N/D Solution of the n-Pole Problem*

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The explicit solution of the n-pole problem in relativistic partial-wave dispersion relations is shown to be a simple generalization of the potential-scattering results of Nyman. The solution and its application to self-consistent S-matrix calculations is discussed.

I. INTRODUCTION

THE N/D method of Chew and Mandelstam[1] has provided the most practical approach to relativistic partial-wave dispersion relations. As is well known, the resulting set of coupled integral equations can be separated to give a Fredholm-like equation in the numerator function N.[2] Since the solution of a Fredholm equation cannot, in general, be expressed in closed form, partial-wave calculations have employed either numerical solutions of the exact equation or one of the many approximation schemes that have been developed.[3–7]

One of the most interesting applications of the partial-wave dispersion relations is found in self-consistent S-matrix calculations[1,8,9] defined by the requirements of analyticity, unitarity, and crossing symmetry. Here the partial-wave amplitudes enter because of the relative simplicity of the partial-wave unitarity condition. In self-consistent calculations, however, the partial-wave driving forces are determined by the physical amplitudes in the crossed channels. As a consequence, numerical solutions of the N/D equations will require rather complicated iteration procedures to handle this interdependence between unitarity and the crossing relations.

It is evident that the complexity of such self-consistent calculations is greatly reduced if an approximate, closed-form solution of the N/D equations is used. But it is also clear that the approximation scheme should possess as many of the properties of the exact solution as possible, and ideally it should be possible to make the approximation arbitrarily accurate. Pagels has developed an approximation technique[7] that promises to be very useful in self-consistent calculations.[10] The purpose of this work is to present another, the explicit solution of the n-pole approximation.

Section II contains the development of the n-pole problem and its solution. Further properties of the solution are discussed in Sec. III, and Sec. IV is devoted to the extension of this solution to more general partial-wave amplitudes. Section V contains the conclusions.

II. THE n-POLE SOLUTION

To begin with, it is assumed that the partial-wave amplitude satisfies an unsubtracted dispersion relation. For convenience it will also be assumed that the amplitude has the analytic structure of pion-pion partial-wave amplitudes in the s plane, that is, a physical branch cut on the positive real axis extending from threshold ($s=1$, say) to $+\infty$ and a dynamic branch cut on the entire negative real axis $(-\infty, 0)$. It will be clear that the solution is trivially extended to the more complicated structure of the pion-nucleon amplitude in the w plane, for example.

On the physical cut the amplitude satisfies a unitarity condition of the form

$$\mathrm{Im}A(s)=\rho(s)\,|A(s)|^2\,,$$

or

$$\mathrm{Im}A^{-1}(s)=-\rho(s)\,, \tag{1}$$

where $\rho(s)$ is the appropriate kinematic factor containing, in principle, the effects of inelastic unitarity. The unsubtracted dispersion relation for $A(s)$ can then be written

$$A(s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds'\,\mathrm{Im}A(s')}{s'-s}+\frac{1}{\pi}\int_{1}^{\infty}\frac{ds'\,\rho(s')|A(s')|^2}{s'-s}\,.$$

The n-pole problem is defined by approximating the discontinuity of $A(s)$ on the dynamic cut by a sum of δ functions:

$$\mathrm{Im}A(s)=-\pi\sum_{i=1}^{n}b_i\delta(s-s_i)\,,\quad s_i<0\,.$$

The driving force, defined as the contour integral of $A(s)$ around the dynamic cuts, is therefore approximated by the sum of n poles with residues b_i:

$$B(s)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{ds'\,\mathrm{Im}A(s')}{s'-s}=\sum_{i=1}^{n}\frac{b_i}{s-s_i}\,. \tag{2}$$

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).
[3] M. Baker, Ann. Phys. (N.Y.) **4**, 271 (1958).
[4] T. Fulton, in *Elementary Particle Physics and Field Theory, 1962 Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. I, p. 55; G. L. Shaw, Phys. Rev. Letters **12**, 345 (1964).
[5] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).
[6] A. W. Martin, Phys. Rev. **135**, B967 (1964).
[7] H. Pagels, Phys. Rev. **140**, B1599 (1965).
[8] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960); G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).
[9] C. F. Kyle, A. W. Martin, and H. Pagels (unpublished); C. F. Kyle and A. W. Martin (unpublished).
[10] See, for example, Ref. 9.

In the manner of Chew and Mandelstam,[1] the partial-wave amplitude is represented by the ratio

$$A(s)=N(s)/D(s),$$

where $N(s)$ contains the dynamic cuts and $D(s)$ the physical cut. The unitarity condition is satisfied by the property of $D(s)$

$$\text{Im}D(s)=-\rho(s)N(s),\quad 1<s.$$

Writing unsubtracted dispersion relations for N and D and carrying through the usual manipulations[2] leads to the integral equation for the numerator function

$$N(s)=B(s)+\frac{1}{\pi}\int_1^\infty \frac{ds'\,\rho(s')}{s'-s}[B(s')-B(s)]N(s'). \quad (3)$$

Because of the pole structure of the driving force $B(s)$, the kernel of this integral equation is degenerate[11] and the solution reduces to an algebraic problem. From (2) and (3) it follows that

$$N(s)=\sum_{i=1}^{n}\frac{b_i}{s-s_i}\left[1-\frac{1}{\pi}\int_1^\infty \frac{ds'\,\rho(s')N(s')}{s'-s_i}\right],$$

and so the solution of (3) must have the form

$$N(s)=\sum_{i=1}^{n}\frac{c_i}{s-s_i},$$

where the constants c_i are the solutions of the algebraic equations

$$c_i=b_i\left[1-\sum_{j=1}^{n}c_j\frac{1}{\pi}\int_1^\infty \frac{ds'\,\rho(s')}{(s'-s_i)(s'-s_j)}\right].$$

Nyman determined the closed-form solutions for N and D in the case of potential scattering,[12] and it is straightforward to generalize his results to the relativistic case. Define the purely kinematic integral

$$K(s_i,s_j)=\frac{1}{\pi}\int_1^\infty \frac{ds'\,\rho(s')}{(s'-s_i)(s'-s_j)}. \quad (4)$$

Then, for the n-pole problem, $N(s)$ and $D(s)$ can be represented as the determinants of $(n+1)$ by $(n+1)$ matrices:

$$N(s)=\det\begin{vmatrix} 0 & -b_1/(s-s_1) & -b_2/(s-s_2) & -b_3/(s-s_3) & \cdots \\ 1 & 1+b_1K(s_1,s_1) & b_2K(s_2,s_1) & b_3K(s_3,s_1) & \cdots \\ 1 & b_1K(s_1,s_2) & 1+b_2K(s_2,s_2) & b_3K(s_3,s_2) & \cdots \\ 1 & b_1K(s_1,s_3) & b_2K(s_2,s_3) & 1+b_3K(s_3,s_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad (5)$$

$$D(s)=\det\begin{vmatrix} 1 & b_1K(s_1,s) & b_2K(s_2,s) & b_3K(s_3,s) & \cdots \\ 1 & 1+b_1K(s_1,s_1) & b_2K(s_2,s_1) & b_3K(s_3,s_1) & \cdots \\ 1 & b_1K(s_1,s_2) & 1+b_2K(s_2,s_2) & b_3K(s_3,s_2) & \cdots \\ 1 & b_1K(s_1,s_3) & b_2K(s_2,s_3) & 1+b_3K(s_3,s_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad (6)$$

and the further terms are easily supplied by inspection. Note that N and D as given above are not the N and D of the integral equation (3) but differ by a common multiplicative constant, leaving the ratio $A(s)$ unchanged.

It is a simple matter to verify that this is indeed the solution of the problem. As a useful convention let the index for the rows and columns of the matrices run over the range 0 to n. Since the matrices for N and D are identical except for the zeroth row, the cofactors of the elements of the zeroth rows are the same for both determinants. Let the cofactor of the ith element be denoted cof_i. Then $N(s)$ and $Ds()$ can be written

$$N(s)=-\sum_{i=1}^{n}\frac{b_i\,\text{cof}_i}{s-s_i},$$
$$D(s)=\text{cof}_0+\sum_{i=1}^{n}b_iK(s_i,s)\,\text{cof}_i. \quad (7)$$

As the cofactors are real constants, the imaginary part of $D(s)$ comes from the integrals $K(s_i,s)$. Note that, by (4),

$$\text{Im}K(s_i,s)=\rho(s)/(s-s_i),\quad 1\leq s,$$

so

$$\text{Im}D(s)=\rho(s)\sum_{i=1}^{n}\frac{b_i\,\text{cof}_i}{s-s_i}=-\rho(s)N(s),$$

and unitarity is satisfied.

To verify that this solution has the correct poles with the specified residues b_i note that for $s\approx s_i$

$$N(s)=-b_i\,\text{cof}_i/(s-s_i)+R(s),$$

where $R(s)$ is regular at $s=s_i$. $D(s)$ is regular everywhere except on the physical cut and so the residue of the pole of $A(s)$ at $s=s_i$ depends only on the constant $D(s_i)$. Since the value of the determinant is invariant under the operation of subtracting one row from another, $D(s_i)$ is most easily calculated by subtracting the ith row from the zeroth row. It is seen from (6) that the result is that the elements of the zeroth row all vanish except for the ith element, which has the value

[11] See, for example, Ref. 6.

[12] E. M. Nyman, Nuovo Cimento 37, 492 (1965).

-1. As a consequence

$$D(s_i)=-\mathrm{cof}_i$$

and $A(s)$ can be written, for $s\approx s_i$,

$$A(s)=b_i/(s-s_i)+\tilde{R}(s)\,,$$

where $\tilde{R}(s)$ is regular at $s=s_i$. The representation therefore reproduces the specified poles and provides the solution of the problem.

III. FURTHER PROPERTIES

In the preceding section the solution of the relativistic n-pole problem was presented and verified. It is now worthwhile to examine some of the further properties of the solution. Two trivial but necessary properties are the following. If the residue of the ith pole vanishes, then the solution reduces to the solution for $(n-1)$ poles. If two pole positions, s_i and s_j, should be identical, then the solution reduces to the solution with $(n-1)$ poles and the residue of the ith pole is (b_i+b_j). Both of these properties can be checked by means of determinant operations. The first is essentially obvious and the second follows from subtracting the jth row from the ith row and then adding the ith column to the jth column.

With the exact solution in hand it is also possible to develop solutions for more general problems. As an example note that with the substitutions

$$b_i\to b_i/\epsilon\,,\quad b_j\to -b_i/\epsilon\,,\quad s_j\to s_i-\epsilon\,,$$

the sum of two simple poles becomes in the limit $\epsilon\to 0$ a double pole

$$\frac{b_i}{s-s_i}+\frac{b_j}{s-s_j}\underset{\epsilon\to 0}{\longrightarrow}\frac{b_i}{(s-s_i)^2}\,.$$

It is well known that if the driving force contains a double pole at $s=s_i$, then the solution of (3) possesses both a double pole and a simple pole at $s=s_i$. This solution is readily obtained from the n-pole solution through determinant operations and the limiting process. Choosing $i=1$ and $j=2$ leads to the result

$$N(s)=\det\begin{vmatrix} 0 & -b_1/(s-s_1) & -b_1/(s-s_1)^2 & -b_3/(s-s_3) & \cdots \\ 0 & 1+b_1K(s_1,s_1,s_1) & b_1K(s_1,s_1,s_1,s_1) & b_3K(s_3,s_1,s_1) & \cdots \\ 1 & b_1K(s_1,s_1) & 1+b_1K(s_1,s_1,s_1) & b_3K(s_3,s_1) & \cdots \\ 1 & b_1K(s_1,s_3) & b_1K(s_1,s_1,s_3) & 1+b_3K(s_3,s_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

$$D(s)=\det\begin{vmatrix} 1 & b_1K(s_1,s) & b_1K(s_1,s_1,s) & b_3K(s_3,s) & \cdots \\ 0 & 1+b_1K(s_1,s_1,s_1) & b_1K(s_1,s_1,s_1,s_1) & b_3K(s_3,s_1,s_1) & \cdots \\ 1 & b_1K(s_1,s_1) & 1+b_1K(s_1,s_1,s_1) & b_3K(s_3,s_1) & \cdots \\ 1 & b_1K(s_1,s_3) & b_1K(s_1,s_1,s_3) & 1+b_3K(s_3,s_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where the notation has been introduced

$$K(s_i,s_j,\cdots,s_k)=\frac{1}{\pi}\int_1^\infty\frac{ds'\,\rho(s')}{(s'-s_i)(s'-s_j)\cdots(s'-s_k)}\,.$$

That these determinants give the solution of the problem can again be verified by means of determinant manipulations. This procedure can clearly be extended to yield the N/D solution for driving forces approximated by sums of poles of arbitrary order.

Another property of the solution concerns the apparent restriction on the asymptotic growth of the kinematic function $\rho(s)$. Equation (4) implies that the solution breaks down unless

$$\rho(s)\underset{s\to\infty}{\longrightarrow}s^{1-\epsilon},\quad \epsilon>0\,,$$

which would rule out the pion-nucleon case, for example, where the usual kinematic factor $q(w)$ is linear in the energy w at high energies. That this restriction does not actually hold is easily seen. Consider the operation of multiplying the zeroth column of the N and D determinants, Eqs. (5) and (6), by $b_iK(s_i,s_j)$ and subtracting the result from the ith column. Doing this for each column and allowing s_j to be arbitrary at each step leads to the expressions

$$N(s)=\det\begin{vmatrix} 0 & -b_1/(s-s_1) & -b_2/(s-s_2) & -b_3/(s-s_3) & \cdots \\ 1 & 1+b_1H(s_1,s_1,s_i) & b_2H(s_2,s_1,s_j) & b_3H(s_3,s_1,s_k) & \cdots \\ 1 & b_1H(s_1,s_2,s_i) & 1+b_2H(s_2,s_2,s_j) & b_3H(s_3,s_2,s_k) & \cdots \\ 1 & b_1H(s_1,s_3,s_i) & b_2H(s_2,s_3,s_j) & 1+b_3H(s_3,s_3,s_k) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

$$D(s)=\det\begin{vmatrix} 1 & b_1H(s_1,s,s_i) & b_2H(s_2,s,s_j) & b_3H(s_3,s,s_k) & \cdots \\ 1 & 1+b_1H(s_1,s_1,s_i) & b_2H(s_2,s_1,s_j) & b_3H(s_3,s_1,s_k) & \cdots \\ 1 & b_1H(s_1,s_2,s_i) & 1+b_2H(s_2,s_2,s_j) & b_3H(s_3,s_2,s_k) & \cdots \\ 1 & b_1H(s_1,s_3,s_i) & b_2H(s_2,s_3,s_j) & 1+b_3H(s_3,s_3,s_k) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},\qquad (8)$$

where

$$H(s_i,s_j,s_k)=K(s_i,s_j)-K(s_i,s_k)$$
$$=\frac{(s_j-s_k)}{\pi}\int_1^\infty \frac{ds'\,\rho(s')}{(s'-s_i)(s'-s_j)(s'-s_k)}. \quad (9)$$

The verification of this form of the n-pole solution is identical to that discussed in Sec. II, and it is evident from (9) that the solution is valid for kinematic functions satisfying

$$\rho(s)\underset{s\to\infty}{\longrightarrow} s^{2-\epsilon}, \quad \epsilon>0.$$

Further manipulations of these forms are possible. For example, the necessary condition for a bound state of the amplitude at $s=s_0$ is that $\mathrm{cof}_0=0$ in (8) where $s_i=s_j=s_k=\cdots=s_0$. It is also possible to express the n-pole solution in terms of the determinants of n-by-n matrices by systematically subtracting the $(i-1)$th row from the ith row with $i=n, n-1, \cdots, 1$. The most suitable form of the solution will depend upon the way in which the numerator and denominator functions are to be evaluated. In any case, since the cofactors are constants, they can be evaluated once and the s dependence of the amplitude is contained in the simpler linear forms of Eq. (7). Also, for a number of problems, the kinematic integrals can be evaluated analytically, again simplifying the calculation.

IV. ONCE-SUBTRACTED AMPLITUDES

One drawback to the n-pole solution presented in Sec. II is the original assumption that the partial-wave amplitude satisfies an unsubtracted dispersion relation. In many problems of physical interest it seems likely that the partial-wave amplitudes do not vanish at infinity and will therefore require subtracted dispersion relations. It is also plausible that once-subtracted dispersion relations will suffice.[13] In this section it will be shown that the n-pole solution goes through for the subtracted case in much the same manner as for the unsubtracted one.

Let the partial-wave amplitude $A(s)$ satisfy the unitarity condition (1) and assume that the once-subtracted dispersion relation holds:

$$A(s)=A(s_0)+\frac{(s-s_0)}{\pi}\int_{-\infty}^0 \frac{ds'\,\mathrm{Im}A(s')}{(s'-s_0)(s'-s)} +\frac{(s-s_0)}{\pi}\int_1^\infty \frac{ds'\,\rho(s')|A(s')|^2}{(s'-s_0)(s'-s)},$$

where $A(s_0)$ is the subtraction constant. The N/D separation of this nonlinear integral equation is analogous to the unsubtracted case except for the fact that once-subtracted dispersion relations must be written for both $N(s)$ and $D(s)$. It is convenient to choose both subtraction points at $s=s_0$. Normalizing $D(s_0)=1$ then gives $N(s_0)=A(s_0)$ and the usual manipulations lead to the linear integral equation[14]

$$N(s)=A(s_0)-B(s_0)+B(s)+\frac{(s-s_0)}{\pi}\times\int_1^\infty \frac{ds'\,\rho(s')}{(s'-s_0)(s'-s)}[B(s')-B(s)]N(s'), \quad (10)$$

with

$$D(s)=1-\frac{(s-s_0)}{\pi}\int_1^\infty \frac{ds'\,\rho(s')N(s')}{(s'-s_0)(s'-s)}, \quad (11)$$

where the driving force $B(s)$ is given by the once-subtracted relation

$$B(s)=B(s_0)+\frac{(s-s_0)}{\pi}\int_{-\infty}^0 \frac{ds'\,\mathrm{Im}A(s')}{(s'-s_0)(s'-s)}. \quad (12)$$

As before, the n-pole problem is defined by approximating the discontinuity of the amplitude on the dynamic cuts by a sum of δ functions

$$\mathrm{Im}A(s)=-\pi\sum_{i=1}^n b_i\delta(s-s_i),$$

so the driving force (12) becomes

$$B(s)=B(s_0)-\sum_{i=1}^n \frac{(s-s_0)b_i}{(s_0-s_i)(s-s_i)}.$$

With this driving-force approximation the kernel of the integral equation is again degenerate and (10) can be written

$$N(s)=A(s_0)-(s-s_0)\sum_{i=1}^n \frac{b_i}{(s-s_i)} \times\left[\frac{1}{(s_0-s_i)}+\frac{1}{\pi}\int_1^\infty \frac{ds'\,\rho(s')N(s')}{(s'-s_0)(s'-s_i)}\right]. \quad (13)$$

Note that the one-subtraction assumption implies

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{b_i}{(s-s_i)}=\infty,$$

whereas

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{b_i}{(s_0-s_i)(s-s_i)}<\infty.$$

The solution of this problem must therefore have the property that whenever a residue b_i appears it is

[13] T. Kinoshita, Phys. Rev. 154, 1438 (1967).

[14] See, for example, C. F. Kyle (unpublished).

accompanied by the inverse square of the pole position s_i.

It is evident from (13) that $N(s)$ must have the form

$$N(s)=A(s_0)-(s-s_0)\sum_{i=1}^{n}\frac{c_i}{(s-s_i)},$$

where the constants c_i are the solutions of the algebraic equations

$$c_i=\frac{b_i}{(s_0-s_i)}+\frac{b_i}{\pi}\int_1^\infty\frac{ds'\,\rho(s')}{(s'-s_0)(s'-s_i)}\times\left[A(s_0)-(s'-s_0)\sum_{j=1}^{n}\frac{c_j}{(s'-s_j)}\right].$$

The full solution of the n-pole once-subtracted problem can be represented as the ratio of determinants of $(n+1)$ by $(n+1)$ matrices as before, where the determinants are

$$N(s)=\det\begin{vmatrix} A(s_0) & -\dfrac{(s-s_0)b_1}{(s_1-s_0)(s-s_1)} & -\dfrac{(s-s_0)b_2}{(s_2-s_0)(s-s_2)} & \cdots \\ 1-(s_1-s_0)A(s_0)K(s_0,s_1) & 1+b_1K(s_1,s_1) & \dfrac{(s_1-s_0)}{(s_2-s_0)}b_2K(s_2,s_1) & \cdots \\ 1-(s_2-s_0)A(s_0)K(s_0,s_2) & \dfrac{(s_2-s_0)}{(s_1-s_0)}b_1K(s_1,s_2) & 1+b_2K(s_2,s_2) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

$$D(s)=\det\begin{vmatrix} 1-(s-s_0)A(s_0)K(s_0,s) & \dfrac{(s-s_0)}{(s_1-s_0)}b_1K(s_1,s) & \dfrac{(s-s_0)}{(s_2-s_0)}b_2K(s_2,s) & \cdots \\ 1-(s_1-s_0)A(s_0)K(s_0,s_1) & 1+b_1K(s_1,s_1) & \dfrac{(s_1-s_0)}{(s_2-s_0)}b_2K(s_2,s_1) & \cdots \\ 1-(s_2-s_0)A(s_0)K(s_0,s_2) & \dfrac{(s_2-s_0)}{(s_1-s_0)}b_1K(s_1,s_2) & 1+b_2K(s_2,s_2) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

and $K(s_i,s_j)$ is given by (4). Again, the additional terms in the determinants can be supplied by inspection, and it should be noted that these determinants differ by a common multiplicative factor (i.e., cof_0) from the N and D of (10) and (11).

The verification of this solution and the discussion of its further properties exactly parallels the development in the preceding sections and will not be repeated here. As before, the solution has the nice property that all of the s dependence is confined to the zeroth row of the determinants so that N and D can be written in linear forms analogous to (7). It is also evident that the dependence on the subtraction constant $A(s_0)$ is highly nonlinear. For partial waves higher than s wave the subtraction point can be chosen to be the physical threshold where $A(s_0)=0$ because of the threshold conditions. The once-subtracted solution then contains no arbitrary parameter and is simpler in form. It is also of interest to note that the constant $B(s_0)$ introduced in (12) never enters the solution.[14] That is, the necessity of writing a once-subtracted dispersion relation for the driving force does not supply a second arbitrary parameter over and above the subtraction constant $A(s_0)$.

V. CONCLUSIONS

One question should be raised whenever approximate solutions of the N/D equations are discussed. Now that it is a relatively simple matter to solve the exact integral equation numerically, why should one worry about approximation schemes? Such schemes would certainly be unnecessary if the exact driving force, or equivalently the discontinuity of the amplitude across the dynamic cuts were known. Unfortunately, little precise information about these discontinuities is available.

For example, it is well known[15] that a very delicate cancellation between the contour integrals around the unphysical and the physical cuts must occur at threshold to ensure the correct threshold behavior of the phase shifts. Until a good deal more is known about the

[15] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

driving forces this cancellation must be handled with adjustable parameters. The n-pole approximation provides a convenient parametrization[16] in that the threshold conditions lead to constraints on the residues of the poles. It is also known[17] that this cancellation has an important effect on partial-wave calculations and cannot be ignored as has been the practice in the past.

Perhaps the most interesting application of the n-pole solution will be in self-consistent S-matrix calculations. Because the partial-wave driving forces are determined by the physical amplitudes in the crossed channels, and these in turn can be found by the N/D method for the crossed-channel partial-wave amplitudes,[18] it is advantageous to have a closed-form solution of the N/D equations. Self-consistent solutions involving more than a few partial-wave amplitudes would be extremely difficult to obtain if the integral equation had to be solved numerically at each step. But with the n-pole solution much of the problem reduces to a complicated but manageable algebraic problem. The n-pole solution, both for the unsubtracted and the once-subtracted case, also has the very important property that as the number of poles is increased the solution converges to the solution of the integral equation.

[16] J. Dilley, Nuovo Cimento (to be published).
[17] R. W. Childers and A. W. Martin (unpublished).
[18] See Ref. 14 for a formulation of the self-consistency problem in terms of convergent sums over partial-wave amplitudes alone.

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Unitarity and the Mandelstam Representation. I. Moment Conditions for the Double Spectral Functions*

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It was shown by Mandelstam that the double spectral function in the elastic strip is determined by double integrals over crossed-channel absorptive parts. The content of this consequence of elastic unitarity can be expressed in a number of different ways, some of which significantly simplify the problem of determining the spectral function for given absorptive parts. Among the results presented are rigorous moment conditions, "open-ended" integrals, and the Laplace transform of the double spectral function. An approximation procedure for the determination of the spectral function in terms of simple, one-dimensional integrals over the absorptive parts is developed.

I. INTRODUCTION

THE Mandelstam representation[1] for two-body scattering amplitudes has provided a powerful tool in the description of properties of scattering processes. As examples one may cite the Froissart bounds[2] for the high-energy behavior of the scattering amplitude and the numerous more recent results[3] of a similar nature. It is also evident that the requirements of crossing symmetry in the S-matrix approach to the strong interactions are most easily satisfied in a framework such as the Mandelstam representation.

But the Mandelstam representation has not been a useful tool in the actual calculation of scattering amplitudes, and the reason is a simple one. It is well known that the requirements of unitarity are of profound importance in the description of the strong interactions and the mathematical problem of imposing unitarity upon the Mandelstam representation is formidable. Not only is it a problem in two variables but also the complications of inelastic unitarity must be overcome.

In the elastic domain, Mandelstam derived[1] rigorous expressions for the double spectral functions in terms of integrals over combinations of crossed-channel absorptive parts. Insofar as these double spectral functions determine the imaginary part of the s-channel scattering amplitude in the elastic region, and hence the partial-wave phase shifts up to a sign, there is a good deal of physical information to be obtained from further investigation of this consequence of elastic unitarity. One practical difficulty with the expressions deduced by Mandelstam is the complicated structure of the double integrals which must be evaluated and their sensitivity to regions of integration in which the integrand becomes singular.

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[3] See, for example, T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. **135**, B1464 (1964); Y. S. Jin and A. Martin, *ibid.* **135**, B1375 (1964).