N/D Effective-Range Theory with J-Independent Short-Range Forces*

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Optical models and Regge-pole theory both suggest that the low partial-wave amplitudes are independent of the angular momentum J at very high energies. This, in turn, implies that short-range forces (faraway cuts) should likewise not depend on J. We therefore set up an N/D representation in which the distant singularities are approximated in terms of J-independent effective-range parameters. Nearby force singularities are obtained from single-particle-exchange graphs in the usual way. Thus, if we know the residue and position of a resonance or bound state, we can adjust our effective-range parameters to reproduce it correctly. Our N/D formulas then enable us to calculate the Regge trajectory on which it lies. If we do this for the ρ meson, we can compute approximately the position of the dip in the πN charge-exchange differential cross section observed near the forward direction at high energies.

I. INTRODUCTION

PERHAPS the greatest inadequacy of most dynamical calculations in strong-interaction physics is the total neglect of short-range forces. This, of course, is due to our inability to calculate such terms explicitly, and the accompanying hope that they may not be too important. While this may be reasonable in certain cases, it more typically yields results in violent disagreement with experiment. It therefore becomes necessary to find some way of taking into account interactions at small distances or, in dispersion language, the singularities in distant regions of the complex plane.

One way of representing distant cuts is to make an effective-range approximation. By this we simply mean that, as long as we are interested in a small enough energy range, we can always replace them with a set of simple singularities (such as poles) which only depend on a few constants.[1] This can be most easily seen in terms of the analogy between the singularities in the complex plane and two-dimensional electrostatic charge distributions. Here, faraway charges always produce a smoothly varying potential in the area of interest, provided that the area is sufficiently small. Such behavior can always be simulated by a simpler equivalent charge distribution (such as a collection of point charges). This, for instance, is what one does when one makes a multipole expansion.

Although an effective-range representation of distant singularities should increase the accuracy of dispersion relations when making detailed calculations, it is, by itself, not very useful if we are only interested in a small number of properties (such as the mass and width of a resonance). In a previous paper,[2] hereafter referred to as S, it was argued that distant cuts should obey internal symmetries, however. It is then possible to at least correlate various low-energy results by imposing symmetries on our effective-range parameters. In particular, one can calculate all the masses and couplings of an $SU(3)$ multiplet in terms of two or three constants. Any symmetry breaking which takes place would arise from nearby singularities (long-range forces), which can be easily obtained from conventional dispersion theory.

In S, the above approach was applied to the vector mesons, which were treated as resonances in the scattering of pseudoscalar mesons. Thus the masses and widths of both the ρ and the ϕ could be found from the same pair of constants. Adjusting these constants to give the correct properties of the ρ, the parameters of the ϕ could then be predicted.

In the present paper we shall extend the above ideas to another type of "symmetry." We shall assume that distant singularities are independent of the angular momentum J, at least for low partial waves. This, again, is suggested by high-energy scattering, where both optical models[3] and Regge analysis[4] suggest that the high-energy amplitude varies slowly with J (see Appendix A). Such a property is not usually thought of as a symmetry in the same sense as, say, $SU(3)$. The main reason is that it is difficult to visualize any kind of limit of unbroken symmetry at low energies, which would, for instance, predict resonances at the same energy for different J. The "symmetry" is very badly broken.[5] However, if we only assume it for faraway cuts, it appears much more natural. As in S, symmetry breaking would come from nearby cuts, which can be readily handled in the usual dispersion theory.

In Sec. II we introduce a formalism in which the nearby singularities are treated in the determinantal approximation. Of course, two-body unitarity is exactly satisfied, but the left-hand cut is only approximately reproduced. On the other hand, it is simpler in many ways than the full N/D method, which is discussed in

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[1] See, e.g., G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), Chap. I.
[2] L. A. P. Balázs, Phys. Rev. **152**, 1512 (1966).
[3] See, e.g., N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).
[4] Rather complete accounts of Regge-pole theory are given in S. C. Frautschi, *Regge-Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963); R. Omnès and M. Froissart, *Mandelstam Theory and Regge Poles* (W. A. Benjamin, Inc., New York, 1963); E. J. Squires, *Complex Angular Momenta and Particle Physics* (W. A. Benjamin, Inc., New York, 1963); R. G. Newton, *The Complex j-Plane* (W. A. Benjamin, Inc., New York, 1964).
[5] The breaking may be more or less linear, however, since many Regge trajectories are almost straight lines when plotted as functions of s. See, e.g., V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966).

Sec. IV. Our treatment of distant singularities differs somewhat from the one given in S, which leads to certain difficulties when one tries to impose a symmetry between states of different angular momenta. Thus, instead of introducing a cutoff on the left and treating it as one of our parameters we shall make a cutoff on the right. This we take to be at about the separation point Λ between the low-energy resonance region and the high-energy diffraction region, i.e., at the strip-width[6] of Chew and Frautschi. We then introduce a pole and a double pole in D at the same point, with J-independent residues. These will be our effective-range parameters.

In Sec. III, an approximate version of the above method is applied to the $I=1$ state of $\pi\pi$ scattering. The nearby left-hand cut is assumed to come from ρ exchange, while the effective-range parameters are adjusted so as to give the correct mass and width of the ρ resonance in the $J=1$ state. Our equations then give the mass of the corresponding resonance or bound state for other values of J. In other words, they enable us to calculate the Regge trajectory on which the ρ meson lies. In particular, we can find the point at which the ρ trajectory passes through zero, which is just the position of the bound state in the $J=0$ unphysical "state." This zero of the ρ trajectory has been a subject of some interest recently,[7] since it can be used to explain the dip observed in the πN charge-exchange cross section near the forward direction at high energies.

II. EFFECTIVE-RANGE APPROXIMATION WITHIN A DETERMINANTAL FRAMEWORK

We will consider $\pi\pi$ scattering for simplicity, although there is nothing to prevent us from applying our approach to more complicated problems. For a given isotopic spin I and angular momentum l, the partial-wave amplitude is

$$A_l(\nu)=[(\nu+1)/\nu]^{1/2}e^{i\delta}\sin\delta\,, \tag{1}$$

where

$$\nu=q^2=\tfrac{1}{4}s-1,$$
$$q=\text{c.m. three-momentum},$$
$$s=(\text{total c.m. energy})^2,$$

and

$$\delta=\text{phase shift}.$$

We are taking the pion mass=1 unless other units are specifically mentioned. The assumption of J independence would now imply that $A_l(\nu)$ is independent of l for large ν.

Suppose we take as our input, or force term, the contribution of ρ exchange $F_l(\nu)$. If we treat the ρ as a stable particle, we would get

$$F_l(\nu)\simeq 12\beta_{I1}\frac{\nu_R\Gamma_1}{\nu}\left(1+2\frac{\nu+1}{\nu_R}\right)Q_l\left(1+2\frac{\nu_R+1}{\nu}\right), \tag{2}$$

where $\nu=\nu_R$ is the position of the ρ, $\nu_R\Gamma_1[\nu_R/(\nu_R+1)]^{1/2}$ is its half-width in the ν variable, and β_{I1} is the crossing matrix element connecting our state to the $I=1$ state in the crossed channel; thus $\beta_{01}=1$, $\beta_{11}=\frac{1}{2}$, and $\beta_{21}=-\frac{1}{2}$. From Eq. (2) we see that $F_l(\nu)$ is independent of l for large values of ν. Unfortunately, it reaches this state only very slowly. If, however, we take a Regge expression for ρ exchange, we find that $F_l(\nu)$ becomes l independent relatively rapidly (see Appendix A). In fact, it has already reached this state in the diffraction region $\nu\gtrsim\Lambda$. We shall take advantage of this fact in setting up our effective-range approximation.

Suppose we assume for the moment that $F_l(\nu)$ gives a good approximation to the singularities of the amplitude everywhere except in the resonance region $\nu\lesssim\Lambda$. Then in the determinantal approximation we can write the amplitude as[8]

$$A_l(\nu)=F_l(\nu)/D_l(\nu)\,, \tag{3}$$

where

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^\Lambda d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{F_l(\nu')}{\nu'-\nu}\,. \tag{4}$$

It is simple to see that this satisfies elastic unitarity exactly for $\nu<\Lambda$, where we would expect it to be at least approximately valid. Of course, $F_l(\nu)$, whether Reggeized or not, leaves out the contribution of many short-range effects. One way of incorporating them is by noting that we can get an exact expression for $A_l(\nu)$ provided that we modify Eq. (4) to read

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^\Lambda d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{F_l(\nu')}{\nu'-\nu}+C(\nu)\,, \tag{5}$$

where

$$C(\nu)=\frac{1}{\pi}\int d\nu'\frac{\mathrm{Im}[A_l^{-1}(\nu')F_l(\nu')]}{\nu'-\nu}\,, \tag{6}$$

with the integral extending over the faraway cuts, both on the right and on the left. Now in these regions, $F_l(\nu)$ is l independent, as we saw in the preceding paragraph; so is $A_l(\nu)$, from our basic assumption of J independence. Hence $C(\nu)$ is likewise independent of l.

Of course, we have no way of caculating the integral (6). However, we are only interested in low energies, i.e., in the region of the nearby cuts. Since Eq. (6) involves faraway cuts we can therefore always approximate it in terms of simpler singularities. The simplest procedure would be just to put

$$C(\nu)\simeq C_0+C_1\nu\,, \tag{7}$$

which corresponds to a pole and double pole at infinity. Since $C(\nu)$ is independent of l, the constants C_0 and C_1 can only depend on I.

[6] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).
[7] F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1045 (1966); S. Frautschi, Phys. Rev. Letters **17**, 722 (1966), and references therein.
[8] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958).

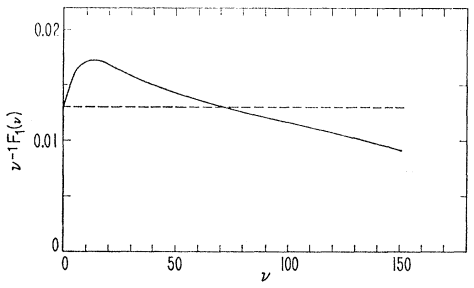

FIG. 1. Plot of $\nu^{-1}F_1(\nu)$ as given by Eq. (2) with $\nu_R=6.5$ and $\Gamma_1=0.17$ (full line) compared with the threshold approximation (11) (dashed line).

Equations (3), (5), and (7) enable us to calculate a Regge trajectory. The position $l=\alpha(\nu)$ of a pole in the complex l plane is given by

$$D_{\alpha(\nu)}(\nu)=0. \tag{8}$$

Now the reduced width, i.e., the residue of the pole in $\nu^{-l}A_l(\nu)$ for a given l, is

$$\Gamma_l=-\nu^{-l}F_l(\nu)/D_l'(\nu). \tag{9}$$

The residue of the corresponding pole in the complex l plane is thus

$$\beta(\nu)=\alpha'(\nu)\Gamma_{\alpha(\nu)}. \tag{10}$$

We can therefore determine our trajectory from the constants C_0 and C_1. These can be fixed if we know the parameters of any resonance or bound state which lies on the trajectory. For instance, a knowledge of the mass and width of the ρ can be used to find C_0 and C_1 through Eqs. (8) and (9) for $l=\alpha=1$.

In the above discussion we assumed that $F_l(\nu)$ is l independent for $\nu>\Lambda$. This, as we have seen, is true for a Reggeized exchange but is not such a good approximation for Eq. (2). However, our final expressions involved an explicit knowledge of $F_l(\nu)$ only for $\nu<\Lambda$. In this region, the Reggeized and un-Reggeized forms do not differ too much from each other.[9] We shall therefore use Eq. (2) in practice.

III. CALCULATION OF THE ZERO OF THE ρ TRAJECTORY

We now apply the above formalism to the ρ trajectory, which occurs in the $I=1$ state. Our equations become difficult to handle for noninteger l, so we shall restrict ourselves to $l=0$ and $l=1$. Since the ρ meson occurs as a resonance in the latter state, we can use its mass and width to fix C_0 and C_1. We can then see where $D_0(\nu)=0$ in the $l=0$ state. This gives the point where the trajectory passes through zero.

In the $l=1$ state, we can make a threshold approximation for $F_l(\nu)$. If, in addition, we take advantage of the fact that $\nu_R\gg1$, we have

$$F_1(\nu)\simeq(\Gamma_1/2\nu_R)\nu. \tag{11}$$

[9] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

From Fig. 1 we see that Eq. (11) is a reasonable approximation to Eq. (2) over a considerable range of ν. If we substitute Eq. (11) into Eqs. (3) and (5) we will have an explicit expression for $A_1(\nu)$. Actually, as long as we are not interested in the region $-1\lesssim\nu\lesssim1$ we can also make the "extreme relativistic" approximation in the D function

$$[\nu'/(\nu'+1)]^{1/2}\simeq1, \tag{12}$$

which leads to

$$D_1(\nu)\simeq1-\frac{1}{\pi}\frac{\Gamma_1}{2\nu_R}\left[\Lambda+\nu\ln\left(\frac{\Lambda-\nu}{-\nu}\right)\right]+C_0+C_1\nu. \tag{13}$$

Taking $\nu_R=6.5$ and $\Gamma_1=0.17$ for the position and reduced width of the ρ meson,[10] we can find C_0 and C_1 by substituting Eqs. (11) and (13) into Eqs. (8) and (9) at $l=\alpha=1$ and $\nu=\nu_R$.

We next turn to the $l=0$ unphysical "state." Here $F_l(\nu)$ has a more complicated shape (see Fig. 2). It was therefore approximated by a constant plus a pole

$$F_0(\nu)\simeq A-\lambda/(\eta+\nu), \tag{14}$$

with $A=3.60$, $\lambda=101.1$, and $\eta=32.72$. Substituting Eq. (14) into Eq. (4) and again making the extreme relativistic approximation (12), we obtain

$$D_0(\nu)=1-\frac{1}{\pi}\left(A-\frac{\lambda}{\eta+\nu}\right)\ln\left(\frac{\Lambda-\nu}{-\nu}\right)-\frac{1}{\pi}\frac{\lambda}{\eta+\nu}\ln\left(\frac{\Lambda+\eta}{\eta}\right)+C_0+C_1\nu. \tag{15}$$

The zero of this function gives the point $\nu=\nu_0$ at which the trajectory goes through zero.

Table I lists the results for various values of Λ, the separation point between the resonance and diffraction regions,[6] which is expected to lie somewhere in the region $200\lesssim\Lambda\lesssim50$.[11] The zero is seen to be relatively insensitive to Λ. It should be compared with the position $t\simeq-0.6$ (GeV/c)^2 of the dip in the high-energy πN charge-exchange cross section, which has been explained as arising from the vanishing of the ρ trajectory.[7] Actually, the position of the dip and the zero of trajectory do not necessarily coincide exactly. The model of Arbab and Chiu,[7] for instance, led to $\alpha=0$ at $t\simeq-0.52$ (GeV/c)^2.

TABLE I. The values ν_0 and t_0 of ν and s at which the ρ trajectory goes through zero for various values of Λ, using the method of Sec. III.

Λ	ν_0	t_0 (GeV/c)^2
75	−6	−0.39
113.3	−7.7	−0.52
300	−9.3	−0.65

[10] We are taking mass=765 MeV and width=105 MeV for the ρ. See A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[11] For a recent discussion of this parameter, see N. F. Bali, Phys. Rev. **150**, 1358 (1966).

IV. GENERALIZATION TO THE FULL N/D FORMALISM

We now discuss the extension of the above approach to a full N/D framework. This has the advantage of reproducing exactly the nearby left-hand cut supplied by the input $F_l(\nu)$. Its main disadvantage is that it is more complicated. The first complication arises from the fact that for noninteger l, $A_l(\nu)$ is singular for $0>\nu>-1$, as can be seen, for instance, from Eq. (2). This prevents us from writing N/D equations for $A_l(\nu)$. We therefore have to consider the modified amplitude

$$B_l(\nu)=\nu^{-l}A_l(\nu), \tag{16}$$

which does not have this difficulty. The corresponding input function is

$$V_l(\nu)=\nu^{-l}F_l(\nu). \tag{17}$$

If we write N/D equations for $B_l(\nu)$, the equations corresponding to (3) and (5) become[12]

$$B_l(\nu)=N_l(\nu)/D_l(\nu), \tag{18}$$

where

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^\Lambda d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{\nu'^l N_l(\nu')}{\nu'-\nu}+C(\nu), \tag{19}$$

and

$$N_l(\nu)=\frac{1}{\pi}\int\frac{\mathrm{Im}V_l(\nu')D_l(\nu')}{\nu'-\nu}, \tag{20}$$

with the integral extending over the singularities of $V_l(\nu)$. As in Sec. II, $C(\nu)$ represents the effect of singularities which are not included in $V_l(\nu)$ or given by unitarity. It has the form

$$C(\nu)=\frac{1}{\pi}\int d\nu'\frac{\mathrm{Im}[B_l^{-1}(\nu')N_l(\nu')]}{\nu'-\nu}, \tag{21}$$

where the integral extends over all faraway cuts. Now it is not immediately obvious how $N_l(\nu)$ varies with l in these regions. If we know it, we would not have to make an effective-range approximation in the first place. We will therefore make the additional assumption that the dynamics of the high-energy region is essentially independent of any effects coming from the direct-channel low-energy region. This is true in a Regge-pole theory, for instance, or, for that matter, any theory which assumes that high-energy effects are governed by the exchanges of low-lying systems in the crossed channel. This means that $B_l^{-1}(\nu)N_l(\nu)$ should be essentially the same for large ν as it would be if we made a calculation which assumed l independence everywhere, both at low and high energies. Therefore, $C(\nu)$ is independent of l.[13]

As in Sec. II, we can make an effective-range approximation for $C(\nu)$ if we are only interested in the amplitude at low energies. In this case, however, a linear approximation leads to certain difficulties in practice. We approximate it instead by a pole and double pole placed at $\nu=\Lambda$, the point at which we expect the singularities of $C(\nu)$ to become important. Thus,

$$C(\nu)\simeq\frac{\alpha}{\Lambda-\nu}\left(1+\frac{\beta}{\Lambda-\nu}\right), \tag{22}$$

where α and β are independent of l.

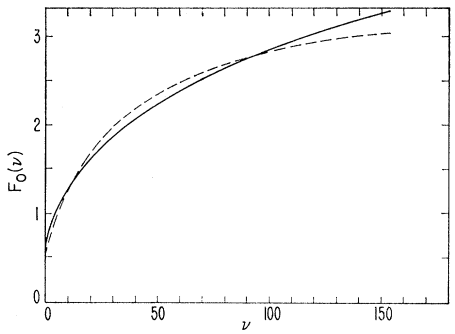

FIG. 2. Plot of $F_0(\nu)$ as given by Eq. (2) with $\nu_R=6.5$ and $\Gamma_1=0.17$ (full line), compared with the approximate form (14) with $A=3.60$, $\lambda=101.1$, and $\eta=32.72$ (dashed line).

In practice, it may sometimes be more convenient to use the Uretsky form of the N/D equations.[14] These can be obtained by substituting Eq. (19) into Eq. (20), which leads to an integral equation for $N_l(\nu)$. In this equation the input is $V_l(\nu)$, which is needed only for $0<\nu<\Lambda$. In this region, as we saw from Sec. II, we can use Eq. (2). Once we know $N_l(\nu)$, we can determine $D_l(\nu)$ from Eq. (19), as before.

To see how an N/D calculation compares with the determinantal results of the preceding section, Eqs. (18) (19), (20), and (22) were used instead of Eqs. (3), (5), and (7) for the case $\Lambda=113.3$. Since, as we noted in the preceding paragraph, they are needed only for $0<\nu<\Lambda$, Eqs. (11) and (14) were again taken as approximate inputs, and the extreme relativistic approximation (12) was used. As in Sec. III, the effective-range parameters α and β were taken so as to reproduce the ρ mass and width in the $l=1$ state, and the vanishing of $D_0(\nu)$ then gave the position of the zero of the trajectory. This occurs at $\nu_0=-6.7$, which is not too different from the value obtained in Table I.

V. CONCLUSION

In S, we saw that it is possible to do calculations in which distant singularities are assumed to obey an internal symmetry, like $SU(3)$. The symmetry breaking then comes from the explicitly calculable nearby singularities. In the present paper this approach is extended to what would normally not be considered a broken symmetry in the first place. We assumed that distant singularities are independent of the angular momentum. Thus the slope of a Regge trajectory, as well as the

[12] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[13] This argument, of course, assumes that D_l is normalized in an l-independent way at $\nu=\infty$, which it certainly is with our equations.

[14] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

variation of the residue β with energy, is assumed to be due to the nearby singularities, which can be handled with conventional techniques.

Both S and the present work are attempts to find a middle ground between the symmetry and bootstrap approaches to strong interactions. The symmetry approach has yielded many relations between experimental quantities. It is, however, severely limited when the symmetry is badly broken, and usually relies on additional assumptions when it is used to get specific results. Many things cannot be predicted at all. The bootstrap approach, on the other hand, can, in principle, predict anything. In practice, however, it is this very "all-or-nothing" feature which has made it so difficult to extract any useful results from it. One is always forced to make approximations which are so drastic that it is not clear whether the model one is using has much relation to the problem one is considering. We have therefore tried to combine various aspects of the two approaches so as to be able to calculate at least as much as would be calculable in an unbroken symmetry. This should make it possible to extend the range of predictions possible in the pure symmetry approach without, hopefully, making the approximations so drastic as to make the calculation useless.

If we include more symmetries, we may be able to reduce the number of parameters needed to describe any set of data even further. Another way of reducing the number of parameters might be to make use of various sum rules. This use of sum rules has, of course, been attempted before in making bootstrap calculations.[15] The results have been similar to those obtained from any other bootstraps. If, however, we first reduce the number of parameters through some symmetry (as we have been doing above or in S), we have to use fewer sum rules to determine the remaining parameters. We could then select those which are more reliable, or less sensitive to approximations. This might lead to better results, perhaps even in a calculation in which there are no remaining free parameters.

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APPENDIX A

To get some idea of the range of validity of J independence we will consider the Regge-pole model of high-energy scattering, which leads to this "symmetry" automatically. As in Sec. II, we shall consider $\pi\pi$ scattering, which is described by a total invariant amplitude $A(s,t)$, where $-t$ is the square of the momentum transfer. We assume that for large s, $A(s,t)$ is dominated by a t-channel Regge pole. If we further assume that the position $\alpha(t)$ and residue of this pole in the angular momentum plane vary slowly with t, we have

$$A(s,t)=\gamma(\tfrac{1}{2}s)^{\alpha(t)}, \tag{A1}$$

with

$$\alpha(t)=\alpha_0+\epsilon t, \tag{A2}$$

where γ, α_0, and ϵ are approximately constant.

Let us project out the $l=0$ and $l=1$ waves from Eq. (A1). We obtain

$$A_0(\nu)=\gamma\frac{(\tfrac{1}{2}s)^{\alpha_0}}{s-4}\frac{1-(\tfrac{1}{2}s)^{-\epsilon(s-4)}}{\epsilon\ln(\tfrac{1}{2}s)}, \tag{A3}$$

and

$$A_1(\nu)=A_0(\nu)-\frac{2\gamma(\tfrac{1}{2}s)^{\alpha_0}}{[\epsilon(s-4)\ln(\tfrac{1}{2}s)]^2}\times\{1-(\tfrac{1}{2}s)^{-\epsilon(s-4)}[\epsilon(s-4)\ln(\tfrac{1}{2}s)+1]\}. \tag{A4}$$

For very large s, Eqs. (A3) and (A4) both reduce to

$$A_l(\nu)=\tfrac{1}{2}\gamma(\tfrac{1}{2}s)^{\alpha_0-1}/\epsilon\ln(\tfrac{1}{2}s), \tag{A5}$$

with a fractional difference

$$R\equiv(A_1-A_0)/A_0\simeq-2/\epsilon s\ln(\tfrac{1}{2}s), \tag{A6}$$

which falls off fairly rapidly with s. We have J independence when $R\ll1$. If, for instance, we take $\epsilon\simeq0.0125$, which is a typical value for the slope of a Regge trajectory, we find that $R=0.1$ when $s\simeq300$, or $\nu\simeq75$. Since this is roughly the separation point Λ between the resonance and diffraction regions,[11] we see that J independence is indeed valid for $\nu>\Lambda$, as we have been assuming above. Of course, this value of R may be somewhat different if γ is a function of t and $\alpha(t)$ is not linear, but it is unlikely that the order of magnitude will be changed.

It is quite straightforward to show that Eq. (A5) is valid for all l at sufficiently large values of s. This is true even if γ and α are more general functions of t, in which case we must replace γ, α_0, and ϵ by $\gamma(0)$, $\alpha(0)$, and $\alpha'(0)$, respectively. In the case of unphysical l, one has to use the representation

$$A_l(\nu)=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)A[s,-\tfrac{1}{2}(s-4)(1-z)]-\frac{\sin\pi l}{\pi}\int_{-\infty}^{-1}dz\,Q_l(-z)A[s,-\tfrac{1}{2}(s-4)(1-z)], \tag{A7}$$

instead of the usual projection formula, which corresponds to the first term of (A7). This representation has been shown by Wong to be equivalent to the usual Froissart-Gribov continuation.[16]

The Regge model is also useful in deciding on the extent of J independence when we bring in additional quantum numbers, such as isotopic spin. For instance, if we assume that high-energy behavior is dominated by the exchange of the Pomeranchuk trajectory (or, for

[15] See, e.g., L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962); **132**, 867 (1963); V. Singh and B. M. Udgaonkar, *ibid.* **130**, 1177 (1963); T. Kanki and A. Tubis, *ibid.* **136**, B723 (1964).

[16] This result is mentioned in G. F. Chew, Phys. Rev. **129**, 2363 (1963).

that matter, by any $I=0$ system), then $A_l(\nu)$ will be independent of both l and I. If, however, we exchange, say, the ρ trajectory, we find that, although $A_l(\nu)$ is independent of l for any given I, it will be different for different I. Thus, J independence is likely to be a good approximation only if we restrict ourselves to a given value of I, although it may be roughly valid even if we ignore isotopic spin, especially for extremely high energies. Needless to say, these arguments apply even if we exchanged more complicated systems, such as Regge cuts.[4]

APPENDIX B

In Secs. III and IV, we restricted ourselves to $l=0$ and $l=1$ in calculating the Regge trajectory. We shall now discuss a simple way of determining the entire Regge trajectory. Although this method is slightly more complicated for $l=0$ and $l=1$, it can be readily extended to noninteger l. For integer l, we can compare the results with the ones obtained in Table I and Section IV. It will be seen that they are not too different.

As in Sec. II we shall start from the determinantal formalism for simplicity. This is given by Eqs. (3) and (5). We saw in Sec. IV that this approximation seems to be a valid one for calculating the ρ trajectory. There is no great simplification resulting from the use of a linearized $C(\nu)$, so we shall use Eq. (22) for $C(\nu)$. In dealing with the D function we shall use the Pagels approximation.[17] This approximation consists of replacing the phase-space factor by a δ function

$$\nu'^l[\nu'/(\nu'+1)]^{1/2}\simeq g_l\delta(\nu-\nu_l). \tag{B1}$$

Equation (5) then becomes

$$D_l(\nu)=1-(1/\pi)g_lV_l(\nu_l)/(\nu_l-\nu)+C(\nu), \tag{B2}$$

where $V_l(\nu)$ is given by Eq. (17).

It can be argued[17] that Eq. (B2) is a good approximation to Eq. (5) provided that

$$I_l(\nu)\equiv\int_0^\Lambda d\nu'\frac{\nu'^l}{\nu'-\nu}\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\simeq\frac{g_l}{\nu_l-\nu} \tag{B3}$$

is valid over the entire left-hand cut.

The approximation (B2) obviously fails on the right-hand cut, where (B3) becomes meaningless. Suppose, however, we rewrite Eq. (5) as

$$D_l(\nu)=1+C(\nu)-V_l(\nu)I_l(\nu)$$
$$-\frac{1}{\pi}\int_0^\Lambda d\nu'\nu'^l\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{V_l(\nu')-V_l(\nu)}{\nu'-\nu}. \tag{B4}$$

The last integral is now nonsingular on the right-hand cut. We can therefore use Eq. (B1) in evaluating it.

[17] H. Pagels, Phys. Rev. 140, B1599 (1965).

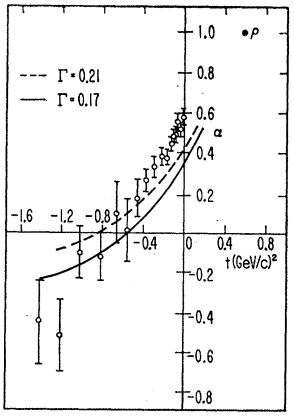

FIG. 3. Plot of $\alpha(t)$ as calculated from Eq. (B5) with $\Gamma_1=0.17$ and $\Gamma_1=0.21$. The corresponding experimental points were taken from Ref. 7. (Here $t=s$.)

This leads to

$$D_l(\nu)=1+C(\nu)-V_l(\nu)I_l(\nu)-\frac{g_l}{\pi}\frac{V_l(\nu_l)-V_l(\nu)}{\nu_l-\nu}, \tag{B5}$$

which can be used for $\nu>0$. We shall use it for $\nu>\frac{1}{2}\nu_L$, where $\nu_L=-\nu_R-1$ is the start of the left-hand cut. For $\nu<\frac{1}{2}\nu_L$ we shall use the simple expression (B2), since Eq. (B3) is valid here.

In practice, g_l and ν_l were determined for each l by requiring that (B3) be exact at $\nu=-2(\nu_R+1)$ and $\nu=-\infty$. For $\Lambda=113.3$, this led to values which enabled (B3) to be satisfied on the average to several percent over the entire left-hand cut, at least for $l=0$ and $l=1$. Equation (B3) thus seems to be quite a reasonable approximation. In evaluating $I_l(\nu)$, we have used the extreme relativistic approximation (12). This was compared with the exact form for $l=0$, $\frac{1}{2}$, and 1, and found to be accurate to a few percent for about the values of ν at which the Regge trajectory intersects these angular momenta.

As in Secs. III and IV, the parameters α and β which occur in $C(\nu)$ were determined by substituting Eqs. (2) and (B5) into (8) and (9) at $l=\alpha=1$ and $\nu=\nu_1$, with $\nu_1=6.5$ and $\Gamma_1=0.17$.[10] Equation (8) then enables us to evaluate the trajectory which is shown in Fig. 3 for $\Lambda=113.3$. We see that the zero of α is at about the same point as in Table I.

We have also shown the results for $\Gamma_1=0.21$, which corresponds to a width of 130 MeV for the ρ meson. Such a larger width for the ρ tends to be increasingly favored by experiment. The experimental points in Fig. 3 are the results of a "model-independent" analysis by Arbab and Chiu,[7] who assumed that the ρ trajectory dominates the $\pi^-p\to\pi^0n$ differential cross section near the forward direction at high energies. It is seen that our trajectory tends to be flatter on the average than the experimental one, but is otherwise approximately consistent with it, especially in view of the uncertainties of the "experimental" analysis.