Low-Energy Constraints on Pion Production Amplitudes in πN Scattering*

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Using the hypothesis of the partially conserved axial-vector currents, and specific assumptions on the equal-time commutators of these currents, a calculation of the pion production amplitude near threshold of the πN system has been attempted. The resulting matrix elements are contrasted with those obtained previously by other means, and a comparison with existing data on the total reaction cross section is made. Agreement is favorable for the energy range considered. Unlike previous studies of the production amplitudes where direct π-π interactions are thought to play a substantial role, the present calculation ignores all possibilities of strong π-π effects, and incorporates only those π-π scattering effects required by the consistency of the current-algebra approach. The evaluation of these contributions requires some assumptions on the equal-time commutators of pseudoscalar and scalar densities. Finally, suggestions are made on extensions of the present approach based on resonance models.

I. INTRODUCTION

THE hypothesis of the partial conservation of the weak axial current (PCAC), when supplemented by specific assumptions on their equal-time commutators (ETC), has yielded remarkably rich information on low-energy pionic amplitudes.[1,2] The procedure in each "soft" pion calculation, as the simultaneous use of the two concepts has come to be called, is to write down an amplitude which, strictly speaking, is only true for zero-mass pions interacting with the system (not including pure π systems) in which we are interested via axial currents. Then proceed to go on mass shell, hoping that the parameters characterizing the zero-mass pions have a weak dependence on the pion energy. For instance, to calculate πN scattering lengths,[2,3] one writes down the amplitudes for zero-mass-πN amplitude at threshold, $s=M^2$, and then proceeds on to the physical threshold at $s=(M+\mu)^2$. (Throughout this paper, M denotes the baryon mass, and μ the meson mass. s stands for the c.m. energy squared.) The assumption here is that the amplitude which characterized πN scattering at zero pion mass, with complete absence of direct and crossed nucleon pole terms, retained its value as $s\rightarrow(M+\mu)^2$. Such extrapolations would be dangerous indeed if π-π scattering at low energies were strong, especially when more than two pions are involved in the extrapolation.

Recently, Weinberg has applied the soft-pion technique to calculate pion scattering lengths on any target, including another pion.[3] The last result was, in a sense, a self-consistency statement: π-π scattering was assumed to be weak to start off with, and, self-consistently, small scattering lengths were obtained. As indicated above, the assumption of weak π-π scattering is not entirely implausible, owing to the successes of soft-pion calculations. However, the π-π scattering lengths obtained were smaller than they were previously thought to be, and it would be useful to see if soft-pion theory can describe physically measurable processes, where the π-π interaction can have significant contributions. Surprisingly, in successful calculations to date, no strong π-π effects have ever been needed. The following reactions serve as examples:

(1) K_{l4} *decay*. This is presumably the best reaction in which to look for π-π interactions. Weinberg[4] has calculated the decay form factors by soft-pion means, and the results agree beautifully with experiment. All π-π final-state interactions were neglected in the calculation.

(2) τ-*decay*. This is another place where the π-π interaction may be of importance. Abarbanel[5] has calculated *its* decay form factors, and they agree very well with experiment. Again, no large π-π effects are needed.

(3) η-*decay*. This is an electromagnetic decay, but the π-π interaction may influence its decay spectrum. A fair amount of success has been attained here,[6] although there are some difficulties too.[7]

(4) $\gamma N\rightarrow 2\pi+N$. To date no soft-pion calculation on this reaction has been attempted. π-π influence here would be comparable to K_{l4}.

(5) $\pi N\rightarrow 2\pi N$. Most of the indications of large π-π scattering lengths come from analysis of this re-

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[1] See for instance, R. F. Dashen, Rapporteur Report at the XIII High-Energy Conference at Berkeley, 1966 (unpublished).

[2] The other low-energy parameters for the πN system obtained to date are the S-wave scattering lengths [S. Weinberg, Ref. 3; also Y. Tomozawa, Nuovo Cimento **46A**, 707 (1966); B. Hemprecht (to be published)], the S-wave effective ranges, and P-wave scattering lengths [K. Raman, Phys. Rev. Letters **17**, 983 (1966); A. P. Balanchandran (to be published); Norman Fuchs, Phys. Rev. **150**, 1241 (1967); H. J. Schnitzer, *ibid.* **158**, 1471 (1967)].

[3] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[4] S. Weinberg, Phys. Rev. Letters **17**, 336 (1966).

[5] H. Abarbanel, Phys. Rev. **153**, 1547 (1967).

[6] R. Ramanchandran, Nuovo Cimento **47A**, 669 (1966); R. H. Graham, L. O'Raifeartaigh, and S. Pakvasa, *ibid.* **48A**, 830 (1967).

[7] D. Sutherland, Phys. Letters **23**, 384 (1966); CERN report (unpublished); S. L. Adler, Phys. Rev. Letters **17**, 519 (1967).

action, especially at high energies, although the conclusions are by no means unambiguous. It is the purpose of this present paper to present a calculation of the total cross section of this reaction near threshold and to illustrate that, once again, large π-π scattering lengths are unnecessary to describe the reaction if one is willing to use the soft-pion formalism as a framework.

In Sec. II we review older attempts at studying this reaction. Section III gives the results of the soft-pion calculation, Sec. IV the numerical comparisons with the data, and we end with some comments on possible extensions of the formalism presented here.

II. DESCRIPTION OF INELASTIC π-N SYSTEM IN NON-"SOFT"-PION THEORIES

The earliest attempt at understanding the reaction has been to extend the theory of Chew and Low, which is highly successful in describingthe πN channel, by incorporating an additional pion into the equations.[8] The source is regarded as static, with the incoming pion interacting directly with the nucleon, and two pions are shaken off. The meson-baryon coupling is essentially pseudovector in character. Because of this, the predominantly S-wave pions are given a very low production cross section. Moreover, all π-π scattering effects are ignored, and the pions are treated asymmetrically.[9]

It was assumed then that proper inclusion of π-π scattering would boost the cross section to the experimental values. Indeed Rodberg[10] subsequently obtained results in agreement with experiment by means of a single-particle-exchange amplitude, with $|a_0| = 0.50\mu^{-1}$, $|a_2| = 0.20\mu^{-1}$, where a_0 and a_2 are the S-wave π-π scattering lengths. No final-state πN scattering was considered. It was subsequently pointed out by Kim and Zoellner[9] that with properly symmetrized treatment of the pions, the static-theory cross sections actually do become larger without the benefit of π-π interaction, although the calculated cross sections were still smaller than experiment. Combining Kim and Zoellner's amplitude with that of Rodberg's would then presumbaly render the cross sections too large. This is also indicated by the rescattering calculations of Goebel and Schnitzer,[11] where, with comparable π-π scattering lengths to Rodberg's, the cross sections did come out to be fairly large.

In addition to the above descriptions, there were also attempts to study the reaction with the isobar production mechanism[12] of Lindenbaum and Sternheimer. Recently, Olsson and Yodh[12] have extended the model and obtained results in good agreement with experiment over a wide energy range. The central feature of the model was to regard all pion production as decay products from isobars; the production matrix elements were then obtained by comparison with some reactions, the results used to predict other reaction rates. No direct π-π interaction was explicitly present.

In summary, we note that the static theory with *symmetric* treatment of the pions provides a fairly accurate picture of the reaction, and may well serve as a means to calculate directly the production matrix element, which can be used later in conjunction with the isobar model to predict cross sections over a range of energies; of course the restriction to static sources might cause trouble at higher energies, but for energies near threshold it should certainly be sufficient. At such energies, the soft-pion technique provides a similar matrix element where the pions are treated symmetrically and where furthermore the sources are not static. The π-π interaction comes out of the formalism automatically, and is small. There are no free parameters in the calculation; in addition a new feature peculiar to soft-pion calculations is present: The ETC provides terms which are linked with isovector photoproduction of pions. These are terms in which two of the π's form an isovector system, which then interacts with the nucleon. The previous theories have included these effects by P-wave scattering lengths to be determined from experiment. We have related these directly to known electromagnetic properties of the baryons.

Previous soft-pion analyses of the same reaction were carried out by Nambu and Lurie,[13] and by Shrauner,[14] but the ETC terms are all assumed to vanish. The resulting cross sections are correspondingly smaller.

III. SINGLE-PION PRODUCTION IN π-N SCATTERING

The production process to be considered here is

$$\pi^\alpha(k)+N(p)\rightarrow\pi^\beta(q_2)+\pi^\gamma(q_3)+N(p') \qquad (1)$$

with the incident pion close to the inelastic threshold. α, β, γ refer to the isospin indices of the pion, and the four-momenta of the various particles are given within the parentheses. Weinberg's calculations suggest that one may hope for a meaningful result if one takes all three of the pions off their mass shells, and replace their field operators by divergences of the corresponding axial currents. This is in the spirit of PCAC; that is, it is assumed here that the divergence of the axial current is a "good" interpolating field for the pion.

It may at first appear that taking the incident π off mass shell is a little too drastic, since the incident π will have, normally, $\sim$200 MeV kinetic energy in the lab

[8] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); J. Franklin, *ibid.* **105**, 1101 (1957); L. Rodberg, *ibid.* **106**, 1090 (1957); E. Kazes, *ibid.* **107**, 1131 (1957).
[9] Kim Tse Peng and W. Zoellner, Nucl. Phys. **34**, 491 (1962).
[10] L. Rodberg, Phys. Rev. Letters **3**, 58 (1959).
[11] C. J. Goebel and H. J. Schnitzer, Phys. Rev. **123**, 1021 (1961); H. J. Schnitzer, *ibid.* **125**, 1059 (1962).
[12] S. Lindenbaum and R. M. Sternheimer, Phys. Rev. **104**, 1723 (1956); more recently, M. Olsson and G. Yodh, *ibid.* **145**, 1309 (1966).
[13] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962).
[14] E. Shrauner, Phys. Rev. **131**, 1847 (1963).

system. However, in taking q_2 and q_3 of the mass shell we have to insert axial spurions into external baryon legs.[15] This automatically forces the incident pion to interact with the external legs as well, but with off-shell coupling constants. The PCAC hypothesis allows us a way of extrapolating these to the on-shell coupling. To exploit this fully, one has to invoke PCAC at the very beginning.[16]

The T matrix for the process is then defined by the equation

$$\int d^4x \int d^4y \int d^4z \langle N(p')|T[\Lambda^\alpha(x)\Lambda^\beta(y)\Lambda^\gamma(z)]|N(p)\rangle\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}\, e^{iq_3\cdot z} = \frac{-(2\pi)^4\delta^4(p'+q_2+q_3-p+q_1)c^3}{(\mu^2-q_1^2)(\mu^2-q_2^2)(\mu^2-q_3^2)} \times \frac{1}{(2\pi)^{9/2}(-8\omega_1\omega_2\omega_3)^{1/2}} \langle N(p')|T_{\alpha\beta\gamma}|N(p)\rangle, \quad (2)$$

$$c=\mu^2 f_\pi = \mu^2 M g_A/g_r,\quad g_A=+1.17,\quad g_r^2/4\pi=14.6,\quad \mathbf{A}(\zeta)\equiv\partial^\mu \mathbf{A}_\mu(\zeta),\quad M=940\text{ MeV},\quad \mu=140\text{ MeV},$$

ω_1, ω_2, ω_3=fourth components of q_1, q_2, and q_3. $\mathbf{A}_\mu(\zeta)$ above is the non-strangeness-changing class-1 weak axial current, while f_π is the weak π-decay constant defined by the equation

$$\langle 0|A_\mu{}^\alpha(\zeta)|\pi^\beta(k)\rangle = \frac{e^{-ik\cdot\zeta}}{(2\pi)^{3/2}(2\omega_k)^{1/2}} i f_\pi k_\mu,$$

$$k^2=\mu^2$$

(see Appendix A).

Notice that for convenience we have let $k\equiv -q_1$. Then over-all momentum conservation implies that

$$p'+q_1+q_2+q_3=p. \quad (3)$$

The left-hand side of the equation above can now be reduced by the familiar process of partial integration. For instance, doing this once gives us

$$\int d^4x d^4y d^4z \langle N(p')|T[\Lambda^\alpha(x)\Lambda^\beta(y)\Lambda^\gamma(z)]|N(p)\rangle e^{iq_1\cdot x}\, e^{iq_2\cdot y}\, e^{iq_3\cdot z}$$

$$= \int d^4x d^4y d^4z\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}\, e^{iq_3\cdot z} \{(-iq_1)^\mu \langle N(p')|T[A_\mu{}^\alpha(x)\Lambda^\beta(y)\Lambda^\gamma(z)]|N(p)\rangle$$

$$-\langle N(p')|T[[A_0{}^\alpha(x),\Lambda^\beta(y)]\Lambda^\gamma(z)]|N(p)\rangle\delta(x_0-y_0) - \langle N(p')|T[[A_0{}^\alpha(x),\Lambda^\gamma(z)]\Lambda^\beta(y)]|N(p)\rangle\delta(x_0-z_0)\}. \quad (4)$$

The process can now be repeated with respect to $(q_2)^\nu$ and Λ^β and $(q_3)^\lambda$ and Λ^γ. So that we may have manifestly Bose symmetry in our final expression, we will pull out the derivatives symmetrically. If we do not follow this prescription, then Bose symmetry is not apparent in the final result. Jacobi's identity will then have to be used to show that Bose symmetry has not been destroyed. Since part of our purpose in doing this calculation is to re-examine the π-π problem, it is more convenient to exhibit Bose symmetry explicitly.

The symmetric pulling out of derivatives takes place in the following manner: Pull out the derivative on $\Lambda^\alpha(x)$, yielding Eq. (4) above; then pull out the derivative on $\Lambda^\beta(y)$ and then that on $\Lambda^\gamma(z)$; now pull out the derivative on $\Lambda^\alpha(x)$ again but this time follow this operation by pulling out the derivative on $\Lambda^\gamma(z)$ and then pull out that on

[15] S. L. Adler, Phys. Rev. **139**, B1638 (1965).

[16] The motivation for this step goes as follows: If the initial pion had been left on its mass shell, the coupling at the π vertex would have been pure pseudoscalar. Extrapolation of this form factor g_r to on-mass-shell coupling would be difficult without PCAC, and we would essentially have to examine the dispersion relation for g_r. With PCAC, the vertex is related to a *weak* vertex whose form factors are, by hypothesis, slowly varying functions. The extrapolation to on-mass-shell is presumably effected smoothly by exploiting the pseudovector coupling that is now present at the vertex. The distinction between an easy extrapolation in the case of the axial-current coupling and a harder extrapolation for pure pseudoscalar coupling is presumably the distinction between "good" and "bad" currents. See S. Fubini, G. Segrè, and J. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966).

$\Lambda^\beta(y)$. The result is that

$$\int d^4x d^4y d^4z \langle N(p')|T[\Lambda^\alpha(x)\Lambda^\beta(y)\Lambda^\gamma(z)]|N(p)\rangle\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}\, e^{iq_3\cdot z}$$

$$=\tfrac{1}{2}\int d^4x d^4y d^4z\, e^{iq_1\cdot x}\, e^{iq_2\cdot y}\, e^{iq_3\cdot z}\{2(-iq_1^\mu)(-iq_2^\nu)(-iq_3^\lambda)\langle N(p')|T[A_\mu{}^\alpha(x)A_\nu{}^\beta(y)A_\lambda{}^\gamma(z)]|N(p)\rangle$$
$$+[-(-iq_1^\mu)(-iq_2^\nu)\langle N(p')|T[[A_0{}^\gamma(z),A_\mu{}^\alpha(x)]A_\nu{}^\beta(y)]|N(p)\rangle\delta(z_0-x_0)-(-iq_1^\mu)(-iq_2^\nu)$$
$$\times\langle N(p')|T[A_\mu{}^\alpha(x)[A_0{}^\gamma(z),A_\nu{}^\beta(y)]]|N(p)\rangle\delta(z_p-y_0)-[(-iq_1^\mu)(-iq_3^\lambda)\langle N(p')|T[[A_0{}^\beta(y),A_\mu{}^\alpha(x)]A_\lambda{}^\gamma(z)]$$
$$\times|N(p)\rangle-(-iq_1^\mu)\langle N(p')|[A_0{}^\gamma(z),[A_0{}^\beta(y),A_\mu{}^\alpha(x)]]|N(p)\rangle\delta(z_0-z_0)]\delta(x_0-y_0)-(-iq_1^\mu)\langle N(p')$$
$$\times|T[A_\mu{}^\alpha(x)[A_0{}^\beta(y),\Lambda^\gamma(z)]]|N(p)\rangle\delta(y_0-z_0)-[(-iq_3^\lambda)\langle N(p')|T[[A_0{}^\alpha(x),\Lambda^\beta(y)]A_\lambda{}^\gamma(z)]]|N(p)\rangle$$
$$-\langle N(p')|[A_0{}^\gamma(z),[A_0{}^\alpha(x),\Lambda^\beta(y)]]|N(p)\rangle\delta(z_0-x_0)]\delta(x_0-y_0)-[(-iq_2^\nu)\langle N(p')|T[A_\nu{}^\beta(y)[A_0{}^\alpha(x),\Lambda^\gamma(z)]]|N(p)\rangle$$
$$-\langle N(p')|[A_0{}^\beta(y),[A_0{}^\alpha(x),\Lambda^\gamma(z)]]|N(p)\rangle\delta(y_0-z_0)]\delta(z_0-x_0)$$
$$+\text{terms with indices }(2,\nu,\beta)\text{ and }(3,\lambda,\gamma)\text{ interchanged}]\}.\quad(5)$$

The process is now repeated with cyclic permutations of the indices (1,2,3), (α,β,γ), and (μ,ν,λ). There will then be an over-all factor of $\frac{1}{6}$.

The various equal-time commutators can be determined within the frame-work of some dynamical models; e.g., the "σ" model or its modification.[17] The terms containing commutators of currents are evaluated following Gell-Mann's assumption that they satisfy the chiral algebra[17]:

$$\delta(\zeta_0-\xi_0)[A_0{}^\alpha(\zeta),A_\mu{}^\beta(\xi)]=i\epsilon_{\alpha\beta\gamma}V_\mu{}^\gamma(\xi)\delta^4(\zeta-\xi),\quad(6)$$

and

$$\delta(\zeta_0-\xi_0)[V_0{}^\alpha(\zeta),A_\mu{}^\beta(\zeta)]=i\epsilon_{\alpha\beta\gamma}A_\mu{}^\gamma(\xi)\delta^4(\zeta-\xi),\quad(7)$$

where $\mathbf{V}_\mu$ are the currents of the generators of SU_2 isotopic group, and whose $(\pm)$ components are also, by the hypothesis of conserved vector current (CVC), the weak vector currents. In addition to these relationships, the following further assumptions are made[17]:

$$\delta(\zeta_0-\xi_0)[A_0{}^\alpha(\zeta),\Lambda^\beta(\xi)]=i\delta_{\alpha\beta}\sigma(\xi)\delta^4(\zeta-\xi),\quad(8)$$

and

$$\delta(\zeta_0-\xi_0)[A_0{}^\alpha(\zeta),\sigma(\xi)]=-i\delta_{\alpha\eta}\Lambda^\eta(\xi)\delta^4(\zeta-\xi).\quad(9)$$

The first equation here can be taken as a definition of a scalar operator $\sigma(\xi)$. The second relation then follows from the Jacobi identity. These commutation relations were abstracted from a quark model (see Ref. 17), in much the same way that the current algebra was obtained. However, while the current algebra may be thought of as an elegant way of expressing universality, there are no similar motivations for the commutators of the scalar and pseudo-scalar densities, except for models like the σ model. Indeed, we do not even require that σ represent a particle of isospin 0. Weinberg's π-π scattering lengths depended of course on the assumption of these commutators, or more explicitly, on the assumption that Λ^α belongs to a chiral quadruplet whose fourth component is an isoscalar σ. This is by no means the only plausible assumption; Λ^α may actually belong to a chiral multiplet with $\tau_{\alpha\beta}$, where $\tau_{\alpha\beta}$ is a symmetric scalar density defined by

$$\delta(\zeta^\nu-\xi^0)[A_0{}^\alpha(\zeta),\Lambda^\beta(\xi)]=i\tau_{\alpha\beta}(\zeta)\delta^4(\zeta-\xi)\quad(10)$$

and

$$\delta(\zeta^0-\xi^0)[A_0{}^\alpha(\zeta),\tau^{\alpha\gamma}(\xi)]=(-i)[\delta^{\gamma\eta}\delta^{\alpha\beta}+\delta^{\beta\eta}\delta^{\gamma\alpha}]\Lambda_\eta(\xi)\delta^4(\zeta-\xi).\quad(11)$$

That is, $\tau^{\alpha\beta}$ transforms like $I=2$. This assumption will lead to a larger scattering in S-wave π-π scattering near threshold and with $I=2$, and much weaker $I=0$ S-wave scattering length than Weinberg's results, although their magnitudes are still small. And in general, we may imagine that $[A_0,\Lambda]$ has both $I=0$ and $I=2$ contributions, giving us a nonet of scalar and pseudoscalar densities. In practice, it would be rather difficult to distinguish between the various forms of the commutators because of the hopefully weak couplings of $\tau^{\alpha\beta}$ and σ to the known particles, and any meaningful distinction between them will occur only in higher orders of soft-pion momenta. In what follows, we will assume the σ commutator. To the order we are working in, our results of the total cross sections are not terribly sensitive to one or the other commutator.

The three momenta of the pions, q_1, q_2, and q_3, are now allowed to go to zero. In this limit, the lower order concontributions (i.e., to order q_1q_2, q_2q_3, q_1q_3) can be isolated and evaluated. For instance, the term free of any

[17] M. Gell-Mann, Physics **1**, 63 (1964); Y. Nambu (unpublished). We only use the $SU_2\times SU_2$ subalgebra of the full chiral $SU_3\times SU_3$ algebra postulated by Gell-Mann. The original σ model is described in M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960). More recently, SU_3 has been incorporated into the model by M. Lévy (to be published).

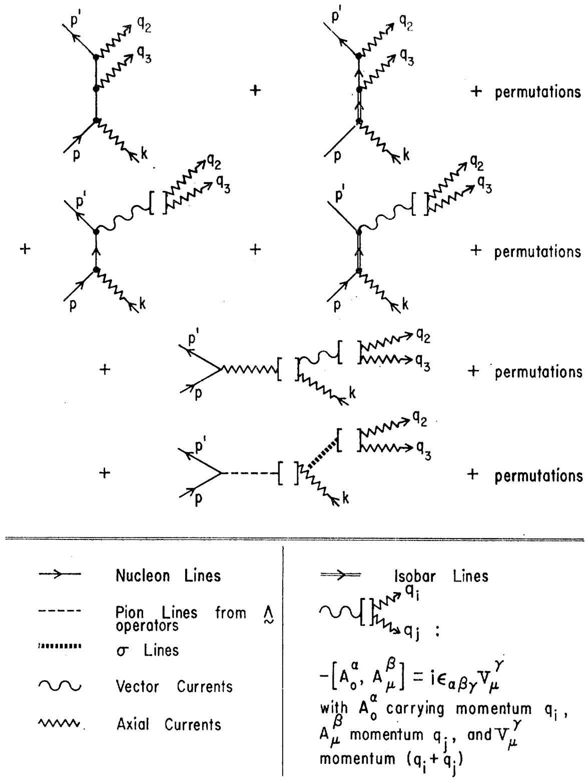

FIG. 1. Diagrammatic representation of Eq. (5).

commutators gets contributions from the nucleon and N^*[18]; those with one commutator involving currents can be similarly evaluated. They are the photoproduction amplitudes. Those with commutators involving the σ operator *once* will be dropped. (If we let all but one of the pions go on mass shell, then these terms contribute and give us the Adler self-consistency condition on π-π scattering.) We expect such contributions to be large only if there are large scalar enhancements in the π-π channel near threshold. The experimental situation regarding this point is not very clear, but the indications are that no such enhancements are present either in the $I=0$ or $I=2$ channels. The check on the self-consistency condition on the πN channel, which depended on $\langle N|\sigma|N\rangle \to 0$, substantiates this claim.[19,20] The double commutators will involve matrix elements of either $\mathbf{A}_\mu$ or $\mathbf{\Lambda}$, and hence are known in terms of weak parameters. They contribute to the peripheral scattering graphs and are responsible for π-π effects.

We may summarize the various contributions by the diagrams in Fig. 1. The explicit evaluation of some of the terms is carried out in Appendix A.

IV. NUMERICAL COMPUTATION

From invariance requirements, we know that there will be 16 independent amplitudes for single pion production; four from space-time invariance, and four from isospin considerations. The scattering amplitude $\mathfrak{M}$ may, for instance, be written as

$$\mathfrak{M}=\sum_{i=1}^{4} \bar{u}(p')\gamma_5[F_1^{(i)}\mathbf{q}_1+F_2^{(i)}\mathbf{q}_2 +F_3^{(i)}\mathbf{q}_3+F_4^{(i)}(\mathbf{q}_1-\mathbf{q}_3)(\mathbf{q}_1-\mathbf{q}_2)], \quad (12)$$

where each F_j can be decomposed in terms of isospin amplitudes

$$F_j=F_j^{(1)}\delta_{\beta\gamma}\tau_\alpha+F_j^{(2)}\delta_{\alpha\gamma}\tau_\beta +F_j^{(3)}\delta_{\alpha\beta}\tau_\gamma+F_j^{(4)}\tau_\beta\tau_\gamma\tau_\alpha. \quad (13)$$

Each invariant amplitude is a function of five independent variables. In contrast to the usual "single-particle-exchange" amplitudes, the functions we consider here will be dependent upon *all* five variables: In other words there will be dependence on the conventionally defined Treiman-Yang angle.[21]

The scattering in various charge states can now be related to the four isospin amplitudes above. To derive the exact relationship is straightforward. The states are characterized by two isospins, which may be chosen to be the initial isospin in the πN system, and the isospin of the final π-π system. The projection operators may be verified to be

$$\mathcal{P}_{1/2,0}=\tfrac{1}{3}\delta_{\beta\gamma}\tau_\alpha, \quad (14a)$$

$$\mathcal{P}_{1/2,1}=[(\tau_\beta\tau_\gamma-\delta_{\beta\gamma})\tau_\alpha](1/3\sqrt{2}), \quad (14b)$$

$$\mathcal{P}_{3/2,1}=\tfrac{1}{2}\sqrt{2}[\tau_\gamma\delta_{\alpha\beta}-\tau_\beta\delta_{\alpha\gamma}+\tfrac{2}{3}(\tau_\beta\tau_\gamma-\delta_{\beta\gamma})\tau_\alpha], \quad (14c)$$

$$\mathcal{P}_{3/2,2}=(1/\sqrt{10})[\tau_\beta\delta_{\alpha\gamma}+\tau_\gamma\delta_{\alpha\beta}-\tfrac{2}{3}\delta_{\beta\gamma}\tau_\alpha]. \quad (14d)$$

The charge states scattering amplitudes can now be expressed in terms of amplitudes with these well-defined isospins. The coefficients are given in Carruther's paper.[22]

The particular scattering that is of interest in the energy range considered would be

$$\pi^- p \to \pi^+\pi^- n. \quad (15)$$

This is by far the most important reaction for $T_\pi \lesssim 250$ MeV, where the branching ratio to other charge states is almost zero. The normal interpretation is that π-π

[18] If we are interested only to *first* order in soft-pion momenta, the N^* makes no contribution. Actually, because the final pions are not too high above threshold, the N^* contributions are relatively small.

[19] In the Adler-Weisberger sum rule, for instance, the scalar contribution is avoided by considering only the isospin antisymmetric amplitude. However, the scalar contribution is expected to be small, at least for low-energy pions, on the basis of the Adler self-consistency condition, which has been shown to check with experiment. See S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[20] K. Kawarabayashi and W. W. Wada, Phys. Rev. **146**, 1209 (1966).

[21] S. B. Treiman and C. N. Yang, Phys. Rev. Letters **8**, 140 (1962).

[22] P. Carruthers, Ann. Phys. (N. Y.) **14**, 229 (1961).

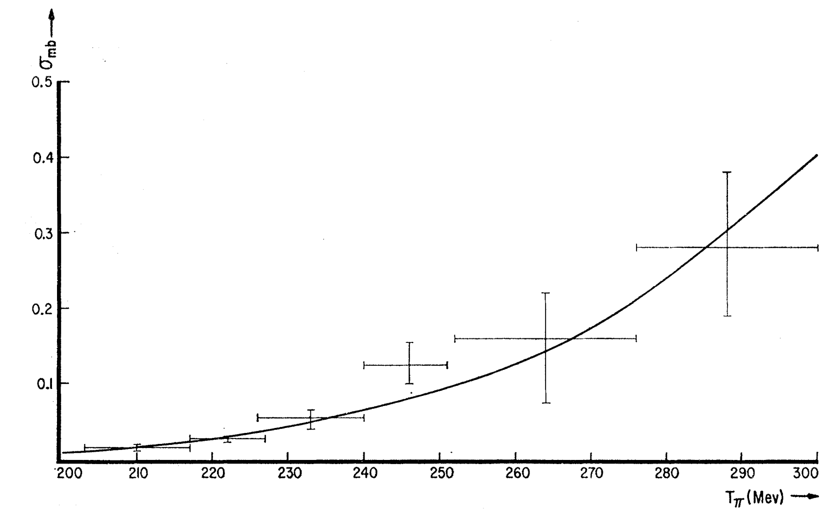

FIG. 2. $\sigma_{\pi^+\pi^-n}$ (mb) plotted as function of T_π, the incident lab energy of the pion. Experimental points are from Y. Batusov *et al.*, Dubna Report No. JINR P-1823 (unpublished).

attraction in $I=0$ state is comparatively stronger than in other channels.

The relevant π-π amplitude is[23]

$$\pi^+\pi^- \to \pi^+\pi^- . \tag{16}$$

The form of its contribution will be examined in the Appendix B. The terms may be evaluated following the steps described in Appendix A using only the known weak and electromagnetic interaction parameters. The actual process of computation is tedious, and merely involves manipulation with the γ matrices. We then obtain that

$$F_1=(1/2M)(a-\tfrac{1}{3}\alpha d), \tag{17a}$$

$$F_2=b+(1/2M)(a-\tfrac{1}{3}\alpha d), \tag{17b}$$

$$F_3=c+(1/2M)(a-\tfrac{1}{3}\alpha d), \tag{17c}$$

$$F_4=\tfrac{1}{3}d, \quad \alpha=3\mu^2+2(p'\cdot q_3-p\cdot q_2), \tag{17d}$$

where a, b, c, and d are functions of the various momenta and weak-interaction parameters. The conventional π-π amplitude is contained in a. The factors a, b, c, and d are given in the Appendix A.

Total reaction cross sections for (15) are available at a number of energy values[24] and give a good check on the present approach. The amplitudes obtained by the soft-pion technique have accordingly been used to calculate these cross sections. A comparison with existing data is made in Table I (Fig. 2). One point to bear in mind in examining the table is that the input amplitudes are presumably only correct to second order in pion momenta. Agreement with experimental data at an energy range considerably removed from threshold is hence not to be expected. We have found that the best agreement is when $T_\pi \lesssim 330$ MeV, where T_π is the incident pion energy in the laboratory.

It is interesting to compare our matrix element with those obtained in earlier studies of the process. For instance, consider the initial attempts involving the extension of the static theory to cover the inelastic process. The amplitudes thus obtained are similar to our direct Born terms: The coupling of the pions to the nucleon is the same in both cases in the nonrelativistic limit. The pions were, however, not treated symmetrically.[9] In any case the predicted cross sections are too small. The situation is improved somewhat when proper symmetrization is included, although the cross section is still smaller than the experimental values.[9] Our amplitudes possess the required crossing symmetry and are similar to those of Kim and Zoellner. However we have more terms in our expression than theirs. As we mentioned before, their study failed to include all π-π scattering. Such is not the case in our amplitudes: All ETC terms contain π-π effects and these furnish a considerable amount of the amplitude (see Appendix B). In this sense, the early Nambu-Laurié[13] and Shrauner[14] considerations are similar in spirit to Kim

TABLE I. Comparison of inelastic π^-p scattering cross sections. T_π is the incident lab energy of the negative pion.

T_π (MeV)	$\sigma_{\pi^-\pi^+n}$ (mb)	$\sigma_{\pi^-p\pi^0}$ (mb)	$\sigma_{\pi^-\pi^+n}{}^{obs}$ (mb)[a]
170	0	0	0
190	0.003	0.000	...
200	0.009	0.001	...
210	0.017	0.003	0.015±0.003
222	0.032	0.006	0.027±0.005
			0.03 ±0.02
233	0.050	0.011	0.053±0.013
245	0.078	0.022	0.10 ±0.04
260	0.128	0.036	0.14 ±0.10
			0.16 ±0.06
290	0.325	0.104	0.16 ±0.013
			0.38 ±0.09
317	0.575		0.71 ±0.17

[a] Reference 20.

[23] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[24] Most of the data here are taken from the compilation of Olsson and Yodh (Ref. 7) and S. A. Bunyatov's report in the *Proceedings of the Twelfth Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).

and Zoellner's[9] calculation. Notice that the vector-current–nucleon scattering effects are a feature peculiar to the current algebra description. A conventional treatment with π-π scattering plus the static-theory contribution will most certainly require fairly large π-π scattering lengths to compensate for the vector-current contributions. The calculation with current algebra therefore shows that large π-π scattering lengths are by no means necessary to interpret these inelastic data, which, indeed, almost consistently demand very small scattering lengths.

Another useful thing that can be done with these amplitudes is to invoke unitarity, and see what constraint it places on elastic πN phase shifts. A program of this sort has been carried out by Arnold and Uretsky[25]; a similar study with the amplitudes found here is now under way.

Perhaps the next interesting thing to do is to include third-order effects by putting in the Roper resonance in addition to the N^*. One can then hope to explain the inelastic data for $T_\pi \gtrsim 330$ MeV. Before any predictions can be made, however, an investigation of the scalar-density contributions must be carried out. If we assume that the scalar density transforms like an isoscalar or isotensor, then we may hope for some information on its contributions by examining πN elastic amplitudes, to which it also contributes, at a finite range above threshold. An indication of the magnitude of this contribution will not only help in computing the inelastic amplitude at higher T_π, but is also of importance in determining high-energy multiple-soft-pion production amplitudes. Further investigations along these lines are now in progress.

Finally, the present calculation tends to strengthen the belief in weak π-π scattering at threshold. Starting with the initial K_{l4} work of Weinberg, there has been increasing evidence to substantiate this belief. The present work represents an attempt to verify the claim in a reaction traditionally used to study π-π interaction. The results obtained here would, indeed, be entirely unfounded if the π-π threshold interaction is large, say of the order conventionally adopted before the soft-pion calculations.

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APPENDIX A

The expansion (5) above can be evaluated explicitly in the limit when all momenta approach zero. One can then classify the expansion by the order of the pion momenta, and the amplitude expressed in terms of weak and electromagnetic form factors. We briefly summarize their values below.

I. The Axial Current Form Factors

The matrix element of the axial current A_λ between pion and vacuum states has already been given[26]:

$$\langle 0|A_\lambda{}^\alpha(\zeta)|\pi^\beta(k)\rangle = \frac{e^{-ik\cdot\zeta}}{(2\pi)^{3/2}(2\omega_k)^{1/2}} i f_\pi(k^2) k_\lambda \delta_{\alpha\beta}, \quad \text{(A1)}$$

where $f_\pi(k^2)$ at $k^2=\mu^2$ can be directly measured from the pion decay rate; numerically, $f_\pi(\mu^2)/\mu = 0.935/\sqrt{2}$. This number can be related to the weak-interaction form factor g_A, defined by

$$\langle N(p')|A_\lambda{}^\alpha(\zeta)|N(p)\rangle = \frac{e^{i(p'-p)\zeta}}{(2\pi)^3}\left(\frac{M^2}{EE'}\right)^{1/2} \bar{u}(p')\tfrac{1}{2}\tau^\alpha \times[\gamma_5\gamma_\lambda g_A(t)+\gamma_5 q_\lambda \beta(t)]u(p), \quad \text{(A2)}$$

$$t=(p'-p)^2\equiv q^2, \quad q_\lambda=(p'-p)_\lambda,$$

where experimentally

$$g_A(0)=1.17. \quad \text{(A3)}$$

The Goldberger-Treiman relation[27] tells us that

$$f_\pi(\mu^2) \cong M g_A(0)/g_r(\mu^2),$$

where $g_r(\mu^2)$ is the $N\bar{N}\pi$ coupling constant. Numerically, $g_r{}^2/4\pi = 14.6$. $\beta(t)$ is the induced pseudoscalar term and contains the one-pion-exchange pole. These quantities alone are sufficient to determine the first term of (5). It corresponds to the "gradient-coupling" Born terms and can be generated by the interaction Lagrangian[28]

$$\mathcal{L}_{\rm int}(x) = (g_r/2m)\bar{\psi}(x)\gamma_5\gamma_\lambda \boldsymbol{\tau}\psi(x)\partial^\lambda \boldsymbol{\varphi}(x), \quad \text{(A4)}$$

if we only retain the nucleon poles. For instance, the first diagram in Fig. 1 gives us a term like

$$\tfrac{1}{8} g_A{}^3 \bar{u}(p')\gamma_5 \mathbf{q}_2 \tau^\beta \frac{p'+q_2+M}{(p'+q_3)^2-M^2}\gamma_5 \mathbf{q}_3 \tau^\gamma \times \frac{p+k+M}{(p+k)^2-M^2}\gamma_5 \mathbf{k} \tau^\alpha u(p),$$

[25] R. C. Arnold and J. I. Uretsky, Phys. Rev. **153**, 1443 (1967).

[26] Unless otherwise stated, the conventions adopted here follow those explained in J. D. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1964), Vol. 1.

[27] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

[28] S. Weinberg, Phys. Rev. Letters **18**, 188 (1967); *ibid.* **16**, 879 (1966).

which may be reduced to

$$\tfrac{1}{8}g_A{}^2\bar{u}(p')\tau^\beta\tau^\gamma\tau^\alpha\gamma_5\left[1+\frac{2M\mathbf{q}_2}{2p'\cdot q_2+q_2{}^3}\right]\times\mathbf{q}_3\left[\frac{2M\mathbf{k}}{2p\cdot k+k^2}-1\right]u(p). \quad \text{(A5)}$$

The other terms will be permutations of this expression, and may more easily be obtained if we use (3), when they are just mere permutations of the indices 1, 2, 3 in (A5). The resulting sum can then be expressed in terms of the five invariants of (12) and (13).

To go one step further in order of "soft" momenta would entail considerations of the contributions from the nonpole terms of the right-hand side of (5). (In dispersion language, we have to include contributions from the cut.) As a first approximation, we have included the contribution from the 3–3 resonance. The intermediate resonance is treated as a particle with complex mass and of spin $\frac{3}{2}$. Using the Rarita-Schwinger formalism, the projection operator for such a particle with momentum p, is $\mathcal{P}_{\mu\nu}$, where

$$\mathcal{P}_{\mu\nu}=\left\{g_{\mu\nu}-\frac{2}{3M^{*2}}p_\mu p_\nu-\tfrac{1}{3}\gamma_\nu\gamma_\mu-\frac{1}{3M^*}(p_\mu\gamma_\nu-p_\nu\gamma_\mu)\right\}\frac{\not{p}+M^*}{2M^*}. \quad \text{(A6)}$$

M^* is a complex quantity. The axial current matrix element is[29]

$$\langle N^*(p')|A_\lambda(\zeta)|N(p)\rangle=e^{i(p'-p)\zeta}\frac{1}{(2\pi)^3}\left(\frac{M^*M}{E'E}\right)^{1/2}\bar{u}_\nu(p')\times\{f_1(t)q^\nu(tp_\lambda-q_\lambda p\cdot q)+if_2(t)q^\nu(\gamma^\alpha\epsilon_{\alpha\beta\gamma\lambda}p^\beta q^\gamma\gamma_5)+if_3(t)\epsilon^{\nu\alpha\beta\gamma}\epsilon_{\gamma\rho\tau\lambda}p_\alpha p^\rho q_\beta q^\tau+ig_A{}^*(t)\delta^\nu{}_\lambda\}u(p), \quad \text{(A7)}$$

where $t=q^2$ and $q_\lambda=(p'-p)_\lambda$. The form factors f_1, f_2, and f_3, are all transverse to q_λ; only $g_A{}^*(t)$ contributes longitudinally. Using PCAC, $g_A{}^*$ can be related to g^*, where g^* describes the $\bar{N}^*N\pi$ coupling through the Lagrangian $\mathcal{L}$

$$\mathcal{L}_{\text{int}}\sim(ig^*/\mu)(\bar{\psi}\psi^\alpha\partial_\alpha\varphi-\partial_\alpha\varphi^*\bar{\psi}^\alpha\psi).$$

Using a width of 125 MeV for the N^* gives $g^*=2.20$, so that the Goldberger-Treiman relation for the N^* reads

$$g_A{}^*=f_\pi g^*/\mu\cong1.44. \quad \text{(A8)}$$

The inclusion of the N^* is now straightforward but tedious. To get an idea of the algebra involved, note that the second diagram of Fig. 1 gives

$$g_Ag_A{}^*\bar{u}(p')\frac{\tau_\beta}{2\gamma_5}\left[1+\frac{2m\mathbf{q}_2}{2p'q_2+q_2{}^2}\right]\mathfrak{M}u(p), \quad \text{(A9)}$$

where, to the order we are working in,[30]

$$\mathfrak{M}=[A^{(+)}+\tfrac{1}{2}(\mathbf{k}+\mathbf{q}_3)B^{(+)}]\delta_{\alpha\gamma}+\tfrac{1}{2}[\tau^\gamma,\tau^\alpha][A^{(-)}+\tfrac{1}{2}(\mathbf{k}+\mathbf{q}_3)B^{(-)}]. \quad \text{(A10)}$$

$A^{(\pm)}$, $B^{(\pm)}$ are functions of $(p+k)^2$ and $(k-q_3)^2$:

$$A^{(+)}=\tfrac{1}{3}g_A{}^*\left\{\left(\frac{1}{(p+k)^2-M^{*2}}+\frac{1}{(p-q_3)^2-M^{*2}}\right)\left[(M^*+M)(k-q_3)^2+2q^{*2}\left[(M^*+M)+\frac{(M^*-M)(E^*+M)}{3(E^*-M)}\right]\right]+\frac{1}{3M^{*2}}[2M^{*3}+2(M^*+M)(M^{*2}+2M^*M-2M^2)+4(M^*+M)q_3{}^2+M[2M^2+k^2+q_3{}^2+2p(k-q_3)]]\right\}, \quad \text{(A11a)}$$

$$B^{(+)}=+\tfrac{1}{3}g_A{}^*\left\{\left(\frac{1}{(p+k)^2-M^{*2}}-\frac{1}{(p-q_3)^2-M^{*2}}\right)\times\left[(k-q_3)^2+2q^{*2}\left(1-\frac{(E^*+M)}{3(E^*-M)}\right)\right]+\frac{[k^2-q_3{}^2+2p(k+q_3)]}{3M^{*2}}\right\}, \quad \text{(A11b)}$$

$$A^{(-)}=-\tfrac{1}{6}g_A{}^*\left\{\left(\frac{1}{(p+k)^2-M^{*2}}-\frac{1}{(p-q_3)^2-M^{*2}}\right)\times\left[(M^*+M)(q_3-k)^2+2q^{*2}\left((M^*+M)+\frac{(M^*-M)(E+M)}{3(E^*-M)}\right)\right]+\frac{M[k^2-q_3{}^2+2p(k+q_3)]}{3M^{*2}}\right\}, \quad \text{(A11c)}$$

$$B^{(-)}=-\tfrac{1}{6}g_A{}^*\left\{\left(\frac{1}{(p+k)^2-M^{*2}}+\frac{1}{(p-q_3)^2-M^{*2}}\right)\left[(k-q_3)^2+2q^{*2}\left(1-\frac{(E^*+M)}{3(E^*-M)}\right)\right]+\frac{1}{3M^{*2}}[2(M^{*2}+2M^*M-2M^2+2q_3{}^2)+(2M^2+k^2+q_3{}^2+2p(k-q_3))]\right\}. \quad \text{(A11d)}$$

[29] J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) 38, 35 (1966); H. J. Schnitzer, Phys. Rev. 158, 1471 (1967).
[30] D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. 12, 419 (1962); H. J. Schnitzer, Ref. 29.

q^* and E^* are the momentum and energy in the c.m. system of the initial πN system (p,k) at the position of the N^*. Again, the other expressions can be obtained from the above by permutation of the indices 1, 2, and 3, with $k\equiv -q_1$.

II. The Vector Current Form Factors

The vector currents considered here are the generators of the SU_2 isospin group. Its matrix elements at zero-momentum transfer are known very well. We will assume that they vary slowly for changes of momenta of order of a pion mass. The nucleon matrix elements are given by

$$\langle N(p')|V_\mu{}^\alpha(\zeta)|N(p)\rangle = e^{i(p'-p)\cdot\zeta}\left(\frac{M^2}{E'E}\right)^{1/2}\frac{1}{(2\pi)^3}\bar{u}(p')\tau^\alpha \times\left(F_1\gamma_\mu + i\frac{\sigma_{\mu\nu}}{2M}q^\nu F_2\right)u(p)\,, \quad \text{(A12)}$$

where $q_\nu=(p'-p)_\nu$ and F_1 and F_2 are analytic functions of t with cuts starting at the 2π-threshold. Their values at zero-momentum transfer are given by

$$F_1(0)=\tfrac{1}{2},\quad F_2(0)=1.85\,. \quad \text{(A13)}$$

Using these, the first diagram on the second line in Fig. 1 gives

$$i\epsilon_{\beta\gamma\eta}\bar{u}(p')\tau^\eta\left(F_1\mathbf{q}_3+\frac{i\sigma_{\mu\nu}}{2M}Q^\nu q_3{}^\mu F_2\right) \times\frac{\mathbf{p}'+\mathbf{Q}+M}{(p+k)^2-M^2}\gamma_5\mathbf{k}\tau^\alpha u(p)g_A\,, \quad \text{(A14)}$$

where $Q_\mu=(q_2+q_3)_\mu$. This can be simplified to give

$$-g_AF_1\bar{u}(p')\gamma_5\mathbf{q}_3\left(1+\frac{2M\mathbf{k}}{2p\cdot k+k^2}\right) \times[i\epsilon_{\beta\gamma\alpha}+i(\delta_{\alpha\beta}\delta_{\sigma\gamma}-\delta_{\beta\sigma}\delta_{\alpha\gamma})\tau^\sigma]u(p) +g_AF_2\bar{u}(p')\gamma_5\frac{\mathbf{Q}\mathbf{q}_3-\mathbf{q}_3\mathbf{Q}}{2M}\left(1+\frac{2M\mathbf{k}}{2p\cdot k+k^2}\right) \times[i\epsilon_{\beta\gamma\alpha}+i(\delta_{\alpha\beta}\delta_{\sigma\gamma}-\delta_{\beta\sigma}\delta_{\alpha\gamma})\tau^\sigma]u(p)\,, \quad \text{(A15)}$$

where $\mathbf{Q}\mathbf{q}_3-\mathbf{q}_3\mathbf{Q}=2q_2\cdot q_3-2\mathbf{q}_3\mathbf{q}_2$.

The second diagram involves the N^*, and so we must examine the amplitude $N^*\rightarrow N\gamma$. This is described by the matrix element[29,31]

$$\langle N^*(p')|V_\mu(\zeta)|N(p)\rangle=\left(\frac{MM^*}{EE'}\right)^{1/2}\frac{1}{(2\pi)^{3/2}}\bar{u}_\nu(p')\gamma_5 \times\left\{\frac{iC_3(t)}{\mu}(\mathbf{q}\delta_\mu{}^\nu-q^\nu\gamma_\mu)-\frac{C_4(t)}{\mu^2}[(p'\cdot q)\delta_\mu{}^\nu-p_\mu{}'q^\nu] -\frac{C_5(t)}{\mu^2}[(p\cdot q)\delta_\mu{}^\nu-p_\mu q^\nu]\right\}u(p)\,,$$
$$q_\mu=(p'-p)_\mu,\quad t=q^2\,. \quad \text{(A16)}$$

The isobar model analysis of photoproduction amplitudes gives[31–33]

$$C_3(0)=0.345\,,\quad C_4(0)=C_5(0)=-0.0035\,. \quad \text{(A17)}$$

We may once again proceed to calculate the N^* contribution by using (A16), (A17), and (A6). Notice that the coupling of the axial spurion is the same as the coupling of the pion to the $\bar{N}N^*$ channel; the isobar terms here then are also the isobar model matrix elements of photoproduction,[34] electroproduction,[35] and production of electron-positron pairs by pions, depending on whether the invariant momentum squared carried by the vector current is near zero, less than zero, or considerably greater than zero. For instance, if we assume that the second diagram of the second line of Fig. 1 is adequately described by a photoproduction amplitude (the momentum squared here is very near the 2π threshold) then it may be represented by the six amplitudes used to parametrize photoproduction amplitudes for off-shell pions[35,36]:

$$i\epsilon_{\beta\gamma\eta}f_\pi\sum_{i=1}^{6}\bar{u}(p')A_i{}^{\eta\alpha}\theta_iu(p)\,, \quad \text{(A18)}$$

where

$$(A_i)_{\eta\alpha}=A_i{}^{(+)}\delta_{\eta\alpha}+A_i{}^{(-)}\tfrac{1}{2}[\tau_\eta,\tau_\alpha] \quad \text{(A19)}$$

are functions of $(p+k)^2$, $(k-Q)^2$, and Q^2, $Q_\lambda=q_2+q_3)_\lambda$ and

$$\begin{aligned}\theta_1&=\tfrac{1}{2}\gamma_5F_{\mu\nu}\gamma^\mu\gamma^\nu\,,\\ \theta_2&=\gamma_5F_{\mu\nu}(p+p')^\mu(\tfrac{1}{2}Q-k)^\nu\,,\\ \theta_3&=-\gamma_5F_{\mu\nu}\gamma^\mu k^\nu\,,\\ \theta_4&=\gamma_5F_{\mu\nu}\gamma^\mu(p+p')^\nu-2m\theta_1\,,\\ \theta_5&=\gamma^5F_{\mu\nu}Q^\mu k^\nu\,,\\ \theta_6&=-\gamma_5F_{\mu\nu}Q^\mu\gamma^\nu\,,\\ F_{\mu\nu}&=-(q_3)_\mu Q\nu+(q_3)_\nu Q_\mu\,.\end{aligned} \quad \text{(A20)}$$

To the order we are interested in, only θ_1 and A_1 matters. A_1 is related to the photoproduction amplitude describing[37]

$$\gamma(k)+N(p)\rightarrow N(p')+\pi(q) \quad \text{(A21)}$$

by the substitution

$$k_\mu\rightarrow -Q_\mu,\quad q_\mu\rightarrow -k_\mu\,. \quad \text{(A22)}$$

A multipole expansion can now be made for A_1, just as in Refs. 35–37. Since we are interested only in the N^*

[31] M. Gourdin and Ph. Salin, Nuovo Cimento **27**, 193 (1963); **27**, 309 (1963).

[32] S. Fubini, C. Rossetti, and G. Furlan, Nuovo Cimento **43A**, 161 (1966).

[33] R. H. Dalitz and D. G. Sutherland, Phys. Rev. **146**, 1180 (1966).

[34] See Ref. 36.

[35] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).

[36] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957); Ph. Dennery, *ibid.* **124**, 2000 (1961).

[37] S. L. Adler and F. J. Gilman, Phys. Rev. **152**, 1462 (1966).

contribution, only the $M_{1+}^{(3/2)}$ and $E_{1+}^{(3/2)}$ terms need be kept. (We follow the convention of Ref. 36 here; $M_{l\pm}$ and $E_{l\pm}$ describe magnetic and electric multipoles of order l, where l is the orbital angular momentum and $j=l\pm\frac{1}{2}$ the total angular momentum of the πN system.) Finally, the $M_{1+}^{(3/2)}$ and $E_{1+}^{(3/2)}$ may be obtained from experiment,[38] and their contributions evaluated for a given value of $(p+k)^2$ and $(k-Q)^2$.

III. The Double-Commutator Terms

These are either matrix elements of A_λ between nucleon states, in which case (A2) will be used to handle them, or matrix elements of $\mathbf{A}$ (ζ) between nucleon states. These can be evaluated by PCAC:

$$\langle N(p')|\Lambda^\alpha(\zeta)|N(p)\rangle = e^{i(p'-p)\cdot\zeta}\left(\frac{MM}{E'E}\right)^{1/2}\frac{1}{(2\pi)^3}\times\frac{\mu^2 f_\pi}{(p'-p)^2-\mu^2}\bar{u}(p')i\gamma_5 u(p)g_r(t). \quad \text{(A23)}$$

Using the quantities defined above, the matrix elements may be written down directly. For the charge states under consideration here, the quantities a, b, c, and d occurring in Eqs. (17) are given by

$$a=g_A\left[\frac{2M(2q_2\cdot q_3)}{2p\cdot q_3+q_3^2}+\frac{2M(2p\cdot q_2-q_2^2)}{2p'\cdot q_1+q_1^2}+\frac{2M(-q_2^2-2p'\cdot q_2-2q_2\cdot q_3)}{-2p\cdot q_1+q_1^2}+\frac{(\sqrt{2})M(2q_2\cdot q_3+2q_1\cdot q_2-4\mu^2)}{(p-p')^2-\mu^2}\right]$$
$$+\frac{g_AF_2}{M}\left\{4M^2(q_2\cdot q_3)\left[\frac{2}{q_3^2-2p\cdot q_3}+\frac{1}{2M^2}+\frac{1}{q_1^2-2p\cdot q_1}+\frac{1}{q_1^2+2p'\cdot q_1}\right]\right\}$$
$$+g_A^3\left[\frac{2M(-2q_2\cdot q_2-2p'\cdot q_2)}{2p\cdot q_1-q_1^2}+\frac{2M(2q_2\cdot q_3)}{2p\cdot q_3-q_3^2}+\frac{16M^3(q_2\cdot q_3)}{(2p\cdot q_3-q_3^2)(2p'\cdot q_3+q_3^2)}-\frac{2M(2p\cdot q_2-q_2^2)}{2p'\cdot q_1+q_1^2}\right], \quad \text{(A24)}$$

$$b=2g_A-g_AF_2\left[\frac{2M(2p'\cdot q_3+q_3^2+2q_1\cdot q_3+q_2\cdot q_3)}{2p'\cdot q_1+q_1^2}-\frac{2(2p'\cdot q_3+q_3^2+q_2\cdot q_3)}{q_1^2-2p\cdot q_1}-4\right]$$
$$-g_A^3\left\{2+4M^2\left[\frac{2q_3\cdot(q_1+q_2)+2p'\cdot q_3+q_3^2}{(2p\cdot q_3-q_3^2)(2p'\cdot q_1+q_1^2)}-\frac{1}{2p\cdot q_1-q_1^2}\right]\right\}, \quad \text{(A25)}$$

$$c=-g_AF_2\left\{(2q_1\cdot q_2)\left[\frac{1}{a_3^2-2p\cdot q_3}-\frac{1}{q_3^2+2p'\cdot q_3}\right]+2\left[\frac{2p'\cdot q_2+q_2^2+2q_2\cdot q_3}{2p'\cdot q_3+q_3^2}-\frac{2p\cdot q_2-q_2^2-2q_2\cdot q_3}{q_3^2-2p\cdot q_3}\right]\right.$$
$$\left.+(2q_2-q_3)\left[\frac{1}{q_1^2+2p'\cdot q_1}-\frac{1}{q_1^2-2p\cdot q_1}\right]+2\left[\frac{2p'\cdot q_2+q_2^2+2q_2-q_3}{q_1^2-2p\cdot q_1}-\frac{2(q_1+q_3)\cdot q_2+2p'\cdot q_2+q_2^2}{q_1^2+2p'\cdot q_1}\right]\right\}$$
$$+g_A^3\left\{4M^2\left[\frac{2q_2\cdot(q_1+q_3)+2p'\cdot q_2-q_2^2+2q_2\cdot q_3}{(2p\cdot q_3-q_3^2)(2p'\cdot q_1+q_1^2)}-\frac{2p'\cdot q_2+q_2^2+2q_2\cdot q_3}{(2p'\cdot q_3+q_3^2)(2p\cdot q_1-q_1^2)}\right]\right\}, \quad \text{(A26)}$$

$$d=2Mg_A\left[\frac{1}{2p'\cdot q_3+q_3^2}-\frac{1}{2p'\cdot q_1+q_1^2}+\frac{1}{-2p\cdot q_1+q_1^2}-\frac{1}{-2p\cdot q_3+q_3^2}\right]$$
$$-4Mg_AF_2\left[\frac{1}{q_3^2-2p\cdot q_3}-\frac{1}{q_1^2-2p\cdot q_1}+\frac{1}{q_1^2+2p'\cdot q_1}-\frac{1}{q_3^2+2p'\cdot q_3}\right]$$
$$+g_A^3\left\{2M\left[\frac{2p'\cdot(q_3-q_1)}{(2p'\cdot q_1+q_1^2)(2p'\cdot q_3+q_3^2)}+\frac{2p\cdot(q_3-q_1)}{(q_3^2-2p\cdot q_3)(q_1^2-2p\cdot q_1)}\right]\right.$$
$$\left.+8M^3\left[\frac{1}{(2p'\cdot q_3+q_3^2)(2p\cdot q_1-q_1^2)}-\frac{1}{(2p\cdot q_3-q_3^2)(2p'\cdot q_1+q_1^2)}\right]\right\}. \quad \text{(A27)}$$

[38] The actual numerical evaluation of the N^* contribution to the ETC terms are based on the assumption that $E_{1+}^{3/2}/M_{1+}^{3/2}=0$ [which is consistent with experiment and $SU(6)$]. The form factor of $N^*\to N\gamma$ now comes directly from the M_1 excitation. The dependence of this form factor on its momentum transferred squared is now assumed to be the same as that for the magnetic moment of the nucleon. [See F. M. Pipkin, in *Proceedings of the Oxford International Conference on Elementary Particles, 1965* (Rutherford High-Energy Laboratory, Harwell, England, 1966), p. 61.] This dependence is suggested by Geshkenbein's analysis of electroproduction [B. V. Geshkenbein, Phys. Letters **16**, 323 (1965)] and also by Dalitz and Sutherland's (See Ref. 33).

To these must now be added the N^* contributions in the way stated above. The result is long and not very illuminating. For the purposes of computation these contributions are taken from experiment using Eqs. (A11) and (A20), and the results added on numerically to Eqs. (A24)–(A27) at any given value of the variables chosen to characterize the system. In this manner, Table I was obtained, where, using the definition of T given in Ref. 26,

$$d\sigma=\frac{M^2}{2[(p\cdot k)^2-\mu^2M^2]^{1/2}}|\bar{T}|^2\frac{d^3\mathbf{q}_2}{2\omega_3(2\pi)^3}\frac{d^3\mathbf{p}'}{2\omega_3(2\pi)^3}\times\frac{d^3\mathbf{p}'}{2E'(2\pi)^3}(2\pi)^4\delta^4(p+k-q_2-q_3-p'),\quad\text{(A28)}$$

where $\omega_i^2=\mathbf{q}_i^2+\mu^2$, $E'^2=\mathbf{p}'^2+M^2$, and the bar over T indicates summation of final spin states and averaging over initial spin states of the nucleons.

APPENDIX B

π-π interactions have been included in the matrix elements of Appendix A, but their effects are not easily identifiable. Each time we have a commutator, of course, we are essentially computing a π-π rescattering effect, and hence, aside from the "gradient-coupling" terms,[3] all of the contributions come from π-π scattering.

Among these ETC contributions are effects coming from a *direct* π-π interaction, corresponding to the single-particle-exchange amplitude, and thus have a pion pole $1/(Q^2-\mu^2)$, where

$$Q_\mu=(q_1+q_2+q_3)_\mu.\quad\text{(B1)}$$

The residue of this pole is then some π-π scattering amplitude appropriately continued off mass shell. If we let $Q^2\rightarrow\mu^2$, then this residue *is* the π-π amplitude and we should recover the amplitudes derived by Weinberg,[3] and we do. Notice that if $Q^2\rightarrow\mu^2$, from Eq. (B1), we find that not all q_i can be zero. This means that some of the pions are not "soft." Some terms which

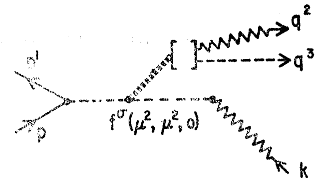

FIG. 3. Diagram giving rise to peripheral π–π scattering. Quantity $f^\sigma(\mu^2,\mu^2;0)$ is defined in Ref. 39 and evaluated there. $f^\sigma(\mu^2,\mu^2;0)=\mu^2/f_\pi$.

vanish in the soft-pion limits now begin to contribute; for instance, the terms containing the scalar density once, which previously went away because of the soft-pion limiting procedure, now contains a pole represented by the Fig. 3. If we sum up all the relevant contributions, we will obtain the amplitude represented by

$$\int\langle\pi^\sigma|T\Lambda^\alpha(x)\Lambda^\beta(y)\Lambda^\gamma(z)|0\rangle\times e^{iq_1\cdot x}e^{iq_2\cdot y}e^{iq_3\cdot z}d^4xd^4yd^4z.\quad\text{(B2)}$$

This is the PCAC expression for π-π scattering and has been studied by Weinberg,[3] and later in greater detail by Khuri,[39] in an evaluation of the scattering lengths; their analysis can be carried out here, and the residue of $1/(Q^2-\mu^2)$, when expressed in terms of their scattering lengths, gives the results of Ref. 3.

In the present case, however, we do not let $Q^2\rightarrow\mu^2$; instead, we are studying the $\pi N\rightarrow 2\pi N$ with all π's "soft." This means that the peripheral contributions arising from the double commutators in Eq. (5), while representing the direct π-π interactions, do not have the on-shell values of the π-π amplitudes. Indeed, as we shall see, it has a lesser contribution than before. We wi.l follow the notation of Chew and Mandelstam,[40] the process to be considered being

$$\pi^\sigma(Q)+\pi^\alpha(-q_1)\rightarrow\pi^\beta(q_2)+\pi^\gamma(q_3),\quad s=(Q-q_1)^2,\quad t=(Q-q_3)^2,\quad u=(Q-q_2)^2.\quad\text{(B3)}$$

The relevant terms in (6) describing this process are, for the A amplitude,[40]

$$\frac{1}{6}\int d^4xd^4yd^4ydz\,e^{iq_1\cdot x}e^{iq_2\cdot y}e^{iq_3\cdot z}\{(-iq_1^\mu)\langle N(p')|[A_0^\gamma(z),[A_0^\beta(y),A_\mu^\alpha(x)]]|N(p)\rangle\delta(z_0-x_0)\delta(x_0-y_0)+(-iq_1^\mu)$$
$$\times\langle N(p')|[A_0^\beta(y),[A_0^\gamma(y),A_\mu^\alpha(x)]]|N(p)\rangle\delta(z_0-x_0)\delta(x_0-y_0)+(-iq_2^\nu)\langle N(p')|A_0^\gamma(z),[A_0^\alpha(x),A_\nu^\beta(y)]]|N(p)\rangle$$
$$\times\delta(z_0-y_0)\delta(y_0-x_0)+(-iq_3^\lambda)\langle(p')|[A_0^\beta(y),[A_0^\alpha(x),A_\lambda^\gamma(z)]]|N(p)\rangle\delta(y_0-x_0)\delta(z_0-x_0)$$
$$+2\langle N(p')|[A_0^\alpha(x),[A_0^\beta(y),\Lambda^\gamma(z)]]|N(p)\rangle\delta(x_0-y_0)\delta(y_0-z_0)$$
$$+2\langle N(p')|[A_0^\alpha(x),[A_0^\gamma(z),\Lambda^\beta(y)]]|N(p)\rangle\delta(x_0-y_0)\delta(y_0-z_0)\}.\quad\text{(B4)}$$

The equation can then be explicitly evaluated using the techniques described in Appendix A, the peripheral graphs being generated by the induced pseudoscalar term. Before explicitly calculating the amplitude, it should

[39] N. Khuri, Phys. Rev. **153**, 1477 (1967).
[40] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

be remembered that the π-π effects come only from the longitudinal part of the axial current, according to the PCAC hypothesis.[41] Hence, we choose to write

$$(q_i)^\lambda = \frac{q_i Q}{Q^2} Q^\lambda + \frac{q_i \epsilon}{\epsilon^2} \epsilon^\lambda , \tag{B5}$$

with $\epsilon^2 = -1$, $\epsilon \cdot Q = 0$. The Q_λ, when dotted with the currents in (B4), will project out the longitudinal part and, thus, the π-π effects. Using Eqs. (6)–(9) and Eqs. (A2) and (A23), one obtains the following result for the A-amplitude[40] contribution:

$$\tfrac{1}{12}\bar{u}(p')\gamma_5 \tau^\sigma u(p) g_r f_\pi \left\{ 8\mu^2 - \frac{(2s-t-u)\mu^2}{s+t+u} \right\} \frac{1}{Q^2-\mu^2} (\delta_{\alpha\sigma}\delta_{\beta\gamma}), \quad s+t+u = Q^2 . \tag{B6}$$

The B and C contributions are obtained by "crossing" (B6). Notice that (B6), together with μ^2, really contributes to second order in soft-pion momenta. As mentioned earlier, the π-π amplitude is presumably the residue of $1/Q^2-\mu^2)$. To compare with the expression in Ref. 3 for π-π scattering, care must be given to the order in which the pion momenta become soft. From Eq. (6) and Eqs. (B4) and (B6) the relevant terms to the A amplitude are in the limit given by

$$\frac{1}{12}\Bigg[\lim_{q_1\to 0}\lim_{q_2\to 0}\lim_{q_3\to 0} \frac{2q_1\cdot Q}{(q_1+q_2+q_3)^2} + \lim_{q_1\to 0}\lim_{q_3\to 0}\lim_{q_2\to 0} \frac{2q_1\cdot Q}{(q_1+q_2+q_3)^2} - \lim_{q_1\to 0}\lim_{q_3\to 0}\lim_{q_2\to 0} \frac{2 ._1\cdot Q}{(q_1+q_2+q_3)^2}$$
$$- \lim_{q_2\to 0}\lim_{q_1\to 0}\lim_{q_3\to 0} \frac{2q_3\cdot Q}{(q_1+q_2+q_3)^2} + \lim_{q_1\to 0}\lim_{q_2\to 0}\lim_{q_3\to 0} (8) \Bigg] \frac{\mu^2}{f_\pi^2}, \tag{B7}$$

and thus to lowest order, the A amplitude approaches $\frac{2}{3}\mu^2/f_\pi^2$. If we assume the Weinberg's expansion[3] still holds this limit[42] (which is at $s\to 0$, $t\to 0$, and $u\to 0$), then we see that we have two-thirds the contribution of an on-shell amplitude. In general, the actual factor of reduction depends on the assumption of the scalar commutators; different factors appear for separate assumptions on these commutators. Such a distinction will be important for higher energies, especially in the computation of the branching ratios. An attempt to study this difference is currently being made for intermediate-energy pions.

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[41] That is, if we assume vector-meson dominance of the current, there will be a portion of A_λ, the longitudinal part, which is not coupled to the vector meson, but is coupled, instead to the pion channel. When a divergence is taken, the transverse components, which are coupled to the vector meson, are eliminated, and we get the PCAC relationships. The vector currents are, of course, by assumption, purely transverse, and only one channel, the ρ channel, is coupled to it. Imposing current algebra on the currents will yield relationships between the two systems of vector mesons. See S. Weinberg, Phys. Rev. Letters **18**, 507 (1967). Also, R. P. Feynman's comment in discussion after R. F. Dashen's report in the *Proceedings of The Thirteenth Annual International Conference on High Energy, Berkeley, California, 1966* (University of California Press, Berkeley, California, 1967).

[42] The expansion in Ref. 3 is of course nonunitary. The justification for using it is that we are demanding weak π–π scattering so that presumably the branch point at $s=4\mu^2$ is weak. If this assumption is not made, fairly large scattering lengths may result [J. Franklin (private communication); J. Sucher and C. H. Woo, Phys. Rev. Letters **18**, 723 (1967)].