Unsubtracted Pion-Pion Dispersion Relation*

JERROLD FRANKLIN†
Lawrence Radiation Laboratory, University of California, Livermore, California
(Received 15 March 1967)

The de Alfaro–Fubini–Furlan–Rossetti assumption on the rapid falloff of amplitudes corresponding to pure $I=2$, t-channel exchange is applied to pion-pion forward dispersion relations, and a sum rule is obtained for the pion-pion scattering lengths. The sum rule is fairly well satisfied using the ρ, f, and low-energy s-wave contributions; but the inclusion of the recently discovered g_1 resonance requires the existence of at least one new $I=0$ resonance to satisfy the sum rule.

It has been suggested by de Alfaro, Fubini, Rossetti and Furlan (AFRF)[1] that linear combinations of forward scattering amplitudes corresponding to pure $I=2$ t-channel exchange will have imaginary parts that vanish like $s^{-\epsilon}$ ($\epsilon>0$) as s (the square of the total center-of-mass energy) becomes infinite. In Regge-pole theory, this would follow from $\alpha_2(0)<0$, where $\alpha_2(0)$ is the zero-momentum-transfer intercept of the leading $I=2$, t-channel Regge trajectory. As AFRF point out, if $\mathrm{Im}f(s)\sim s^{-\epsilon}$ as $s\rightarrow\infty$, then an unsubtracted dispersion relation can be written for $f(s)$. This idea has been applied by several authors.[2]

In this paper we apply the AFRF reasoning to the linear combination

$$f(s)=2A_0(s)-3A_1(s)+A_2(s) \tag{1}$$

of pion-pion elastic forward scattering amplitudes, corresponding to pure $I=2$, t-channel exchange. The amplitude $f(s)$ is proportional to the backward π^--π^+ elastic scattering amplitude. Using the AFRF assumption, we find sum rules for the $I=0$ and $I=2$ scattering lengths which are reasonably well satisfied keeping "well established" contributions (the ρ and f resonances, low-energy $I=0$ s-wave scattering, and Reggeized ρ exchange). We find no reason to exclude higher π-π resonances, however. In particular, inclusion of the recently discovered $I=1$ g_1 resonance[3,4] would require a corresponding $I=0$ resonance of high spin (probably $L=4$) and additional resonances would have to occur in $I=0$, $I=1$ combinations to preserve the sum rule.

The AFRF assumption allows us to write the unsubtracted dispersion relation[5]

$$f(\nu)=\frac{1}{4\pi^2}\int_1^\infty \frac{\nu'(\nu'^2-1)^{1/2}d\nu'}{\nu'^2-\nu^2}\times[2\sigma_0(\nu')-3\sigma_1(\nu')+\sigma_2(\nu')], \tag{2}$$

where

$$\nu=\tfrac{1}{2}(s-2), \tag{3}$$

and we have used unitarity to introduce $\sigma_I(\nu')$, the total cross section in isotopic spin state I, under the integral. The AFRF assumption implies that the linear combination of cross sections under the integral goes like $\nu^{-1-\epsilon}$ as $\nu\rightarrow\infty$, so that the integral will converge.

If we evaluate Eq. (2) at threshold ($\nu=1$), we get

$$2a_0+a_2=\frac{1}{4\pi^2}\int_1^\infty \frac{\nu d\nu}{(\nu^2-1)^{1/2}}\times[2\sigma_0(\nu)-3\sigma_1(\nu)+\sigma_2(\nu)], \tag{4}$$

where a_0 and a_2 are the $I=0$ and 2 pion-pion scattering lengths, respectively. We can also write the well-known sum rule

$$2a_0-5a_2=\frac{1}{4\pi^2}\int_1^\infty \frac{d\nu}{(\nu^2-1)^{1/2}}\times[2\sigma_0(\nu)+3\sigma_1(\nu)-5\sigma_2(\nu)]. \tag{5}$$

In the spirit of AFRF, the integral in Eq. (5) converges because the linear combination of cross sections under the integral corresponds to pure $I=1$ exchange in the t channel and the leading $I=1$ trajectory (the ρ trajectory) has $\alpha_\rho(0)<1$. Equations (4) and (5) can be solved for the individual scattering lengths, yielding

$$a_0=\frac{1}{8\pi^2}\int_1^\infty \frac{d\nu}{(\nu^2-1)^{1/2}}[\sigma_1(\nu)-\sigma_2(\nu)]+\frac{1}{48\pi^2}\times\int_1^\infty \frac{(5\nu+1)d\nu}{(\nu^2-1)^{1/2}}[2\sigma_0(\nu)-3\sigma_1(\nu)+\sigma_2(\nu)], \tag{6}$$

---

* Work performed under the auspices of the U. S. Atomic Energy Commission.

† Present address: Department of Physics, Temple University, Philadelphia, Pennsylvania.

[1] V. de Alfaro, S. Fubini, G. Rossetti, and G. Furlan, Phys. Letters 21, 576 (1966).

[2] See, e.g., H. Harari, Phys. Rev. Letters 17, 1303 (1966); 18, 319 (1967); B. Sakita and K. C. Wali, *ibid.* 18, 31 (1967).

[3] M. Goldberg, F. Judd, G. Vegni, H. Winzeler, P. Fleury, J. Huc, R. Lestinne, G. DeRosny, R. Vanderhaghen, J. Allard, D. Drijard, J. Hennessy, R. Huson, J. Six, J. Veillet, A. Floret, P. Musset, G. Bellini, M. DiCorato, E. Fiorini, P. Negri, M. Rollier, J. Crussard, J. Ginestet, and A. H. Tran, Phys. Letters 17, 354 (1965). The resonance observed in Ref. 3 is broader (180 MeV) and somewhat shifted (1670 MeV) from that in Ref. 4 and could include another effect as well.

[4] D. J. Crennell, P. V. C. Hough, G. R. Kalbfleisch, K. W. Lai, J. M. Scarr, T. G. Schumann, I. O. Skillicorn, R. C. Strand, M. S. Webster, P. Baumel, A. H. Bachman, and R. M. Lea, Phys. Rev. Letters 18, 323 (1967).

[5] We use the pion mass as our unit.

$$a_2=\frac{-1}{4\pi^2}\int_1^\infty \frac{d\nu}{(\nu^2-1)^{1/2}}[\sigma_1(\nu)-\sigma_2(\nu)]+\frac{1}{24\pi^2}\times\int_1^\infty \frac{(\nu-1)d\nu}{(\nu^2-1)^{1/2}}[2\sigma_0(\nu)-3\sigma_1(\nu)+\sigma_2(\nu)], \quad (7)$$

as sum rules for the pion-pion scattering lengths. The sum rules are sensitive to low-energy and high-energy contributions, so we propose to use them as tests of the high-energy contributions and the AFRF assumption rather than to determine the scattering lengths.

Contributions to the integrals in Eqs. (6) and (7) from the following sources are considered and are listed in Table I:

TABLE I. Contributions to the scattering lengths a_0 and a_2 given by Eqs. (6) and (7) for the listed sources described in the text. The low-energy $I=0$ contributions were calculated for a range of $0.2\leq a_0\leq 2.0$ and $-20\leq r_0\leq -1$. Those cases listed include the smallest and largest contributions to a_0. Positive r_0's give smaller contributions than the negative values listed. All lengths are in units of the pion Compton wavelength.

Contribution		a_0	a_2
Direct ρ resonance		−1.39	−0.60
Direct f resonance		+0.83	+0.32
Direct g_1 resonance		−1.10	−0.42
Regge ρ exchange		+0.01	−0.01
Low-energy s wave			
a_0	r_0	I_0	I_2
0.2	−1.0	+0.34	+0.11
0.2	−20.0	+0.21	+0.04
1.0	−1.0	+1.40	+0.32
1.0	−20.0	+1.02	+0.05
2.0	−1.0	+2.40	+0.38
2.0	−20.0	+1.93	+0.06

(1) The well-established ρ and f resonances for which we use the latest data compilation[6] and the zero-width approximation.

(2) The recently observed g_1 resonance[3,4] for which we use the values[4] $M_{g_1}=1630$ MeV, $\Gamma_{g_1}=100$ MeV, $J^P=3^-$, and the zero-width approximation, assuming the g_1 to be mainly elastic.

(3) The contribution of Reggeized ρ exchange to the first integral in Eqs. (6) and (7), for which we use parameters from the analysis of Barger and Olsson.[7]

(4) A low-energy $I=0$ contribution using the effective-range approximation[8] and a range of parameters.

We have left out

(1) Low-energy $I=2$ contributions. All indications are that these are small. Also, they tend to cancel between the first and second integrals of Eq. (6) for a_0, although the net contribution would be positive.

(2) The asymptotic contribution from $I=2$ exchange to the second integral in Eqs. (6) and (7). The AFRF assumption is that this contribution is small. A reasonable upper limit can be estimated by taking $\alpha_2(0)=-0.5$, with a coupling strength equal to that of the ρ. This would contribute 0.01 to a_0. It is difficult to see how this contribution could be much larger without the existence of a low-energy $I=2$ resonance for which there is no evidence.[9]

(3) Higher direct-channel pion-pion resonances. There are no convergence factors in the second integral in Eqs. (6) and (7), so that there is no *a priori* reason, other than lack of knowledge about them, to leave out other resonances. Their contribution would go like $xL\Gamma/W$, where L is the spin of the resonance, x is the branching ratio to two pions, Γ is the full width in energy, and W is the total energy of the resonance. The experimental indications and suggestions from Regge theory are that L is roughly proportional to W^2, so that only increasing inelasticity (which is to be expected eventually) or the total absence of higher resonances would justify leaving them out. We will later discuss higher resonances and the probable need for them.

We concentrate on the $I=0$ scattering length a_0, for which the indications are that it is positive with estimates ranging from the current algebra, partially conserved axial-vector current (PCAC) result[10] of at least 0.2 up to some experimental estimates of the order of 1.[11] The indications for a_2 are that it is not large and is possibly negative. Keeping all contributions (1,2,4) to Eq.

[6] A. H. Rosenfeld, Angela Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Roos, and W. J. Willis, Rev. Mod. Phys. **39**, 1 (1967).

[7] V. Barger and M. Olsson, Phys. Rev. **146**, 1080 (1966).

[8] We use $[(s-4)/s]^{1/2}\cot\delta=1/a+\frac{1}{8}(s-4)r$. This implies that $\delta=90°$ at $s=-8/ar+4$.

[9] Most experimental statements in this paper are derived from G. Goldhaber, B. C. Shen, N. P. Samios, A. Astier, and K. W. Lai, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967). I would also like to thank Professor Gerson Goldhaber for a copy of his rapporteur talk at that conference and for a useful discussion.

[10] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966); N. N. Khuri, Phys. Rev. **153**, 1477 (1967); K. Kang and T. Akiba, Phys. Letters **25B**, 35 (1967).

[11] L. W. Jones, D. O. Caldwell, B. Zacharov, D. Harting, E. Bleuler, W. C. Middelkoop, and B. Elsner [Phys. Letters **21**, 590 (1966)] find $|a_0|=0.78\pm0.15$. They also observe a change in sign of the forward-backward asymmetry, one explanation for which would be a negative scattering length and the implication of a negative $I=0$ phase shift throughout the range 280–750 MeV, since they see no indication of a zero in the phase shift. However, L. D. Jacobs and W. Selove, Phys. Rev. Letters **16**, 669 (1966); W. D. Walker, J. Carroll, A. Garfinkel, and B. Y. Oh, *ibid* **18**, 630 (1967) do not see a change in sign of the asymmetry and indicate a positive phase shift. R. W. Birge, R. P. Ely, Jr., G. Gidal, V. Hagopian, G. E. Kalmus, W. M. Powell, K. Billing, F. W. Bullock, M. J. Esten, M. Govan, C. Henderson, W. J. Knight, D. J. Miller, F. R. Stannard, S. Tovey, O. Treutler, U. Camerini, D. Cline, W. F. Fry, H. Haggerty, R. H. March, and W. J. Singleton [in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967)] find $a_0=1.2\pm1.0$ from angular correlations in K_{e4} decays. They find $\delta_0-\delta_1=27\pm18°$, averaged over the $\pi-\pi$ decay spectrum, which is the most significant experimental evidence that a_0 is positive.

(6) but the g_1 resonance results in

$$a_0=-0.55+\frac{1}{24\pi^2}\times\int_1^\infty\frac{d\nu(5\nu+1)}{(\nu^2-1)^{1/2}}\sigma_{00}{}^{\text{el}}(\nu)=-0.55+I_0, \quad (6')$$

$$a_2=-0.29+\frac{1}{12\pi^2}\times\int_1^\infty\frac{d\nu(\nu-1)}{(\nu^2-1)^{1/2}}\sigma_{00}{}^{\text{el}}(\nu)=-0.29+I_2, \quad (7')$$

where $\sigma_{00}{}^{\text{el}}(\nu)$ is the elastic s-wave, $I=0$ cross section. It is possible to almost satisfy Eq. (6′) for the range of $I=0$ scattering lengths from $a_0=0.2$ to $a_0=2.0$ (adjusting r_0),[12] although $a_0\gtrsim 0.5$ would imply a low-energy σ resonance to satisfy the sum rule. The $I=2$ scattering length turns out to be $-0.3\leq a_2\leq +0.1$, which is not unreasonable.

If the g_1 contribution[13] is added, however, then

$$a_0=-1.65+I_0, \quad (6'')$$

$$a_2=-0.72+I_2. \quad (7'')$$

Now Eq. (6″) cannot be satisfied for any reasonable positive a_0. Our conclusion, assuming the AFRF convergence and considering the weight of evidence that a_0 is positive,[10,11] is that the existence of the g_1 meson requires the existence of at least one additional $I=0$ π-π resonance,[14] if the sum rule of Eq. (6) is to be satisfied. Also, any new $I=1$ resonances[15] would have to be "balanced" in Eq. (6) by new $I=0$ resonances.

The AFRF assumption on the convergence of the $I=2$ exchange amplitude is, of course, crucial to the existence of the sum rule given by Eq. (4). Recently, Muzinich[16] has argued that the existence of a Regge-cut contribution from double ρ exchange would lead to an "effective" asymptotic power (to within a logarithm) for the cut contribution of $\alpha_{\text{eff}}=2\alpha_\rho(\frac{1}{4}t)-1$ with an undetermined coupling coefficient. Analyses of $\pi^- - p$ charge-exchange scattering[16] indicate that $\alpha_\rho(0)\simeq\frac{1}{2}$, so that including the cut contribution could invalidate $I=2$ exchange sum rules. However, if the coupling coefficient of the cut contribution is small, the unsubtracted dispersion relation could be written for an approximate amplitude which does not include the cut contribution.[17] The importance of the undetermined cut contribution can in principle be determined by experimental tests of sum rules like Eq. (4) and $I=2$ superconvergence relations.

We have assumed positive a_0 because that is the experimental indication,[11] but this assumption is not necessary. Goebel[18] has derived a rigorous extreme lower limit of $a_0>-0.56$. If reasonable assumptions about low-energy (280–750 MeV) π-π scattering are made, especially the one that the $I=0$ scattering is larger than the $I=2$ scattering in this energy range, which is indicated by experiment, then Goebel's lower limit becomes $a_0>-0.2$. Thus, even if a_0 were negative, it could not be large enough in magnitude to satisfy Eq. (6″).

[12] The scattering length a_0 as given by Eq. (6′), can be brought within from 0.1 to 0.3 of the value used in the effective-range formula under the integral. While not completely self-consistent, this result might be considered acceptable in view of the large cancelations between other contributions.

[13] The g_1 contribution given in Table I assumes a purely elastic g_1 resonance. The experiment of Ref. 4 only observes the g_1 in its elastic mode and puts about a 40% lower limit to the elasticity. This lower limit would still give a large enough contribution to significantly affect the sum rule.

[14] Unless its width is abnormally large, relatively high spin and elasticity would be required for a resonance to affect the sum rule appreciably. There is some evidence in the $\pi^- - \pi^+$ spectrum of Ref. 4 for an $I=0$ enhancement of undetermined parameters at 1750 MeV.

[15] M. N. Focacci, W. Kienzle, B. Levrat, B. C. Maglić, and M. Martin [Phys. Rev. Letters **17**, 890 (1966); **17**, 1205(E) (1966)] find considerable structure in the singly-charged-boson higher mass spectrum which, if $I=1$, could further unbalance the sum rule. However, these peaks seem to have low elasticity and might not be important. See also Ref. 9.

[16] I. J. Muzinich, Phys. Rev. Letters **18**, 381 (1967).

[17] See Keiji Igi (to be published) for a discussion of this possibility. I would like to thank Professor Stanley Mandelstam for a useful discussion of this point.

[18] C. Goebel, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967). Goebel uses Eq. (5) and a dispersion relation for the inverse of A_0+2A_2 which can be written if a_0 is negative.