Goldberger-Treiman Relation for a Composite Pion*

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We consider the problem of obtaining the Goldberger–Treiman relation for a composite pion. We use the N/D formalism so that our basic assumption is that the pion shows up as a dynamical pole in the pion-channel strong-interaction D matrix. We formally obtain the Goldberger–Treiman relation by examining the residue of the dynamical pion pole in the pion-channel $l\nu \rightarrow i$ reaction (i labels the ith strong channel). With an approximation suggested by the small mass of the pion, and results of a bootstrap-theory calculation of Dashen and Frautschi, we suggest that the experimental success of the Goldberger–Treiman relation may be understood in a bootstrap theory.

I. INTRODUCTION

It is our purpose in this paper to discuss the Goldberger-Treiman (G.T.) relation[1] with the explicit assumption that the pion is a composite particle. The possibility that the pion is composite was raised by Fermi and Yang,[2] and, of course, in a bootstrap universe[3] the pion will be composite. The G.T. relation is often written in the form[4]

$$f=-\sqrt{2}Mg_A/[g_{\pi N}K_{N\bar{N}}(0)], \tag{1}$$

where M is the nucleon mass, g_A is the ratio of the nucleon–axial-vector weak coupling to the nucleon-vector weak coupling, $g_{\pi N}{}^2/4\pi\approx 15$, and $K_{N\bar{N}}(0)$ is the pionic form factor of the nucleon, normalized to $K_{N\bar{N}}(\mu^2)=1$ with μ the pion mass, evaluated at zero momentum transfer. The quantity f is defined by

$$\langle 0|P_\mu(0)|\pi^+\rangle = i\pi_\mu[(2\pi)^3 2E_\pi]^{-1/2} f, \tag{2}$$

where P_μ is the axial-vector weak current,[5] π_μ is the pion four-momentum, and E_π is the pion energy. In terms of f the decay rate for $\pi^+ \rightarrow l\nu$ is

$$\tau_{\pi l\nu}{}^{-1}=(Gf)^2M^2(\mu^2-M^2)^2(8\pi\mu^3)^{-1}, \tag{3}$$

where G is the weak-coupling constant, $G\approx 10^{-5}M^{-2}$, and m is the charged lepton mass. The form of the G.T. relation first written down by Goldberger and Treiman has $K_{N\bar{N}}(0)=1$,

$$f=-\sqrt{2}Mg_A/g_{\pi N} \tag{4}$$

and this value of f agrees with experiment to $\approx 10\%$.

There have been two main approaches to the G.T. relation; the original one of Goldberger and Treiman, and an alternate approach, which we will call the PCAC (partially conserved axial-vector current) approach, due to Gell-Mann[6] and Nambu.[7]

The characteristic assumption of the G.T. approach is that $F(s)$, the analytic continuation in the pion mass of the invariant amplitude associated with $\langle 0|P_\mu(0)1\pi\rangle$, satisfies an unsubtracted dispersion relation; this dispersion relation is then used to calculate $f=F(\mu^2)$. The alternate approach stresses the axial-vector nucleon form factors, and has led to the partially conserved axial-vector current hypothesis (PCAC)[4,6,7] which has played an important role in current algebra calculations.[8] Our calculations are in the spirit of the bootstrap approach,[3,9] as this approach allows us to formulate our basic assumption of a composite pion. For example, we assume that the pion shows up as a dynamical pole in the $^1S_0$, $T=1$, $N\bar{N}$ scattering amplitude, and then we obtain the G.T. relation from the residue of the dynamical pole in the $l\nu \rightarrow N\bar{N}$ reaction amplitude. We also consider the case when the pion is a bound state in a coupled two-body multichannel system. Our approach is thus more closely related to the PCAC approach than the Goldberger-Treiman approach.

Our results are: (i) that the G.T. relation for a composite pion can be derived from once-subtracted dispersion relations for the appropriate reaction amplitudes describing $l\nu \rightarrow i$, where i is a strong-interaction channel, whereas previous work based on dispersion relations assumes unsubtracted dispersion relations for these amplitudes; (ii) that with an approximation suggested by the fact that the pion has a small mass in the scale of strong interactions, and with results of a bootstrap calculation by Dashen and Frautschi,[10] we can obtain the multichannel generalization of Eq. (4) with $K_i(0)=1$. Result (ii) suggests that the success of the G.T. relation, Eq. (4), can be understood in a bootstrap theory.

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[1] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

[2] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).

[3] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); for a review and a large bibliography see F. Zachariasen, 1965 Pacific International Summer School in Physics, Honolulu, Hawaii (unpublished). Simple dispersion-theory calculations [R. C. Arnold, Nuovo Cimento **37**, 589 (1965); J. S. Ball, A. Scotti, and D. Y. Wong, Phys. Rev. **142**, 1000 (1966)] provide some support for composite pions.

[4] See, for example, S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[5] For simplicity we will not write isotopic spin indices for P_μ.

[6] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, *ibid.* **17**, 755 (1960).

[7] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).

[8] See, for example, W. F. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); S. L. Adler, *ibid.* **14**, 1051 (1965); S. S. Weinberg, *ibid.* **17**, 616 (1966).

[9] The point of view in the present paper is similar to that of R. F. Dashen and S. C. Frautschi, Phys. Rev. **143**, 1171 (1966); **137**, B1318 (1965).

[10] See the first reference in Ref. 9.

In the next section we give some definitions, establish notation, and briefly review earlier work on the G.T. relation. In Sec. III we present our calculations, and in Sec. IV we summarize our work.

II. DEFINITIONS AND NOTATIONS

In this section we will establish notation and define various amplitudes. Then we will briefly discuss earlier approaches to the G.T. relation.

For the sake of simplicity we will omit spinorial and kinematic factors when we define invariant amplitudes. Also, we will formally treat all strong channels as two-body channels, as has previously been done by Ida,[11] and Nishijima[12] in their work on the G.T. relation; we will not consider any difficulties associated with anomalous thresholds and overlapping cuts. The basic amplitudes of interest describe the strong-interaction part of the $l\nu \to i$ reaction generated by the axial-vector current in the pion channel; these amplitudes are

$$R_i(s)=\langle i|\partial_\mu P^\mu(0)|0\rangle, \tag{5}$$

where s is the center-of-mass energy squared. For actual two-body channels, R_i can be expressed in terms of weak-interaction form factors; for example,

$$R_{N\bar{N}}(s)=2MF_A(s)+sH_A(s), \tag{6}$$

where $F_A(s)$ is the nucleon axial-vector form factor, and $H_A(s)$ is the induced pseudoscalar form factor. Our normalization is such that

$$R_{N\bar{N}}(0)=2Mg_A. \tag{7}$$

We note the important fact that R_i has a pole at $s=\mu^2$,

$$R_i(s) \underset{s\to\mu^2}{\longrightarrow} \frac{\mu^2 f g_{\pi i}}{s-\mu^2}. \tag{8}$$

The quantity $g_{\pi i}$ is the "coupling" of the pion to the strong channel i; when $i=N\bar{N}$, $g_{\pi N\bar{N}}=\sqrt{2}g_{\pi N}$, where $g_{\pi N}$ is the usual renormalized pion-nucleon coupling. We define the pionic form factors $K_i(s)$ by

$$g_{\pi i}K_i(s)=\langle i|-(\Box+\mu^2)\phi_\pi(0)|0\rangle. \tag{9}$$

The unitarity relations for R_i and K_i are

$$\begin{aligned}\mathrm{Im}\,g_{\pi i}K_i(s)&=\sum_i g_{\pi j}K_j(s)T_{ji}^*(s)\rho_{jj}(s),\\ \mathrm{Im}R_i(s)&=\sum_i R_j(s)T_{ji}^*(s)\rho_{jj}(s),\end{aligned} \tag{10}$$

where $\boldsymbol{\rho}$ is a diagonal phase-space matrix such that ρ_{ii} is nonzero only for $s>$ threshold for channel i, and T_{ij}^* is the complex conjugate of the pion-channel T matrix for the strong reaction $j\to i$.

Now we will briefly review work on the G.T. relation. The original G.T. approach assumes an unsubtracted dispersion relation for $F(s)$, the analytic continuation in the pion mass of the invariant amplitude associated with $\langle\pi|P_\mu(0)|0\rangle$. With the standard normalization $F(\mu^2)=f$, we have

$$f=\frac{1}{\pi}\int ds'\frac{\mathrm{Im}F(s')}{s'-\mu^2}. \tag{11}$$

Goldberger and Treiman kept only the $N\bar{N}$ intermediate state in $\mathrm{Im}F(s)$, and they then made numerous approximations in order to calculate f; their result is given by Eq. (2). More detailed investigations of this approach have been made by Barret and Barton,[13] Ida,[11] Nishijima,[12] and Saito.[14] The key question in this approach is whether the integral for f is actually convergent, and, as it turns out, two cases must be considered: (i) $Z_\pi\neq0$ and (ii) $Z_\pi=0$, where Z_π is the wave-function renormalization of the pion. Goldberger and Treiman's assumptions are consistent with $Z_\pi\neq0$. Saito has shown, with reasonable assumptions and without the restriction of only $N\bar{N}$ intermediate states in $\mathrm{Im}F(s)$, that if $Z_\pi\neq0$ the integral for f is convergent; and further, that if the G.T. relation with $K_{N\bar{N}}(0)=1$ is to hold, then $Z_\pi\ll1$. On the other hand, if $Z_\pi=0$, then in general the integral for f is divergent. However, $\mathrm{Im}F(s)$ explicitly depends on f, and neglecting the possibility of oscillatory behavior for $\mathrm{Im}F(s)$ as $s\to\infty$, the convergence condition $\mathrm{Im}F(s)\to0$ as $s\to\infty$ determines f, as was first pointed out by Barret and Barton. Ida showed that use of the convergence condition allows the G.T. approach to be used when $Z_\pi=0$. Furthermore, as noted by Ida, and particularly emphasized by Nishijima, the requirement, in the case $Z_\pi=0$, that the convergence condition and the dispersion relation give the same f can give rise to consistency requirements among the strong-interaction parameters. We note that one possible way of having $Z_\pi=0$ is to have a composite pion,[15] so that considerable care must be taken with the G.T. approach with a composite pion. Our approach to the G.T. relation for a composite pion goes through in a more straightforward manner than does the G.T. approach.

Now we come to the PCAC approach, due primarily to Gell-Mann[6] and Nambu.[7] There are two distinct arguments associated with PCAC, and we will call these arguments PCAC(1) and PCAC(2).

PCAC(1). Here we require $R_i(s)\to0$ as $s\to\infty$; that is, we require P_μ to be conserved in the limit of large momentum transfer. Then we can write an unsubtracted dispersion relation, say for $i=N\bar{N}$ for R_i

$$R_{N\bar{N}}(s)=\frac{\mu^2 f\sqrt{2}g_{\pi N}}{s-\mu^2}+\frac{1}{\pi}\int ds'\frac{\mathrm{Im}R_{N\bar{N}}(s')}{s'-s}. \tag{12}$$

[11] M. Ida, Phys. Rev. **132**, 401 (1963).
[12] K. Nishijima, Phys. Rev. **133**, B1092 (1964).
[13] B. Barrett and G. Barton, Nuovo Cimento **29**, 703 (1963).
[14] P. Saito, Phys. Rev. **140**, B957 (1965).
[15] See, for example, B. W. Lee, K. T. Mahanthappa, I. S. Gerstein, and M. L. Whippman, Ann. Phys. (N. Y.) **28**, 466 (1964).

When we set $s=0$ we have

$$2Mg_A=-\sqrt{2}g_{\pi N}f+\frac{1}{\pi}\int ds'\frac{\mathrm{Im}R_{N\bar{N}}(s')}{s'}. \qquad (13)$$

When the integral is negligible, pion-pole dominance, Eq. (13) gives the G.T. relation in the form of Eq. (4). The essence, then, of what we call the PCAC(1) argument is that $R_i(s)$ obeys a π-pole-dominated unsubtracted dispersion relation.[16]

PCAC(2). Here we assume the relation, sometimes called the PCAC hypothesis,[6,4]

$$\partial_\mu P^\mu(x)=\mu^2 f\phi_\pi(x)\,, \qquad (14)$$

where $\phi_\pi(x)$ is the pion field. The G.T. relation, in the form Eq. (1), follows immediately by taking the vacuum-$N\bar{N}$ matrix element of Eq. (14). However, this derivation is empty in as much as Eq. (14) does not specify $K_{N\bar{N}}(0)$; in fact, the Haag-Nishijima theorem[17] (H.N.T.) says that Eq. (14) may be considered a definition of the pion field. The H.N.T. says, if $B(x)$ is almost local field with the quantum numbers of the pion such that

$$\begin{aligned}&\langle 0|B(x)|0\rangle=0\,,\\ &\langle 0|B(0)0\rangle=a[(2\pi)^{3/2}2E_\pi]^{-1}\,,\end{aligned} \qquad (15)$$

then $B(x)/a$ may be used as the pion field, in as much as $B(x)/a$ will have the asymptotic properties we require of the pion field. To make an argument based on Eq. (14) go past mere definition, we must make some assumption about $K_{N\bar{N}}(s)$, for example that $K_{N\bar{N}}(s)$ is slowly varying for $0\gtrsim s\gtrsim\mu^2$.

In practice, what PCAC has come to mean is that[18]

$$\langle A|\partial_\mu P^\mu(0)|B\rangle\approx\frac{\mu^2 f}{t-\mu^2}\times\langle A|-(\Box+\mu^2)\phi_\pi(0)|B\rangle|_{t=\mu^2}\,, \qquad (16)$$

where $t=(P_A{}^\mu-P_B{}^\mu)^2$. We shall refer to Eq. (16) as PCAC. In PCAC (1), Eq. (16) provides the pole dominance condition; in PCAC(2) ,Eq. (16) is equivalent to the assumption $K_i(\mu^2)\approx K_i(0)$. If Eq. (16) is true, then we have

$$f=-R_i(0)/g_{\pi i}\,, \qquad (17)$$

which we shall refer to as the PCAC result for f. PCAC, expressed by Eq. (16), is by itself strong enough to guarantee the G.T. relation with $K_i(0)=1$. It is reasonable, but by no means necessary that Eq. (16) will be true when $\langle A|\partial_\mu P^\mu(0)|B\rangle$ obeys an unsubtracted dispersion relation in t; it is, in fact, possible for Eq. (16) to be true for matrix elements with $|B\rangle=|0\rangle$ when the $R_i(t)$ obey once-subtracted dispersion relations as we will see in the next section.

To conclude this section we note that the dispersion-theoretic approaches to the G.T. relation described here require unsubtracted dispersion relations. We will see in the next section that with the assumption of a composite pion we will be able to derive the G.T. relation with once-subtracted dispersion relations.

III. CALCULATIONS

In this section we will discuss the G.T. relation with the specific assumption that the pion is composite. To handle this assumption it is most convenient to use the N/D formalism.[19] We will consider the strong reactions $i\to j$, where i and j stand for strongly interacting channels with the quantum numbers of the pion, and the weak reactions $l\nu\to i$. Our basic assumption is that the pion shows up as a dynamical pole in $\mathbf{t}(s)$, the strong-interaction T matrix describing $i\to j$. We will obtain f by calculating the residue of the pion pole in $R_i(s)$, the amplitude which describes the $l\nu\to i$ reaction, using the fact that $R_i(s)$ is coupled to $\mathbf{t}(s)$ through unitarity.

Specifically, we will consider the coupled-channel T matrix $\mathbf{T}(s)$ which has the matrix form

$$\mathbf{T}(s)=\begin{pmatrix}\mathbf{t}(s) & G\mathbf{R}(s)\\ G\mathbf{R}^T(s) & W(s)\end{pmatrix}, \qquad (18)$$

where $\mathbf{R}(s)$ is the vector whose components are $R_i(s)$, and $W(s)$ describes $l\nu\to l\nu$. We assume that $\mathbf{T}$ has only dynamical singularities, and we write

$$\mathbf{T}(s)=\mathbf{N}'(s)\mathbf{D}'^{-1}(s)\,, \qquad (19)$$

where, apart from possible subtractions for $\mathbf{N}$,

$$\begin{aligned}\mathbf{N}'(s)&=\frac{1}{\pi}\int\frac{\mathbf{L}(s')\mathbf{D}'(s')ds'}{s'-s}\,,\\ \mathbf{D}'(s)&=\mathbf{1}-\frac{s}{\pi}\int\frac{\boldsymbol{\varrho}(s')\mathbf{N}'(s')ds'}{s'(s'-s)}\,,\end{aligned} \qquad (20)$$

with $\mathbf{L}(s)$ the left-hand discontinuity of $\mathbf{T}(s)$. The factor $\boldsymbol{\varrho}(s)$ is the diagonal phase-space factor in the unitarity relation

$$\mathrm{Im}\mathbf{T}(s)=\mathbf{T}^*(s)\boldsymbol{\varrho}(s)\mathbf{T}(s)\,. \qquad (21)$$

The equation for $\mathbf{D}'$ shows we assume that $\mathbf{D}'$ has no

[16] In the case that $|i\rangle$ contains more than three particles, $R_i(s)$ depends on variables other than s, such as relative c.m. energies of two particles in $|i\rangle$, Adler (see Ref. 4) points out that, in general, different sets of variables lead to different pion-pole residues of $R_i(s)$. Thus, strictly speaking, PCAC(1) must contain a precise specification of the variables on which R_i depends.

[17] R. Haag, Phys. Rev. **112**, 699 (1958); K. Nishijima, *ibid.* **111**, 995 (1958); W. Zimmermann, Nuovo Cimento **10**, 597 (1958). For a discussion of the H.N.T. and its relation to current algebra calculations see A. P. Balachandran, M. G. Gundzik, and F. Nicodemi, in Proceedings of the Boulder Conference on Particle Physics, 1966 (to be published). We wish to thank Professor Balachandran for a report of this work prior to publication which brought the H.N.T. to our attention.

[18] See, for example, J. S. Bell, CERN Report No. CERN 66–29 (unpublished).

[19] For details of the N/D formalism see the Hawaii lectures of Zachariasen, Ref. 3.

poles; such poles represent the presence of elementary particles,[15] and we want the pion to be a pure composite particle. It is straightforward to show that the calculation which follows is independent of subtraction point. Finally, we note again that we will neglect quadratic and higher terms in G, the weak coupling.

Because the lowest nontrivial order of G in $\mathbf{t}(s)$ is quadratic we have

$$\mathbf{t}(s)=\mathbf{N}(s)\mathbf{D}^{-1}(s)\,, \qquad (22)$$

where $\mathbf{N}$ and $\mathbf{D}$ are the solutions of Eq. (20) when $G=0$. Our basic assumption that the pion is composite is that $\mathbf{D}^{-1}$ has a simple, dynamically induced pion pole,

$$\mathbf{D}^{-1}(s)=\mathbf{d}(s)/(s-\mu^2)\,, \qquad (23)$$

where $\mathbf{d}(\mu^2)\neq 0$. We assume, of course, that the pion pole is the only pole of $\mathbf{t}$.

It is straightforward matter to show that to first order in G,

$$R_i(s)=\sum_k N_k(s)d_{ki}(s)/(s-\mu^2)\,, \qquad (24)$$

where

$$N_k(s)=R_k(0)+\frac{s}{\pi}\sum_j\int\frac{L_{l\nu j}(s')D_{jk}(s')ds'}{s'(s'-s)}. \qquad (25)$$

We use a subtraction for N_k because the well-known point structure of the basic V–A leptonic coupling guarantees that R_k has no left-hand cut so that $L_{l\nu k}(s)=0$. Thus we have for R_i

$$R_i(s)=\sum_j R_j(0)d_{ji}(s)/(s-\mu^2)\,. \qquad (26)$$

To calculate f we equate the pion-pole residues obtained from Eq. (8) and from Eq. (26), and we get

$$fg_{\pi i}=\mu^{-2}\sum_k R_k(0)d_{ki}(\mu^2)\,. \qquad (27)$$

Before we treat the general multichannel case, we will discuss the calculation of f keeping only the $N\bar{N}$ strong channel; while this restriction is no doubt physically unreasonable, it has the virtue of simplicity. In this case we have only a single strong-interaction channel label which, for the sake of convenience, we will often omit. Equation (27) reduces to

$$\sqrt{2}fg_{\pi N}=-2Mg_A d^{-1}(0)d(\mu^2)\,, \qquad (28)$$

if we recall that $R_{N\bar{N}}(0)=2Mg_A$, and $g_N=\sqrt{2}g_{\pi N}$. We can get a recognizable answer if we bring in the pionic form factor of the nucleon. The simplest way of satisfying the unitarity relation, Eq. (10), and the normalization condition $K(\mu^2)=1$, is to choose

$$K(s)=d(s)/d(\mu^2)\,, \qquad (29)$$

in which case

$$f=-\sqrt{2}Mg_A/[g_{\pi N}K(0)]. \qquad (30)$$

Furthermore, with our choice of $K(s)$ we have

$$R(s)=\sqrt{2}\mu^2 fg_{\pi N}K(s)/(s-\mu^2)\,, \qquad (31)$$

which is the vacuum-$N\bar{N}$ matrix element of Eq. (14). We see that, given a definite model in which the pion is an $N\bar{N}$ bound state, the assumption that $R_i(s)$ needs at most one subtraction leads to a prediction for f. We will defer for the moment any discussion of the condition for which we can expect $d(\mu^2)\cong d(0)$. We also note that here we can have Eq. (14) hold for vacuum-$N\bar{N}$ matrix elements with a simple and natural choice of $K_{N\bar{N}}(s)$.

Now let us return to the general case. A most important requirement is that f, when calculated from Eq. (27), should be independent of the strong-channel index i. To see that this requirement is satisfied we need to examine the properties of the matrix $\mathbf{d}(\mu^2)$. Let us assume that $\mathbf{t}(s)$ has been normalized so that

$$t_{ij}(s)\xrightarrow[s\to\mu^2]{}\frac{g_{\pi i}g_{\pi j}}{s-\mu^2}. \qquad (32)$$

The key property of $\mathbf{d}(\mu^2)$ is that it is of rank one. The assumption that $\mathbf{t}$ has a simple dynamical pole at $s=\mu^2$ guarantees that $\mathbf{D}(\mu^2)$ has rank $n-1$ for an n-channel problem; then an application of Sylvester's law of nullity[20,21] suffices to show that $\mathbf{d}(\mu^2)$ is of rank one. Now an arbitrary rank-one $n\times n$ matrix can always be written as $\{A_iB_j\}$,[21] where A_i and B_j are n-dimensional vectors. This last-mentioned fact, along with Eq. (32), tells us that there exists a vector a_i such that

$$d_{ij}(\mu^2)=a_i g_{\pi j}. \qquad (33)$$

In particular, Eq. (33) shows that $g_{\pi i}$ is an eigenvector of $\mathbf{d}(\mu^2)^T$, and that for any vector b_i

$$\sum_j b_j d_{jk}(\mu^2)=(\text{const})g_{\pi k}. \qquad (34)$$

From Eq. (34) we can see that our calculation does indeed give an f independent of strong-channel label.

Let us investigate what relation, if any, exists between $R_i(s)$ and $K_i(s)$ when K_i is expressed in terms of $\mathbf{d}(s)$. Because of Eq. (33), we find the curious fact that

$$g_{\pi i}K_i(s)=\lambda^{-1}\sum_j g_j d_{ji}(s)\,, \qquad (35)$$

$$\lambda=\sum_i a_i b_i\,,$$

satisfies the unitarity relation for K_i, and the normalization condition, $K_i(\mu^2)=1$, for arbitrary b_i as long as $\lambda\neq 0$. Thus contrary to the case of the pion as an $N\bar{N}$ bound state, we do not have unique expressions for the K_i in terms of a once-subtracted D matrix. In view of the H.N.T. this lack of uniqueness is not disturbing.

[20] J. M. Charap and E. J. Squires, Phys. Rev. **127**, 1387 (1962).
[21] L. Mirsky, *An Introduction to Linear Algebra* (Oxford University Press, New York, 1955).

Clearly there is no necessary relationship between the R_i and the K_i as given above; to obtain the formal expression for f implied by Eq. (14) we must properly define $K_i(0)$. If we choose b_i such that

$$g_{\pi i}K_i(0)=\beta^{-1}R_i(0)\,, \tag{36}$$

where β is a constant, then $-\mu^2\lambda^{-1}b_i=\beta^{-1}R_i(0)$, and Eq. (27) then gives us the G.T. relation in the form

$$f=-\beta=-R_i(0)/[g_{\pi i}K_i(0)]\,. \tag{37}$$

Also, the choice of Eq. (36) for b_i leads to Eq. (14) as far as vacuum-state i matrix elements are concerned,

$$g_{\pi i}K_i(s)=(\mu^2 f)^{-1}(s-\mu^2)R_i(s)\,. \tag{38}$$

Thus we see in detail how to make the choice the H.N.T. says we can make, in the case of a multichannel bound pion, in order to guarantee that Eq. (14) is true for vacuum-state i matrix elements.

We see that the assumptions of a composite pion and once-subtracted dispersion relations for $R_i(s)$ are not strong enough to get us past the formal result of Eq. (37). To go further we need another assumption to play the role of PCAC, Eq. (16). Naively, we would expect that PCAC is equivalent to $\mathbf{d}(\mu^2)\approx\mathbf{1}$ in Eq. (27); however, because det $\mathbf{d}(\mu^2)=0$, while det$\mathbf{1}=1$, $\mathbf{d}(\mu^2)\approx\mathbf{1}$ is evidently not a reasonable approximation. Nevertheless, we can obtain the PCAC result for f, given certain assumptions which we will now discuss. A striking feature of the pion is that it is the least-massive known strongly interacting particle; this has suggested the approximation that in some cases at least the pion mass is negligible. Such an approximation is clearly in the spirit of PCAC. For our purposes we shall assume that a reasonable quantitative statement of the smallness of the pion mass is the linear D approximation (LDA) for $0\lesssim s\lesssim\mu^2$:

$$\begin{aligned}\mathbf{D}(s)&\approx\mathbf{D}_L(s)\,,\\ \mathbf{D}_L(s)&\approx\mathbf{1}-s\mathbf{\Gamma}\,,\end{aligned} \tag{39}$$

where $\mathbf{\Gamma}$ is independent of s. Let us investigate the consequences of the LDA. Of course we require

$$\det\mathbf{D}_L(\mu^2)=0\,, \tag{40}$$

and we require $\mathbf{D}_L{}^{-1}(s)$ to have a simple pole,

$$\mathbf{D}_L{}^{-1}(s)=\mathbf{d}_L(s)/(s-\mu^2)\,, \tag{41}$$

with $\mathbf{d}_L(\mu^2)$ having rank one. An important result for us, which we prove in the Appendix, is that

$$\mathbf{d}_L(\mu^2)\mathbf{d}_L(\mu^2)=-\mu^2\mathbf{d}_L(\mu^2)\,. \tag{42}$$

If we write

$$(d_L)_{ij}(\mu^2)=C_i g_{\pi j}\,, \tag{43}$$

we have from Eq. (42),

$$\sum_j g_{\pi j}(d_L)_{ji}(\mu^2)=\mathbf{g}_\pi\cdot\mathbf{C}g_{\pi i}=-\mu^2 g_{\pi i}\,. \tag{44}$$

From Eq. (27) we get for f

$$f=[-\mathbf{R}(0)\cdot\mathbf{g}_\pi/|\mathbf{g}_\pi|^2]-\mathbf{R}(0)\cdot\mathbf{\Delta}\,, \tag{45}$$

when we write

$$C=-\mu^2(\mathbf{g}_\pi/|\mathbf{g}_\pi|^2)-\mu^2\mathbf{\Delta} \tag{46}$$

with

$$\mathbf{\Delta}\cdot\mathbf{g}_\pi=0\,. \tag{47}$$

Note that the first term on the right-hand side of Eq. (45) has the general structure of (weak coupling/strong coupling) which we associate with the G.T. relation. Clearly the geometrical relation between $\mathbf{R}(0)$ and $\mathbf{g}_\pi$ is of utmost importance for the G.T. relation. In particular, if

$$\begin{aligned}R_i(0)&=\zeta g_{\pi i}\,,\\ \zeta&=\text{const.}\,,\end{aligned} \tag{48}$$

then we get the PCAC result for f,

$$f=-\zeta=-R_i(0)/g_{\pi i}\,. \tag{49}$$

Furthermore, with the LDA and Eq. (48) we get, essentially, PCAC, as far as the R_i are concerned, for a composite pion. With Eqs. (48) and (49) we have

$$R_i(s)=-f\sum_j g_{\pi j}d_{ji}(s)/(s-\mu^2)\,. \tag{50}$$

Now, because of our normalization for D,

$$R_i(0)=-fg_{\pi j}\,, \tag{51}$$

while, because of Eq. (44),

$$R_i(\mu^2)=\lim_{s\to\mu^2}\mu^2 fg_{\pi i}/(s-\mu^2)\,. \tag{52}$$

Thus our $R_i(s)$ satisfies Eq. (16), PCAC, for $s=0$. Equivalently, we have with

$$K_i(s)=-\mu^{-2}\sum_j g_{\pi j}d_{Lji}(s)\,, \tag{53}$$

$K_i(0)=K_i(\mu^2)=g_{\pi i}$. Without a detailed analysis of $\mathbf{\Gamma}$ we cannot assert that for $0\lesssim s\lesssim\mu^2$, $\sum g_{\pi j}d_{Lji}(s)\approx g_{\pi i}$, although such an assertion seems reasonable; the point is that a linear approximation is not necessarily valid for $\mathbf{D}^{-1}(s)$ or $\mathbf{d}(s)$ when we require det$\mathbf{D}(\mu^2)=0$. For many practical purposes the statement

$$\begin{aligned}\langle A|\partial_\mu P^\mu(0)|B\rangle|_{t=0}&=-f\\ &\times\langle A|-(\Box+\mu^2)\phi_\pi(0)|B\rangle|_{t=\mu^2}\end{aligned} \tag{54}$$

is as powerful as PCAC, Eq. (16); hence our use of the word "essential" in the first sentence of this paragraph.

Finally, we note that Dashen and Frautschi[10] have done approximate bootstrap calculations which suggest that Eq. (48) is true at least as far as the baryon-antibaryon channels are concerned. Thus we have the distinct possibility that the success of the G.T. relation can be understood in a bootstrap theory.

IV. SUMMARY

In this paper we have investigated the G.T. relation for a composite pion by means of the multichannel

N/D formalism. We have found that with the assumption of a composite pion we can derive a formal expression for f, Eq. (37), from once-subtracted dispersion relations for $R_i(s)$, whereas previous dispersion-theory work requires unsubtracted dispersion relations. With two additional assumptions, (1) the LDA suggested by the small mass of the pion and (2) the condition suggested by bootstrap theory, $g_{\pi i}/R_i(0)=\text{constant}$ independent of channel label i, we can derive the PCAC result for f, Eq. (17).

We conclude that the G.T. relation can be obtained for a composite pion, and that, assuming assumptions (1) and (2) above are reasonable, the experimental success of the G.T. relation can very likely be understood in a bootstrap theory.

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APPENDIX

Consider a matrix

$$\mathbf{D}(s)=\mathbf{1}-\mathbf{\Gamma}s\,, \tag{A1}$$

where $\mathbf{\Gamma}$ is independent of s. Suppose that $\det\mathbf{D}(s)$ has a simple zero at $s=\mu^2$,

$$\det\mathbf{D}(\mu^2)=0\,, \tag{A2}$$

so that

$$\mathbf{D}^{-1}(s)=\mathbf{d}(s)/(s-\mu^2)\,, \tag{A3}$$

where $\mathbf{d}(\mu^2)\neq 0$. We will show that

$$\mathbf{d}(\mu^2)\mathbf{d}(\mu^2)=-\mu^2\mathbf{d}(\mu^2)\,. \tag{A4}$$

We can write

$$\mathbf{D}^{-1}(s)=\mathbf{A}(s)/[(s-\mu^2)f(s)]\,, \tag{A5}$$

where $f(\mu^2)\neq 0$, and

$$\sum_k D_{lk}(s)A_{km}(s)=\delta_{lm}(s-\mu^2)f(s)=\delta_{lm}\det\mathbf{D}(s)\,. \tag{A6}$$

First we show

$$f(\mu^2)=-\mu^{-2}\sum_i A_{ii}(\mu^2)\,. \tag{A7}$$

We have

$$\frac{\partial(\det\mathbf{D}(s))}{\partial s}=\sum_{ij}\frac{\partial D_{ij}(s)}{\partial s}A_{ji}(s)=-\sum_{ij}\Gamma_{ij}A_{ji}(s)\,. \tag{A8}$$

Thus

$$\left.\frac{\partial(\det\mathbf{D}(s))}{\partial s}\right|_{s=\mu^2}=\sum_{ij}D_{ij}(\mu^2)A_{ji}(\mu^2)-\mu^{-2}\sum_{ij}\delta_{ij}A_{ji}(\mu^2)\,. \tag{A9}$$

The first term on the right of (A9) is zero because $\det\mathbf{D}(\mu^2)=0$, and, as $\det\mathbf{D}(s)=(s-\mu^2)\,\partial(\det\mathbf{D}(s))\partial s|_{s=\mu^2}+\cdots$, we have demonstrated (A7).

Now we show (A4). We have

$$\mu^{-4}\mathbf{d}(\mu^2)\mathbf{d}(\mu^2)=\mathbf{A}(\mu^2)\mathbf{A}(\mu^2)/[\sum_i A_{ii}(\mu^2)]^2\,. \tag{A10}$$

Now we use the fact, discussed in Sec. III, that $\mathbf{d}(\mu^2)$ is of rank one; hence $\mathbf{A}(\mu^2)$ is of rank one. In particular, all 2×2 minors of $\mathbf{A}(\mu^2)$ vanish so that

$$A_{mn}(\mu^2)A_{nl}(\mu^2)=A_{nn}(\mu^2)A_{ml}(\mu^2)\,. \tag{A11}$$

Thus

$$(\mathbf{A}(\mu^2)\mathbf{A}(\mu^2))_{ij}=\sum_k A_{ik}(\mu^2)A_{kj}(\mu^2)=\sum_k A_{kk}(\mu^2)A_{ij}(\mu^2)\,, \tag{A12}$$

or

$$\mathbf{A}(\mu^2)\mathbf{A}(\mu^2)=\mathbf{A}(\mu^2)\sum_k A_{kk}(\mu^2)\,, \tag{A13}$$

and (A4) follows since

$$\mathbf{d}(\mu^2)=-\mu^2\mathbf{A}(\mu^2)/[\sum_k A_{kk}(\mu^2)]\,. \tag{A14}$$