}{4s}\right).\quad(33)$$

But the confluent hypergeometric functions are entire functions of their argument so (33) is an impossible equation. These examples support the intuitively obvious conclusion that the moment integrals of the absorptive parts must decrease with increasing n but cannot decrease exponentially.

As the final example of the application of the moment conditions consider the more realistic case in which the absorptive part is approximated by means of the optical theorem. Since $A_t(s,x)$ is the imaginary part of the amplitude in the crossed channel in which x is the square of the c.m. energy and s is the momentum transfer, the optical theorem gives

$$A_t(0,x)=2[x(x-1)]^{1/2}\sigma_{\text{tot}}(x).$$

With the assumption that the absorptive part does not vary radically as s goes from zero to the range $1\leq s<4$ and with the further approximation of replacing $\sigma_{\text{tot}}(x)$ by a constant average value σ_{tot}, the integral over the absorptive part yields a β function

$$\frac{1}{\pi}\int_1^{\infty}\frac{dx\,A_t(s,x)}{x^{n+1}}\approx\frac{2\sigma_{\text{tot}}}{\pi}\int_1^{\infty}\frac{dx\,[x(x-1)]^{1/2}}{x^{n+1}}=\frac{\sigma_{\text{tot}}\Gamma(n-1)}{\pi^{1/2}\Gamma(n+\frac{1}{2})}.\quad(34)$$

The integral exists only for $n\geq2$, so we make the replacement $n=k+2$ and insert (34) into (29) to obtain

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x^{k+3}}\approx\frac{\sigma_{\text{tot}}^2}{16\pi^{1/2}s}\left(\frac{s-1}{4s}\right)^{5/2}\times\frac{\Gamma^2(k+1)\Gamma(k+3)}{\Gamma^2(k+\frac{5}{2})\Gamma(k+\frac{7}{2})}\left(\frac{s-1}{4s}\right)^k.\quad(35)$$

The next step is to sum (35) over t^k and identify the discontinuities. But in this case the sum is a generalized hypergeometric function, and while multiple integral representations exist for these functions it is more convenient to make the simplifying approximation

$$\frac{\Gamma^2(k+1)\Gamma(k+3)}{\Gamma^2(k+\frac{5}{2})\Gamma(k+\frac{7}{2})}\approx\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+4)}.\quad(36)$$

This approximation is excellent for large k and adequate for smaller k. It remains to be verified that the qualitative behavior of the resulting double spectral function is not appreciably affected by such an approximation. This point will be examined at a later date.

Inserting (36) into (35) and summing over t^k leads to the result

$$\int_1^{\infty}\frac{dx\,\rho(s,x)}{x^2(x-t)}\approx\frac{\sigma_{\text{tot}}^2}{96s}\left(\frac{s-1}{4s}\right)^{5/2}F\left(1,\tfrac{1}{2};4;\frac{(s-1)t}{4s}\right).$$

The integral representation (32) for the hypergeometric function then gives the estimate for the spectral function

$$\rho(s,t)\approx\frac{2\sigma_{\text{tot}}^2(st-4s-t)^{5/2}}{15\pi(4s)^3(s-1)^{1/2}t}.$$

This spectral function has the correct boundary behavior and is not patently wrong. In view of the approximations that went into the derivation, however, it should be viewed as at best an averaged expression for the double spectral function in the elastic strip.

The primary purpose of these examples is to indicate the usefulness of the moment conditions as a tool in the determination of the spectral functions.The necessary integrals over the absorptive part are of a simple form and in a first approximation they are determined by the gross structure of the absorptive part. The method also allows symmetric improvement of the estimate of the double spectral function. Applications of interest would

[11] A. W. Martin, Bull. Am. Phys. Soc. 11, 902 (1966); also (unpublished).

be the use of Regge theory to improve the optical theorem approximation studied above together with Breit-Wigner forms to include crossed-channel resonance contributions to the total cross section.

VII. CONCLUSIONS

The rigorous expression for the double-spectral function in the elastic strip in terms of finite two-dimensional integrals over absorptive parts has been known for some time.[1] Because reasonable estimates for the absorptive parts can be made and because the double-spectral function determines the imaginary part of the scattering amplitude in the elastic region, it is of interest to pursue this consequence of elastic unitarity. The practical difficulty with Mandelstam's expression is the complexity of the double integral and its sensitivity to certain regions of integration.

In addition to the problem of determining the elastic phase shifts to within a sign there is the larger problem of the self-consistent determination of the scattering amplitude itself. While this problem requires the specification of inelastic unitarity, it is likely that alternative approaches to the elastic case will be useful in the more general situation. The central purpose of this work has been the development of different ways of expressing the content of Mandelstam's result.

The results divide roughly into three (not uncorrelated) categories. First, the observation that an intermediate step in the derivation of the double spectral function involves the generating function for the Legendre functions of the second kind led to "moment conditions" for the spectral function. These conditions relate inverse moments of the spectral function to integrals over absorptive parts and Legendre functions of integer order. The application of known sums for Legendre functions then leads to a number of further identities.

The second category of results follows from the simple operation of integrating the explicit t dependence of the spectral function in the elastic strip with arbitrary functions of t. In particular, the choice of a finite range of integration yields "open-ended" identities. These identities avoid the sensitive integration of the original expression for the double spectral function because the inverse-square-root singularity has been integrated out. The third category of identities evolves from the use of the integration procedure to deduce the Laplace transform of the double spectral function.

With the Laplace transform in hand it is possible to obtain a number of further identities for the spectral function. As one example, a generalization of the moment conditions follows immediately from the transform formulas. This derivation indicates that the moment conditions hold for the n-times subtracted Mandelstam representation, the necessary number of subtractions being determined in turn by the convergence of the absorptive part integrations. Another simply obtained result is the well-known generalization of the partial-wave unitarity condition to complex values of l.

Finally, as a practical approach to the problem of determining the double spectral function in the elastic strip, given the absorptive parts, it was shown that the rigorous moment conditions lead to a simple approximation procedure. In this approximation the inverse moments of the spectral function are related to the modulus squared of the inverse moments of the absorptive parts. Because the integrals are of a relatively simple form, it is possible to estimate the dominant n-dependence of the moment integrals and hence to reconstruct the general behavior of the double spectral function. The application of this approach to physically reasonable models for the absorptive parts will be presented at a later time.