Relativistic Parametrization of Resonances: the ρ Meson

JAMES S. BALL AND MICHAEL PARKINSON
Department of Physics, University of California, Los Angeles, California
(Received 11 April 1967; revised manuscript received 26 July 1967)

We construct a simple formula for a resonant scattering amplitude which employs relativistic phase space and possesses reasonable analytic properties in the total-energy variable. This formula is a direct analog of the nonrelativistic Breit-Wigner formula and contains the same number of parameters. With it, one can immediately calculate electromagnetic form factors and determine the relative contribution of each channel to the formation of the resonance. Application to the ρ meson indicates that the ρ resonance alone is not sufficient to explain the isovector form factor even when many channels are taken into account. It is also noted that the dominant channels for making the ρ are the $N\bar{N}$ and $\pi\omega$, but not the $\pi\pi$. The coupling of the $\pi\omega$ channel to the ρ produces a $\pi\pi\rightarrow\pi\omega$ cross section which has an enhancement near the B-meson mass due solely to the decrease of the ρ pole with increasing energy and the phase space of the $\pi\omega$ channel. The value of the isovector form factor in the timelike region is also calculated. Values for the p-wave $\pi\pi$ phase shift are computed and compared with the nonrelativistic prediction.

I. INTRODUCTION

THE dominance of nearby singularities in a scattering amplitude has been exploited extensively in S-matrix theory treatments of strong interactions. In particular the dominance of a nearby pole plus the requirements of unitarity have been used to describe the behavior of the scattering amplitude in the neighborhood of that pole. The size of the region in which the scattering amplitude is accurately approximated is roughly bounded by the nearest singularity that is incorrectly treated. The most commonly used parametrization in the case that the pole corresponds to a resonance is the Breit-Wigner (BW) formula. This treatment, while exact at the pole, suffers from several deficiencies: (1) Since only open channels enter into the BW formula it cannot describe a resonance with energy just below a closed-channel threshold as it neglects this nearby singularity. (2) It produces poles on all sheets of the scattering amplitude. (3) The position of the pole is an independent parameter which is not related to the partial widths and hence the BW formula says nothing of the role each channel plays in the dynamical origin of a resonance. Relativistic kinematics produce numerous singularities making the analytic properties of the amplitude complicated if not incorrect. (5) The BW formula gives no direct information about form factors that may contain the resonance as an intermediate state.

The treatment of resonances in the K-matrix formalism overcomes the first difficulty, and has been used extensively in treating resonances of poles near thresholds.[1] It also can describe the case where a pole exists only on certain sheets of the scattering amplitude.[2] Deficiencies 3–5 still exist in the usual K-matrix formulation.

The ND^{-1} formulation with the left-hand singularities approximated by a pole, as described by Shaw and Nath,[3] overcomes the remaining difficulties but at considerable cost in complication.

In this paper we will obtain a parametrization of a resonant-scattering amplitude which overcomes all of the above-mentioned difficulties but is still rather similar to the BW formula and contains no more free parameters.

II. SIMPLE ND^{-1} DESCRIPTION OF A RESONANCE

At a fixed energy an n-channel scattering T matrix is completely determined by specifying the elements of the K matrix which is real and symmetric. Thus T depends in general on $\frac{1}{2}n(n+1)$ parameters. For the special case of a first order pole in the T, the residue is a factorizable matrix depending on only n parameters. The BW formula is obtained by assuming that the residues of the pole are independent of energy except for phase-space factors.

If we consider the matrix ND^{-1} equations in which the left-hand singularities have been approximated by a pole, we find that the integral equations for N and D reduce to quadrature. The following results are obtained for N and D:

$$N_{ij}=B_{ij}/(s+s_p)\,,$$
$$D_{ij}=\delta_{ij}-\frac{s+s_p}{\pi}\int_{t_i}^{\infty}\frac{\rho_i(s')B_{ij}ds'}{(s'+s_p)^2(s'-s)}\,,$$
$$=\delta_{ij}+B_{ij}R_i(s)\,,$$

where

$$M_{ij}=(ND^{-1})_{ij}=T_{ij}/(\rho_i\rho_j)^{1/2}\,,$$

B_{ij} is the residue of the left-hand pole at $-s_p$ in M_{ij}, t_i is the threshold of channel i, and $\rho_i(s)$ is the phase-space factor for the ith channel.

With the above definition,

$$\sigma_{ij}^{(J)}=[4\pi(2J+1)/k^2]\rho_i\rho_j|M_{ij}|^2\,,$$

[1] R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, Oxford, England, 1962), Chaps. 6 and 7; F. Uchiyama-Campbell, Phys. Letters **18**, 189 (1965); A. W. Hendry and R. G. Moorhouse, *ibid.* **18**, 171 (1965); P. N. Dobson, Phys. Rev. **146**, 1022 (1966).

[2] W. Frazer and A. W. Hendry, Phys. Rev. **134**, B1307 (1964).

[3] P. Nath and G. L. Shaw, Phys. Rev. **138**, B702 (1965).

where $\sigma_{ij}^{(J)}$ is the total cross section for the process $i \to j$ with total angular momentum J, and k is the incident momentum in channel i in the center-of-mass system.

Thus we see that for even the simplest phenomenological interaction the ND^{-1} equations depend on $\frac{1}{2}n(n+1)$ parameters, the residues of the interaction pole. It is clear that in a phenomenological description of a resonance, only n parameters and the position of the interaction pole can be determined from the partial widths and the position of the resonance, and the remaining $\frac{1}{2}n(n-1)$ parameters can only be determined by the detailed energy dependence of the partial wave in question. The number of undetermined parameters is quite acceptable for $n=1$ or 2, but in strong coupling physics where all channels with the same quantum numbers are essentially equal, one may easily imagine seven important channels coupled to the ρ meson. In this case we are left with 21 parameters after the first 8 are fixed by the data.

In view of the disparity mentioned above, we propose the following parametrization of a resonance containing no undetermined parameters, but possessing all of the advantages of the ND^{-1} parametrization.

If we assume that the input residue is factorizable, we obtain

$$N_{ij}=C_iC_j/(s+s_p),$$

and

$$D_{ij}=\delta_{ij}+C_iC_jR_i(s),$$

where we have set $B_{ij}=C_iC_j$ and

$$R_i(s)=-\frac{s+s_p}{\pi}\int_{t_i}^{\infty}\frac{\rho_i(s')ds'}{(s'+s_p)^2(s'-s)}. \tag{1}$$

It is easy to show that

$$\det D=1+\textstyle\sum_i R_i(s)C_i^2,$$

and

$$(D^{-1})_{ij}=\delta_{ij}-R_i(s)C_iC_j/\det D.$$

Thus, $M=ND^{-1}$ yields

$$M_{ij}=C_iC_j/[(s+s_p)(1+\textstyle\sum_i R_i(s)C_i^2)]. \tag{2}$$

The simplicity of this result is due to the algebraic simplicity of N_{ij}. This is seen explicitly in the Appendix.

It is clear that the n residue parameters and the pole position can be determined from the partial width and resonance energy. The ratios M_{ij}/M_{ik} are independent of energy because of the simple form of D^{-1}. Furthermore, in the nonrelativistic limit, the above expressions reduce to the BW formula.

By employing the above formula we obtain a parametrization with the following properties:

(a) Relativistic kinematics are employed without introducing unwanted singularities.

(b) The scattering amplitude is completely determined by the resonance parameters.

(c) Since the amplitude is resolved into N and D^{-1}, the D functions can be identified and the form factors resulting from considering the resonance as an intermediate state in the coupling of each channel to the photon (provided the resonance is 1^- of course) are as follows:

$$F_i(s)=\sum_{j,k}F_j(0)D_{jk}(0)D_{ki}^{-1}(s). \tag{3}$$

(d) The expression for D^{-1} is sufficiently simple so that it can easily be generalized to a continuously infinite set of channels. The sum over i goes over into an integral. Thus it is possible to handle many-body thresholds in this approximation. For example, an s-wave 3π threshold can be represented by

$$R_3=\frac{s+s_p}{\pi}\int_{(3m_\pi)^2}^{\infty}ds'\int_{(2m_\pi)^2}^{(\sqrt{s'}-m_\pi)^2}\frac{\rho_t(s')dt}{(s'+s_p)^2(s'-s)},$$

where

$$\rho_t=\left(\frac{s-2[(\sqrt{t}-m_\pi)^2+m_\pi^2]+[(\sqrt{t}-m_\pi)^2-m_\pi^2]^2/s}{4s}\right)^{1/2}.$$

(e) If threshold properties are expressed as linear relations between T-matrix elements, the threshold behavior is guaranteed automatically by the input residues.

(f) Since the value of s_R is determined by

$$\textstyle\sum_i C_i^2R_i(s_R)+1=0,$$

the relative contributions of each channel toward producing the resonance (in this model) is given by

$$C_i^2R_i(s_R).$$

This at least gives a rough indication of the relative importance of each channel in the dynamical origin of the resonance.[4]

With $\rho_i(s)=p^3/E=[(s-t_i)^3/(16s)]^{1/2}$, appropriate for a p-wave channel, we find that as $t_i\to\infty$, $R_i(s_R)\sim 1/t_i$, which is encouraging. This means that, all other things being equal, the more distant the threshold, the smaller the contribution to the formation of the resonance, as we would expect.

It is interesting to note, however, that while thresholds above the resonance always produce positive contributions to the formation of the resonance, the same is *not* true of all the thresholds *below* the resonance. In fact, if the resonance is far enough above the threshold t_i, then $\mathrm{Re}[R_i(s_R)]<0$. In such a case, it would appear that such a channel would have to be considered as impeding rather than helping the formation of the resonance.

We would also like to point out that the use of a factorizable residue for the interaction pole is not without some theoretical justification. If one considers a

[4] We wish to thank G. F. Chew for calling to our attention the above interpretation of the relative contributions of each channel to the formation of a resonance.

TABLE I. The necessary data for the parametrization of the ρ meson as described in the text. The t_i and u_i are the thresholds. For any channel $t=(m_1+m_2)^2$ and $u=(m_1-m_2)^2$, where m_1 and m_2 are the masses of the particles involved.

Channel (i)	Particles	Masses	Thresholds (t_i)	(u_i)	Coupling constants (C_i)	Photon coupling constants	Functions (ρ_i)
1	$\pi\pi$	$m_\pi=1$	4.0	0	0.90	e	$[(s-t_1)^3/16s]^{1/2}$
2	$\pi\omega$	$m_\omega=5.61$	43.7	21.2	0.47	$0.56e$	$[(s-t_2)^3(s-u_2)^3]^{1/2}/(16s)$
3	$K\bar{K}$	$m_K=3.56$	50.7	0	0.45	$e/2$	$[(s-t_3)^3/16s]^{1/2}$
4	$\rho\eta$	$m_\rho=5.47$	88.4	2.37	0.27	$0.33e$	In ρ_2, let $m_\pi\rightarrow m_\rho$, $m_\omega\rightarrow m_\eta$
5	$K^*\bar{K}$	$m_\eta=3.93$	98.8	7.90	0.47	$0.56e$	In ρ_2, let $m_\pi\rightarrow m_K$, $m_\omega\rightarrow m_{K^*}$
6	$N\bar{N}(++)$	$m_{K^*}=6.38$	180.6	0	$0.64(1+3.7s/4m_N^2)$	$e/2$	$m_N^2[(s-t_6)/s]^{1/2}$
7	$N\bar{N}(+-)$	$m_N=6.72$	180.6	0	3.00	$2.35e$	$[(s-t_7)s/4]^{1/2}$

dynamical model for n channels, each with two equal mass particles (i.e., m_i for the ith channel), in which the interaction arises from the exchange of a single particle of mass m, the coupling-constant matrix for that pole in the momentum-transfer variable will be of rank one (factorizable) in the crossed channel. The multiplication by the crossing matrix to obtain the direct-channel interaction cannot increase the rank of this matrix. The resulting interaction is of the form

$$B_{ij}^l(s)=\frac{G_{ij}}{(q_iq_j)^{l+1}}Q_l\left(\frac{m^2-m_i^2-m_j^2+\frac{1}{2}t_i}{2q_iq_j}\right).$$

If one now approximates B_{ij} by a pole and adjusts the residue so that the pole approximation agrees with the Born term at the threshold $t_i=4m_i^2$, then

$$C_{ij}=(s_i+s_p)B_{ij}^l(t_i)\propto G_{ij}\frac{1}{(m^2-m_i^2-m_j^2+\frac{1}{2}t_i)^{l+1}}.$$

Thus, if $|m_i^2-m_j^2|\ll m^2$, C_{ij} is factorizable. So, we see that for single-particle exchange, C_{ij} is factorizable provided the mass of the exchanged particle is large compared to the energy separation of the channels involved.

III. PARAMETRIZATION OF THE ρ MESON

Let us consider the 7-channel problem consisting of the $J^P=1^-$, $I=1$, states of $\pi\pi$, $\pi\omega$, $K\bar{K}$, $\rho\eta$, $K^*\bar{K}$, $N\bar{N}(++)$, $N\bar{N}(+-)$. The $(+\pm)$ refers to the helicity values of the $N\bar{N}$ system, properly symmetrized to give the correct parity states.[5] This is a situation with one known resonance—the ρ meson. The equations developed in Sec. II may be applied directly to give a one-resonance approximation to the scattering amplitude.

A. Calculational Details

In Table I is to be found a synopsis of all the numbers that enter into the calculation $C'^2=g^2/12\pi$ and we have used $g_{\rho\pi\pi}^2/4\pi=2.0$, $g_{\rho K\bar{K}}^2=\frac{1}{2}g_{\rho\pi\pi}^2$, $g_{\rho\omega\pi}^2/4\pi=0.67m_\pi^{-1}$, $g_{\rho\rho\eta}^2=\frac{1}{3}g_{\rho\omega\pi}^2$, and $g_{\rho K^*+K^-}^2=\frac{1}{4}g_{\rho\omega\pi}^2$. The relations were obtained from $SU(3)$, the values for $g_{\rho\pi\pi}^2$ and $g_{\rho\omega\pi}^2$ from the widths of the ρ and ω mesons.[6] The couplings we have used are $V_\mu[(\partial_\mu P^\dagger)P-P^\dagger(\partial_\mu P)]$, $\epsilon_{\mu\nu\lambda\sigma}\partial_\mu V_\nu\partial_\lambda V_\sigma'P$, and $\frac{1}{2}V_\mu\bar{N}[a\gamma_\mu+(b/2m_N)\sigma_{\mu\nu}P_\nu]N$, where V is the vector meson, P is the pseudoscalar meson, N is the nucleon, and $P=p_N+p_{\bar{N}}$. The $N\bar{N}(++)$ coupling constant is $\frac{1}{2}(a+bs/4m^2)$, that for $N\bar{N}(+-)$ is $\frac{1}{2}(a+b)$. It is known that $a^2/4\pi\simeq2.0$ and the anomalous isovector magnetic moment of the nucleon suggests that $(a+b)/a=2.35/0.5$.

The coupling constants in Table I do not go directly into formula (2), but must be multiplied by a factor depending on s_p and the $R_i(s)$. This factor will be discussed in a moment. Table I also gives the ρ functions chosen. In order to make the integrals over channels 2, 4, and 5 converge, the following convergence factor was used:

$$c(s,s_0)=(s_0/s)\theta(s-s_0)+\theta(s_0-s).$$

This yields the same asymptotic behavior for each $R_i(s)$, and was chosen for that reason.

B. Nucleon-Coupling Constants

The $NN(++)$ amplitude leads to minor difficulties. One would expect that $C_6(s)$ should be evaluated at $s=m_\rho^2$. This causes two problems:

(1) The $^3D_1$ threshold behavior requires $C_6=C_7$.

(2) $G_M^V(0)/G_E^V(0)=2.35/0.5$ suggests $(C_6+C_7)/C_6=2.35/0.5$. [$G_M^V(q^2)$ and $G_E^V(q^2)$ are the isovector electromagnetic form factors of the nucleon.] Allowing C_6 to vary with s as shown in the table solves both these problems but introduces two more:

(3) Two unwanted poles appear on the physical sheet, fairly far away, however.

(4) $G_M(q^2)/G_E(q^2)\simeq G_M(0)/G_E(0)$ for $0\leq q^2\lesssim50m_\pi^2$.

The unwanted poles are not a serious difficulty; they are far away and we are only approximating an analytic function anyway. Point (4) was solved at the cost of some more analyticity, by taking

$$\begin{aligned}C_6&=0.64, & s&\leq0\\ &=0.64(1+3.7s/4m_N^2), & s&\geq0.\end{aligned}$$

The approximation still has all the features it had before, plus threshold behavior and a nice form-factor relationship. These adjustments are not severe ones, and it is hard to believe that they seriously affect the results

[5] M. Jacob and G. Wick, Ann. Phys. (N. Y.) 1, 427 (1959).

[6] See Appendix II of Ref. 13 for a discussion of the details of these determinations.

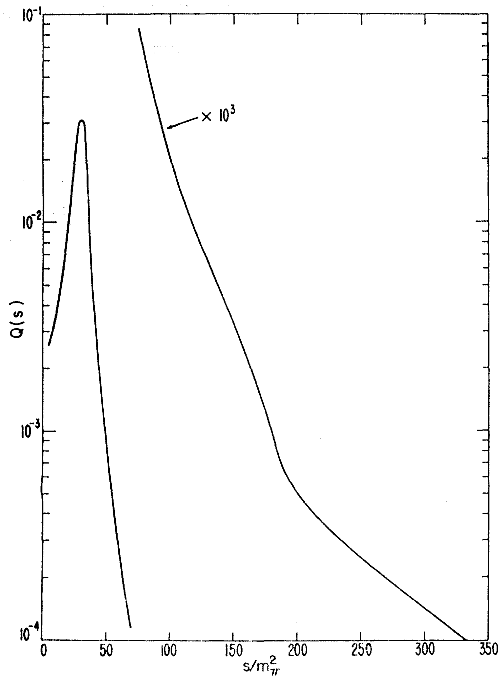

FIG. 1. The function $Q(s)$, as described in the text.

of the calculation. That this is so was checked by calculations with C_6=constant; the essential features of the results were unchanged.

Using the parameters and ρ functions of Table I and the nucleon-coupling constants as described above, the procedure of the Appendix was carried out by a computer, which found

$$s_p=+50.9m_\pi^2,\quad k_f=1.08,$$

with

$$s_0=120m_\pi^2.$$

As described in the Appendix, the C_i which appears in (2) are related to the C_i' of Table I via the relation $C_i=k_fC_i'$.

Having determined s_p and the C_i, one may use (2) immediately to calculate cross sections, and (3) to calculate form factors.

The cross sections all depend on the function

$$Q(s)=|1/[(s+s_p)(1+\sum_{i=1}^{7}R_i(s)C_i^2)]|^2$$

in the following way:

$$\sigma_{ij}=(12\pi/k^2)\rho_i\rho_jC_i^2C_j^2Q(s),$$

where the ρ's are as given in Table I and k is the incident momentum in the center of mass. This function $Q(s)$ is plotted in Fig. 1 for s_p, s_0, and the C_i as given in the previous paragraph. Thus, if desired, it is a simple matter to calculate the cross sections we do not explicitly discuss in the next section.

C. Results

Some of the results have been selected and presented in Figs. 2–4. The $\pi\pi$ cross section shows the expected peak at the ρ mass (Fig. 2). The $\pi\pi\to\pi\omega$ cross section (Fig. 3) shows a bump near the B-meson mass, although *no resonance* has been inserted near this peak. This is entirely an effect due to the decreasing contribution of the ρ pole and the threshold behavior of the channel. Comparison with the experimental curve of Abolins *et al.*[7] (see Fig. 4) shows some similarity. No parameters have been juggled here; and it is clear that one can obtain a "B" meson merely from the decay of the off-the-mass-shell ρ meson.[8]

In Fig. 4 is seen the nucleon form factor for spacelike q^2, normalized to 1 at $q^2=0$. The available experimental information has also been indicated, also normalized to 1 at $q^2=0$. As one can see from inspection, the agreement, while qualitatively good, is not good in a quantitative sense.

The curve of Fig. 4 was produced after many trial values of the ρNN-coupling constants. The reader may have noticed in Table I that $g_{\rho\pi\pi}/g_{\gamma\pi\pi}=g_{\rho\omega\pi}/g_{\gamma\omega\pi}=\cdots=g_{\rho K^*K}/g_{\gamma K^*K}=0.90/e$, but that $g_{\rho NN}(s=0)/g_{\gamma NN}=1.4/e$. The reason for the difference is that $g_{\rho NN}(s=0)/g_{\gamma NN}>0.90/e$ makes the nucleon form factor fall off more quickly with increasing q^2. The final choice for $g_{\rho NN}$ reflects this fact. If one chooses $g_{\rho NN}(s=0)/g_{\gamma NN}=0.90/e$, then the form factors for all channels are proportional, and are given by

$$F_i(s)=F_i(0)/[1-\sum_{i=1}^{7}R_i(s)C_i^2],$$

which simply says that they all look like $1/(q^2+m_\rho^2)$.

In the timelike region, preliminary results indicate that for $s=350m_\pi^2$, $|G_M|\lesssim 0.2$.[9] The theory here yields $|G_M^V|\cong 0.5$. Not knowing the isoscalar G_M, there is not much more one can say.

D. Relative Contributions to ρ Meson

By examining the various $R_i(s)C_i^2$ at $s=m_\rho^2$, one can see what channels most contribute to the formation of the ρ meson in this calculation. One finds that the

$$\rho\simeq(10\%)\pi\pi+(30\%)\pi\omega+(20\%)K^*K+(40\%)\bar{N}N(+-).$$

The actual numbers, as obtained from the computer

[7] M. Abolins *et al.*, Phys. Rev. Letters 11, 381 (1964).

[8] For a more detailed and somewhat different discussion of the $\rho\to\pi\omega$ decay and the B meson, see M. Parkinson, Phys. Rev. Letters, 18, 270 (1967).

[9] A. Tollestrup *et al.*, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, California, 1966* (University of California Press, Berkeley, California, 1967). We would like to thank Professor Tollestrup for a discussion of his results.

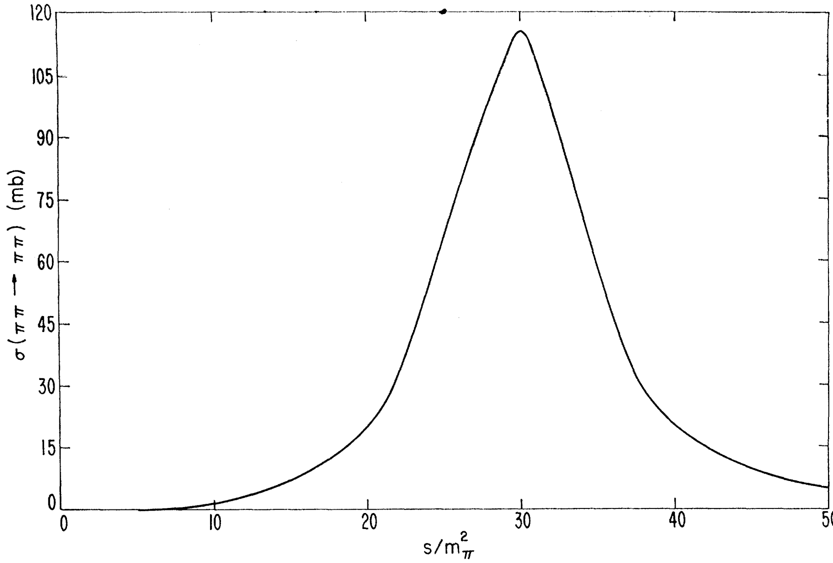

FIG. 2. The $\pi\pi \rightarrow \pi\pi$ total cross section.

run, were:

$\pi\pi$	9%
$\pi\omega$	26%
$K\bar{K}$	1%
$\rho\eta$	6%
$K^*\bar{K}$	16%
$N\bar{N}(++)$	1%
$N\bar{N}(+-)$	42%.

One sees that, if anything, the ρ should be considered as a $N\bar{N}$ bound state rather than a $\pi\pi$ resonance, even though it is only seen directly in the $\pi\pi$ channel.[4]

One cannot help noticing the smallness of the $\pi\pi$ contribution to the ρ. It turns out that by increasing slightly the coupling constants involved in this calculation, which shifts s_p to smaller values, one can make the $\pi\pi$ contribution zero or even negative. This is because

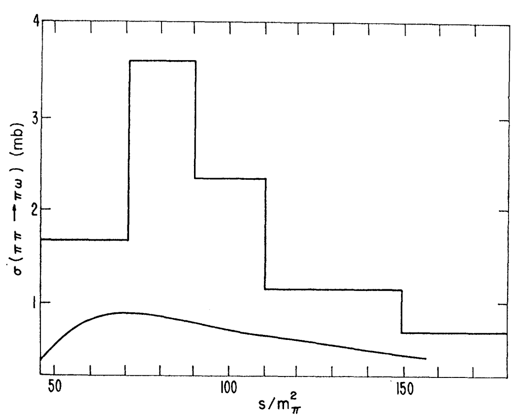

FIG. 3. The $\pi\pi \rightarrow \pi\omega$ total cross section. The solid curve is the theoretical prediction. The histogram is from Abolins *et al.* (Ref. 7), who constructed it by Chew-Low extrapolation of their $\pi p \rightarrow \pi\omega p$ data. For another, somewhat different, discussion of this process, see also Ref. 8.

$\text{Re}[R_{\pi\pi}(s)]$ just happens to have its second zero very close to $s=m_\rho^2$ (one zero is at $s=-s_p$). This is an explicit example of the remark made at the end of Sec. II.

IV. SIMPLE ANALYTIC FORM FOR ρ RESONANCE

In view of the fact that experimentalists are now attempting to determine $\pi\pi$ phase shifts,[10] it would seem to be useful to have a closed and relatively simple analytic form that describes the ρ meson. The expression to be derived here is suggested by K-matrix theory rather than the N/D method, although both are equivalent.

First of all, we have from unitarity

$$T_{ij}^{(l)-1}=K_{ij}^{(l)-1}-i\rho_{ij}^{(l)}\theta(s-t_i)\,, \tag{4}$$

where $\rho_{ij}^{(l)}=\rho_i^{(l)}(s)\delta_{ij}$, which gives the usual definition of the K matrix; as before, t_i is the ith threshold, and $\rho_i(s)$ is the phase-space function for the ith channel.

However, Eq. (4) is not analytic, due to the step function. This can be remedied by replacing $\rho_{ij}(s)\theta(s-t_i)$ by

$$R_{ij}(s)=\int_{t_i}^{\infty}\frac{1}{\pi}\frac{\rho_{ij}(s')ds'}{s'-s'-i\epsilon}$$

(the limit $\epsilon\rightarrow 0$ is understood), and now defining the K matrix by the equation

$$T^{-1}(s)=K^{-1}(s)-R(s)\,. \tag{5}$$

If the integral defining R does not converge, we can make subtractions, as in dispersion theory; e.g.,

$$R(s)=a+\frac{s-s_0}{\pi}\int_t^{\infty}\frac{\rho(s')ds'}{(s'-s_0)(s'-s'i\epsilon)}$$

[10] W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967); P. Schlein and E. Malamud (to be published). We would like to thank Professors Schlein and Malamud for a discussion of their work.

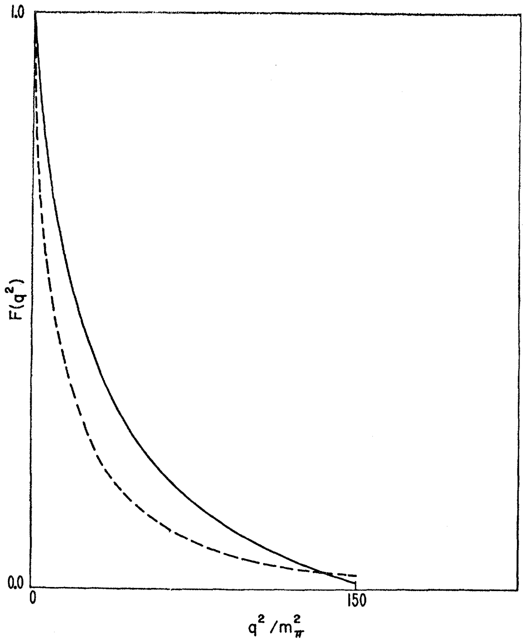

FIG. 4. The nucleon isovector form factor, normalized to 1 at $q^2=0$, shown for spacelike q^2. The dashed curve is the (very good) fit to experiment given by the formula $m^4/(q^2+m^2)^2$ with $m^2=36m_\pi^2$ [see K. W. Chen *et al.*, Phys. Rev. **141**, 1267 (1966)]. The theoretical curve (solid line) was produced using (3) and the parameters of Sec. III of the text. The experimental error bars associated with the dashed curve are much smaller than the maximum spacing between the two curves.

if one subtraction is enough. Next, at s_R, the resonance position, we know the $T^{-1}(s_R)$ has a simple zero; furthermore, K^{-1} contains the coupling constants and must also have a simple zero.[11] Therefore, as $s\to s_R$, $K^{-1}\to C^{-1}\times(s_R-s)$ where C factorizes (i.e., $C_{ij}=c_ic_j$), since it is just the matrix of the coupling constants. In order that $R(s)$ make no contribution at $s=s_R$, we want $R(s)\propto(s-s_R)^2$ near $s=s_R$. Thus, we want *two* subtractions at $s=s_R$ in the definition of R:

$$R(s)=\frac{(s-s_R)^2}{\pi}\int_t^\infty\frac{\rho(s')ds'}{(s'-s_R)^2(s'-s'-i\epsilon)}. \quad (6)$$

Therefore,

$$T=C\left[(s_R-s)\left(1+\frac{s-s_R}{\pi}\int\frac{\rho(s')C}{(s'-s_R)^2(s'-s'-i\epsilon)}\right)\right]^{-1}.$$

Calculating the inverse of the bracket and multiplying by C, we again obtain a very simple result due to the fact that the C factorizes:

$$T=C\bigg/(s_R-s)\left[1+\frac{s-s_R}{\pi}\times\sum_{i=1}^n\int\frac{\rho_i(s')ds'}{(s'-s_R)^2(s'-s'-i\epsilon)}C_i^2\right], \quad (7)$$

a formula whose similarity to the Breit-Wigner formula is more easily seen if placed in the following form:

$$T=\frac{C}{\tilde{D}(s)-i\sum_{i=1}^n\rho_i(s)\theta(s-t_i)C_i^2}, \quad (8)$$

where

$$\tilde{D}(s)=s_R-s-\sum_{i=1}^n(C_i^2/\pi)I_i(s),$$

and

$$I_i(s)=(s-s_R)^2\mathrm{P}\int\frac{\rho_i(s')ds'}{(s'-s_R)^2(s'-s)},$$

where $\mathrm{P}\int$ indicates a principal-value integral.

Equation (8) is the same formula as Eq. (2), except that $s-s_R$ is factored out instead of $s-s_p$. Other than that difference in form, they are the same analytic function. It remains to explicitly evaluate the various integrals involved. From Table I we take the parameters to use in the evaluation of Eq. (8). We will not include the the $K\bar{K}$ and $N\bar{N}(++)$ channels because they do not contribute significantly to the formation of the ρ meson, as shown at the end of Sec. III.

Let us define the following functions:

$$f(s,t,u,\Lambda)=\mathrm{P}\int_t^\Lambda\frac{ds'}{[(s'-u)(s'-t)]^{1/2}}\frac{1}{s'-s}, \quad (0<u<t)$$

for $t<s<\Lambda$,

$$f=\frac{-2}{[(s-u)(s-t)]^{1/2}}\coth^{-1}\left[\left(\frac{s-u}{s-t}\frac{\Lambda-t}{\Lambda-u}\right)^{1/2}\right],$$

for $u<s<t$,

$$f=\frac{2}{[(t-s)(s-u)]^{1/2}}\tan^{-1}\left[\left(\frac{s-u}{t-s}\frac{\Lambda-t}{\Lambda-u}\right)^{1/2}\right],$$

for $s<u$,

$$f=\frac{2}{[(s-u)(s-t)]^{1/2}}\tanh^{-1}\left[\left(\frac{s-u}{s-t}\frac{\Lambda-t}{\Lambda-u}\right)^{1/2}\right],$$

[11] A nonrelativistic derivation similar to the relativistic one presented here may be found in R. H. Dalitz, *Strange Particles and Strong Interactions* (Oxford University Press, Oxford, England, 1962), p. 64.

$$f'(s,t,u,\Lambda)=\frac{\partial}{\partial s}f(s,t,u,\Lambda)=\frac{1}{(s-t)(s-u)}\left\{\frac{[(\Lambda-t)/(\Lambda-u)]^{1/2}(u-t)}{(t-u)-\{1-[(\Lambda-t)/(\Lambda-u)]^{1/2}\}(s-u)}+\left(\frac{t+u}{2}-s\right)f(s,t,u,\Lambda)\right\},$$

$$H_1(s,t)=\mathrm{P}\int_t^\infty\frac{(s'-t)^2}{[s'(s'-t)]^{1/2}}\frac{ds'}{(s'-m_\rho^2)^2(s'-s)}=\frac{(s-t)^2}{(s-m_\rho^2)^2}f(s,t,0,\infty)$$
$$+\frac{\partial}{\partial m_\rho^2}\left(\frac{(m_\rho^2-t)^2}{m_\rho^2-s}\right)f(m_\rho^2,t,0,\infty)+\frac{(m_\rho^2-t)^2}{m_\rho^2-s}f'(m_\rho^2,t,0,\infty),$$

$$H_2(s,t)=\int_t^\infty\frac{(s'-t)(s')}{[s'(s'-t)]^{1/2}}\frac{ds'}{(s'-m_\rho^2)^2(s'-s)}=\frac{(s-t)(s)}{(s-m_\rho^2)^2}f(s,t,0,\infty)$$
$$+\frac{\partial}{\partial m_\rho^2}\left(\frac{(m_\rho^2-t)(m_\rho^2)}{m_\rho^2-s}\right)f(m_\rho^2,t,0,\infty)+\frac{(m_\rho^2-t)(m_\rho^2)}{m_\rho^2-s}f'(m_\rho^2,t,0,\infty),$$

$$H_3(s,t,u,\Lambda)=\mathrm{P}\int_t^\Lambda\frac{(s'-t)^2(s'-u)^2}{[(s'-t)(s'-u)]^{1/2}}\frac{ds'}{s'(s'-m_\rho^2)(s'-s)}=\left[\frac{t^2u}{s(m_\rho^2)^2}+\frac{\partial}{\partial m_\rho^2}\left(\frac{(m_\rho^2-t)^2(m_\rho^2-u)}{m_\rho^2(m_\rho^2-s)}\right)\right.$$
$$\left.+\frac{(s-t)^2(s-u)}{s(m_\rho^2-s)^2}\right]\cosh^{-1}\left(\frac{\Lambda-\frac{1}{2}(u+t)}{\frac{1}{2}(t-u)}\right)-\frac{t^2u^2}{s(m_\rho^2)^2}f(0,t,u,\Lambda)+\frac{\partial}{\partial m_\rho^2}\left[\frac{(m_\rho^2-t)^2(m_\rho^2-u)^2}{m_\rho^2(m_\rho^2-s)}\right]f(m_\rho^2,t,u,\Lambda)$$
$$+\frac{(m_\rho^2-t)^2(m_\rho^2-u)^2}{(m_\rho^2-s)m_\rho^2}f'(m_\rho^2,t,u,\Lambda)+\frac{(s-t)^2(s-u)^2}{s(s-m_\rho^2)}f(s,t,u,\Lambda).$$

Then $\tilde{D}(s)$ is explicitly

$$\tilde{D}(s)=m_\rho^2-s-\frac{(s-m_\rho^2)^2}{\pi}[(C_1^2/4)H_1(s,t_1)+(C_2^2/16)H_3(s,t_2,u_2,\Lambda)$$
$$+(C_4^2/16)H_3(s,t_4,u_4,\Lambda)+(C_5^2/16)H_3(s,t_5,u_5,\Lambda)+(C_7^2/2)H_2(s,t_7)],\quad(9)$$

where the C_i, t_i, and u_i are taken from Table I. In the above formula, we have used a sharp cutoff Λ on the vector-pseudoscalar channels rather than the gradual one used before. We expect (9) to be a good approximation to reality only in the region $|s|\lesssim\Lambda$. Since

$$T=e^{i\delta}\sin\delta/\rho=1/[\rho\cot\delta-i\rho],$$

we may use the formula

$$\delta(s)=\tan^{-1}\left(\frac{\Gamma(s)}{\tilde{D}(s)}\right),\quad(10)$$

where

$$\Gamma(s)=\sum_{i=1}^{n}\rho_i(s)C_i^2\theta(s-t_i),$$

to get the phase shift. $\delta(s)$ is tabulated in Table II for a value of the cutoff $\Lambda=300m_\pi^2$. One expects a value for Λ between 200 and $400m_\pi^2$, as in calculations of this sort, this generally is the range in which a reasonable Λ will fall. $\delta(s)$ is not too sensitive to Λ in that range; for example, at $s=18m_\pi^2$, δ changes by 8% as Λ goes from 200 to $400m_\pi^2$.

Also tabulated in Table II is the function $\delta'(s)$ de-

TABLE II. Numerical values for the p-wave $\pi\pi$ phase shift from Eqs. (7) and (8) of the text.

s/m_π^2	$\delta'(s)$ {nonrelativistic} (radians)	$\delta(s)$ {relativistic} (radians)
18.0	0.209	0.244
19.6	0.270	0.308
21.2	0.350	0.388
22.8	0.458	0.495
24.4	0.607	0.640
26.0	0.814	0.839
27.6	1.093	1.105
29.2	1.424	1.426
30.8	1.747	1.749
32.4	2.008	2.020
34.0	2.199	2.223
35.6	2.337	2.371
37.2	2.437	2.480
38.8	2.512	2.562
40.4	2.569	2.626
42.0	2.615	2.677

fined by the nonrelativistic formula

$$\delta'(s)=\tan^{-1}\left[\frac{\Gamma'(s)}{m_\rho^2-s}\right], \tag{11}$$

where

$$\Gamma'(s)=\frac{2C_1^2}{1+(p/p_r)^2}\frac{p^3}{m_\rho},$$

where

$$p=[(s-4m_\pi^2)/4]^{1/2},$$

and $p_r=p$ evaluated at $s=m_\rho^2$. The $1+(p/p_r)^2$ factor in the denominator is a guess at the correct "barrier penetration" factor.[12] As can be seen, $\delta'(s)$ and $\delta(s)$ deviate away from the resonance position in such a way that $\delta(s)>\delta'(s)$ on both sides of the resonance. No choice of barrier penetration factor can increase $\delta'(s)$ on both sides of the resonance. Although the difference between the two sets of phase shifts is not large, it is not insignificant, since it consistently goes one way. One can hope that, with sufficiently good data, it might be possible to favor one set over the other. On the other hand, in the event the difference is not discernible, one can regard the relativistic theory as determining what the barrier penetration factor of the nonrelativistic theory must be.

V. CONCLUSIONS

From the above analysis, we learn that the following is true within the one resonance approximation given by (2):

(a) Matrix elements differ only by the ρ-meson coupling constants involved. This would explain why a previous calculation[13] which used only the ρ-pole term and did not satisfy unitarity could produce reasonable branching ratios.

(b) The form factor for spacelike q^2 will be very similar to those given by the usual ρ dominance in the $\pi\pi$ channel alone, the other channels being neglected. This follows from the dispersion formula

$$F(s)=\frac{1}{\pi}\int\frac{\mathrm{Im}F(s')ds'}{s'-s}$$

plus possible subtractions, in conjunction with

$$\mathrm{Im}F_i(s)=F_j^*(s)\rho_{jk}(s)T_{ki}(s),$$

where $\rho_{jk}(s)=\rho_j(s)\delta_{jk}$.

From this one finds

$$F(q^2)\cong m_\rho^2/(q^2+m_\rho^2) \tag{12}$$

[we normalize $F(0)=1$]. Experimentally,

$$F(q^2)\cong[m_\rho^2/(q^2+m_\rho^2)]^2.$$

Although we can produce somewhat different behaviors for F depending on how we choose the photon and ρ-meson coupling constants, we still cannot escape the fundamental behavior given by (12), and thus we cannot reproduce the experimental behavior.

(c) The experimental form factor for timelike q^2 is not inconsistent with the one resonance approximation.

(d) An enhancement is seen in the $\pi\pi\to\pi\omega$ cross section (Fig. 3) which is somewhat like the B meson. The theoretical cross section is, however, a factor of 4 below the Chew-Low extrapolation for the $\pi\pi\to\pi\omega$ cross section (Fig. 4). Nevertheless, it would seem that the $\rho\to\pi\omega$ decay probably contributes a substantial part of the B-meson enhancement.[8]

(e) The relativistic p-wave $\pi\pi$ phase shift is greater by a few percent or more than the phase shift given by the nonrelativistic formula [Eq. (11)] on *both* sides of the resonance.

The parametrization of other $N\bar{N}$ channels which are coupled to a resonance or single-particle state in the way described in Sec. II is also possible, although there are complications for most cases. The ω and ϕ cannot be treated independently. The π-meson bound state would require three-body channels. The K^*(890 MeV), however, could be parametrized just as we have done here. And the η meson might well be approximately treated with only the $N\bar{N}$ channels.

The advantages of this parametrization are that it is simple and contains the minimum number of parameters to determine, uses relativistic kinematics, has the correct analyticity except that a pole replaces the left-hand cut (effective-range approximation), and reduces to the Breit-Wigner formula in the nonrelativistic limit. For these reasons, Eq. (2) would appear to be a leading candidate for the most natural relativistic generalization of the Breit-Wigner formula.

We must keep in mind, however, the following limitations: (1) We have used a cutoff on the vector-pseudoscalar channels; (2) we have made some adjustments in order to obtain desirable properties for the nucleon channel (namely, correct threshold behavior and the magnetic to electric form factor ratio). Both of these modifications derive from the use of so simple an N function, and in the correct theory would not be necessary.

ACKNOWLEDGMENTS

Our thanks to the Western Data Processing Center for access to their WDCOM system, to Matts Roos and M. J. Moravcsik for stimulating correspondence, and to L. A. P. Balázs for useful discussions.

APPENDIX

The actual calculation using Eq. (2) of the text may be broken into two steps, which will be illustrated by the case of the ρ meson: (1) Find s_p so that there is a

[12] M. Gell-Mann and K. Watson, Ann. Rev. Nucl. Sci. 4, 231 (1954); K. A. Brueckner, Phys. Rev. 86, 106 (1952); E. P. Wigner and L. Eisenbud, *ibid.* 72, 29 (1947); H. Feshbach, D. C. Peaslee, and V. F. Weisskopf, *ibid.* 71, 145 (1947).

[13] M. Parkinson, Phys. Rev. 143, 1369 (1966).

resonance at $s=m_\rho^2$; (2) find the C_i so that the resonance widths are correct.

Step (1) is accomplished by first finding the matrix θ that diagonalizes the matrix $M_{ij}=C_i'C_j'$ where the C_i' are from Table I. Define $R_{ij}(s)=R_i(s)\delta_{ij}$. Then $\tilde{M}=\theta^T M\theta$ is a diagonal matrix; in fact,

$$\tilde{M}=\begin{bmatrix} C'^2 & & & & & & \\ & 0 & & & 0 & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & 0 & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

where

$$C'^2=\sum_{i=1}^{7} C_i'^2.$$

Let $\tilde{R}(s)=\theta^T R\theta$ and $\tilde{R}'(s_p)=\theta^T R'(s_p)\theta$, where $R'(s_p)=\partial/\partial s[R(s,s_p)]|_{s=m_\rho^2}$.

Then we find s_p by solving

$$\tilde{M}\tilde{R}'(s_p)=1.$$

Step (2) is accomplished by setting $\tilde{N}_{11}=-1/\tilde{R}_{11}$; we recover $B_{ij}=C_iC_j$ by using $B=\theta\tilde{N}\theta^T$. Notice that this prescription implies that the C_i of formula (2) are proportional to the C_i' of Table I; $C_i=k_fC_i'$, where k_f depends on the $R_i(s)$ and s_p.

PHYSICAL REVIEW VOLUME 162, NUMBER 5 25 OCTOBER 1967

Spin-Parity-$\frac{3}{2}^-$ Baryons and Singlet-Octet Mixing*

N. Masuda and S. Mikamo
Department of Physics, University of Wisconsin, Madison, Wisconsin
(Received 25 May 1967)

Recent experimental data are used to examine the effects of possible $SU(3)$ mixing among the spin-parity-$\frac{3}{2}^-$ baryons. It is found that the mixing model is a quite reasonable one to account for the various decay widths of the members. With a mixing angle $\theta=22.5°$, the following optimum values of the parameters are obtained: $F/D\approx1.1$, the ratio of the coupling of $\{8\}\otimes\{8\}$ to $\{1\}$ and $\{8\}\otimes\{8\}$ to $\{8\}\approx2.5$, and inverse interaction radius $X\approx1000$ MeV. Predicted decay widths of the members are also presented.

The recently observed $Y_0^*(1700)$[1,2] appears to be a neutral member of a spin-parity-$\frac{3}{2}^-$ baryon octet which is completed by the $N_{1/2}^*(1525)$, $Y_1^*(1660)$, and $\Xi^*(1815)$. It has also been speculated that the $Y_0^*(1518)$ belongs to a unitary singlet state. The experimental situation concerning some members of this multiplet is still unclear according to Rosenfeld *et al.*[3] There are large contradictions among the measured branching ratios of the $Y_1^*(1660)$, and poorly observed branching ratios of the $\Xi^*(1815)$. It also seems that the branching ratio for the $Y_0^*(1518)$ is not as well confirmed as indicated by the earlier experiments.

The $Y_0^*(1518)$ has been thought of as a unitary singlet because the earlier data for the branching ratio[4] $(\bar{K}N)/(\Sigma\pi)$ were compatible with the $SU(3)$ prediction, which is about 0.43.[5–9] The ratio from the recent compilation is 39/51 (=0.76),[3] and it may be even higher if the recently observed value of 52/37 by Dauber *et al.*[10] is taken into account. This suggests that the $Y_0^*(1518)$ can hardly be assigned to a pure unitary singlet state. On the other hand, if we assign the newly observed $Y_0^*(1700)$ as the neutral member of the pure octet state, the mass (1700 MeV) departs by 30 MeV from the computed mass based on the Gell-Mann–Okubo mass formula.[11,12] The presently observed evidence that the $Y_0^*(1700)$ couples strongly to the $\Sigma\pi$ channel ($\sim50\%$) and weakly to the $\bar{K}N$ ($\sim20\%$) contradicts the assignment of the $Y_0^*(1700)$ to a pure octet state.[13] These deviations suggest that there might be

* Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, COO-881-108.

[1] J. D. Davis, J. D. Dowell, P. M. Hattersley, R. J. Homer, A. W. O'Dell, A. A. Carler, K. F. Riley, R. J. Tapper, D. V. Bilgg, R. S. Gilmore, K. M. Knight, D. C. Salter, G. H. Stafford, and E. J. N. Wilson, Phys. Rev. Letters **18**, 62 (1967).

[2] M. Ferro-Luzzi, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, California, 1967), p. 183; R. Armenteros, M. Ferro-Luzzi, D. W. G. Leith, R. Levi-Setti, A. Minten, R. D. Tripp, H. Filthuth, V. Hepp, E. Kluge, H. Schneider, R. Barloutaud, P. Grant, J. Meyer, and J. P. Porte, Phys. Letters **24B**, 198 (1967).

[3] A. H. Rosenfeld, A. Barbaro-Galtieri, W. J. Podolsky, L. R. Price, P. Soding, C. G. Wohl, M. Ross, and W. J. Willis, Rev. Mod. Phys. **39**, 1 (1967).

[4] A. H. Rosenfeld, A. Barbaro-Galtieri, J. Kirz, W. J. Podolsky, M. Roos, W. J. Willis, and C. G. Wohl, University of California Radiation Laboratory Report No. UCRL-8030, August 1966 Revision (unpublished).

[5] J. J. Sakurai, Phys. Letters **10**, 132 (1964).

[6] A. W. Martin, Nuovo Cimento **32**, 1645 (1964).

[7] R. Dalitz, in *Proceedings of the Oxford International Conference on Elementary Particles, September, 1965* (Rutherford High-Energy Laboratory, Chilton, Berkshire, England, 1966), p. 157.

[8] J. J. Coyne, S. Meshkov, and G. B. Yodh, Phys. Rev. Letters **17**, 666 (1966).

[9] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).

[10] P. M. Dauber, E. I. Malamud, P. E. Schlein, W. E. Slater, and D. H. Stork, University of California at Los Angeles Report No. UCLA-1015, 1967 (unpublished).

[11] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[12] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[13] See Table I (part of no mixing).