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Quantum Theory of Superfluid Vortices. I. Liquid Helium II*

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A Hamiltonian formulation is constructed for the classical dynamical equations of slightly deformed rectilinear vortices. The system is then quantized by interpreting the conjugate variables as quantum-mechanical operators that obey canonical commutation relations. A linear canonical transformation diagonalizes the Hamiltonian in terms of operators that create and destroy single quanta of vortex vibrations. The theory is applied to two distinct configurations in He II: a single vortex and a rotating vortex lattice. The specific heat associated with the vortex waves varies approximately as $T^{1/2}$ at low temperatures. Quantum-mechanical and thermal fluctuations produce a finite mean-square displacement of the vortex core, which is studied both at $T=0$ and at $T>0$.

I. INTRODUCTION

THE concept of quantized vortices[1–3] has successfully explained many of the properties of superfluid helium and type-II superconductors.[4] For most purposes, these vortices may be treated classically, with the quantized circulation as the only remnant of their fundamental quantum-mechanical origin. Such an approach cannot be wholly satisfactory, however, for a fundamental difficulty of principle remains unsolved: the localization of the vortex axis necessarily leads to zero-point motion, which cannot be included in the classical framework. A related question is the calculation of the mean-square displacement of the cores in a vortex lattice. In the quantum theory of a crystal consisting of point masses,[5,6] the mean-square displacement depends on the frequency spectrum and on the temperature. The analogous frequency spectrum of a vortex lattice has been derived for He II,[7,8] for bulk type-II superconductors,[9,10] and for thin superconducting films.[11] Direct substitution into the standard expressions[5,6] apparently predicts a divergent mean-square displacement at any finite temperature, which suggests that a more fundamental approach is needed. Finally, the thermal excitation of vortex waves contributes to the specific heat, and it is interesting to consider the possibility of experimental detection of vortices by calorimetric techniques.

In order to answer the above questions, a fully quantum-mechanical treatment is required; the present paper therefore proposes a quantum theory of slightly deformed rectilinear vortices.[12] In the approximation of small bending, the classical equations of vortex motion may be cast in a Hamiltonian formulation (Sec. II), in which the x and y components of the displacements act as conjugate variables. The system is quantized by interpreting these variables as quantum-mechanical operators subject to the canonical commutation relations. For the special case of a vortex lattice, the translational invariance allows the introduction of plane-wave states, which simplifies the Hamiltonian considerably (Sec. III). A linear transformation diagonalizes the Hamiltonian in terms of operators that create and destroy single quanta of vortex waves (Sec. IV). It is then straightforward to calculate both the specific heat associated with the excited states of the vortex lattice (Sec. V) and the mean-square displacement of the vortex cores (Sec. VI). For simplicity, the present work is restricted to a vortex lattice in He II; most of the formulas remain valid for a general vortex lattice, however, and a subsequent paper will treat the situation in bulk type-II superconductors.

II. HAMILTONIAN FORMULATION

It has been known for nearly a century[13–15] that the motion of a system of rectilinear vortices allows a Hamiltonian formulation; the Hamiltonian is proportional to the interaction energy of the vortex array, and the x and y coordinates of the ith rectilinear vortex constitute the ith pair of conjugate variables. This simple dynamical approach depends crucially on the

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1 L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

2 R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

3 A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

4 See, for example, W. F. Vinen, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Company, Amsterdam, 1966), p. 74.

5 R. E. Peierls, *Quantum Theory of Solids* (Oxford University Press, Oxford, 1956), pp. 63–68.

6 J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1964), pp. 51–63.

7 V. K. Tkachenko, Zh. Eksperim. i Teor. Fiz. **50**, 1573 (1966) [English transl.: Soviet Phys.—JETP **23**, 1049 (1966)].

8 D. Stauffer, Phys. Letters **24A**, 72 (1967) and (private communication).

9 P. G. de Gennes and J. Matricon, Rev. Mod. Phys. **36**, 45 (1964); J. Matricon, Phys. Letters **9**, 289 (1964).

10 A. L. Fetter, P. C. Hohenberg, and P. Pincus, Phys. Rev. **147**, 140 (1966).

11 A. L. Fetter and P. C. Hohenberg, Phys. Rev. **159**, 330 (1967).

12 The only previous attempt at a quantum-mechanical treatment is by H. E. Hall, Proc. Roy. Soc. (London) **A245**, 546 (1958), whose approach is rather different from that used here.

13 G. Kirchhoff, *Vorlesungen über Mathematische Physik: Mechanik* (B. G. Teubner, Leipzig, 1883), 3rd ed., pp. 251–272.

14 H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., p. 230.

15 C. C. Lin, *On the Motion of Vortices in Two Dimensions* (University of Toronto Press, Toronto, Canada, 1943).

two-dimensional nature of the motion and is incorrect for an arbitrary three-dimensional vortex system. Nevertheless, it is possible to extend the Hamiltonian formulation to include a restricted class of three-dimensional configurations, in which rectilinear vortices are slightly bent from their original straight form. Since such deformations are precisely those of interest in the small oscillations of a vortex lattice, the resulting theory is able to treat interesting physical questions, in spite of its limited range of validity.

The energy of a system of vortices in an incompressible fluid is the total kinetic energy of the moving fluid. Each vortex contributes linearly to the velocity field, so that the kinetic energy contains two distinct contributions: the self-energy of each vortex, and the interaction energy between each pair of vortices. It is therefore sufficient to construct a Hamiltonian for a system consisting of only two separate vortices; the general theorem then follows from the linearity of the dynamical equations. Furthermore, the self-induced motion does not require a separate treatment because a single vortex may be considered as a bundle of parallel elementary filaments. This last picture is identical with that used in calculating the self-inductance of a current-carrying wire.[16]

Hence we shall temporarily study an infinite incompressible fluid of density ρ, containing two vortices with circulation κ. The vortex axes are specified by the three-dimensional vector functions $\mathbf{R}_1$ and $\mathbf{R}_2$; these functions depend on a single parameter, which is usually chosen as the arc length along the vortex. It is convenient to resolve all vectors in cylindrical polar coordinates: $\mathbf{R}_1=(\mathbf{r}_1,z_1)$ and $\mathbf{R}_2=(\mathbf{r}_2,z_2)$, where $\mathbf{r}$ is a two-dimensional vector in the xy plane, perpendicular to the undeformed vortex axis. A straightforward calculation shows that the fluid velocity at $\mathbf{R}_1$ due to the presence of the second vortex is given by[17]

$$\mathbf{v}(\mathbf{R}_1)=\frac{\kappa}{4\pi}\int\frac{d\mathbf{s}_2\times(\mathbf{R}_1-\mathbf{R}_2)}{|\mathbf{R}_1-\mathbf{R}_2|^3},\qquad(1)$$

where the line integral is along the length of the second vortex. In addition, a basic theorem of classical hydrodynamics[17] states that each element of a vortex core moves with the local fluid velocity at that point; Eq. (1) therefore provides the dynamical equation for the first vortex, apart from self-induced effects, which are considered in Sec. III. A second fundamental result of classical hydrodynamics expresses the interaction energy of two vortices as[17]

$$E_{12}=\frac{\rho\kappa^2}{4\pi}\int\int\frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{|\mathbf{R}_1-\mathbf{R}_2|}.\qquad(2)$$

Equations (1) and (2) are direct analogs of the Biot-Savart law and of Neumann's formula for the mutual inductance of two current-carrying wires.[16]

For a general three-dimensional configuration, Eqs. (1) and (2) are not directly related. In the special case of rectilinear vortices subject to small deformations, however, the equations may be greatly simplified. Let the axes of the undeformed rectilinear vortices be specified by the vector functions $\mathbf{R}_1^0(z)=(\mathbf{r}_1,z)$ and $\mathbf{R}_2^0(z)=(\mathbf{r}_2,z)$, where the coordinate $z(-\infty<z<\infty)$ has been chosen as the parametric variable. When the vortices are deformed, the corresponding axes are given by the vectors $(\mathbf{r}_1+\mathbf{u}_1(z),z)$ and $(\mathbf{r}_2+\mathbf{u}_2(z),z)$. Here, the small displacements $\mathbf{u}_i$ are confined to the xy plane, since it can be shown that displacements along the vortex axes do not contribute to the linearized equations of motion.[18] It is natural to take z as the integration variable, so that $d\mathbf{s}_i=(d\mathbf{s}_i/dz)dz=dz[\hat{z}+(d\mathbf{u}_i/dz)]$. Hence, the translational velocity of the vortex element originally at R_1^0 is given by

$$\dot{\mathbf{u}}_1(z_1)=\frac{\kappa}{4\pi}\int_{-\infty}^{\infty}dz_2\frac{[\hat{z}+(d\mathbf{u}_2/dz_2)]\times[\mathbf{R}_1^0+\mathbf{u}_1-\mathbf{R}_2^0-\mathbf{u}_2]}{|\mathbf{R}_1^0+\mathbf{u}_1-\mathbf{R}_2^0-\mathbf{u}_2|^3}.\qquad(3)$$

This equation may be linearized in the small displacements, yielding[18]

$$\dot{\mathbf{u}}_1(z_1)=(4\pi)^{-1}\kappa\int_{-\infty}^{\infty}dz_2\{|\mathbf{R}_{12}^0|^{-3}\times[\hat{z}\times\mathbf{r}_{12}+\hat{z}\times(\mathbf{u}_1-\mathbf{u}_2)+(d\mathbf{u}_2/dz_2)\times\hat{z}(z_1-z_2)]-3|\mathbf{R}_{12}^0|^{-5}\hat{z}\times\mathbf{r}_{12}[\mathbf{r}_{12}\cdot(\mathbf{u}_1-\mathbf{u}_2)]\},\qquad(4)$$

where $\mathbf{R}_{12}^0=\mathbf{r}_{12}+\hat{z}(z_1-z_2)$ and $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$.

In a similar manner, the interaction energy between the deformed lines may be written *exactly* as

$$E_{12}=\frac{\rho\kappa^2}{4\pi}\int\int dz_1dz_2\left[1+\frac{d\mathbf{u}_1}{dz_1}\cdot\frac{d\mathbf{u}_2}{dz_2}\right]|\mathbf{R}_{12}^0+\mathbf{u}_1-\mathbf{u}_2|^{-1}.\qquad(5)$$

This expression must be expanded to *second* order in the small displacements, and we find

$$E_{12}=\frac{\rho\kappa^2}{4\pi}\int\int dz_1dz_2\left\{|\mathbf{R}_{12}^0|^{-1}+|\mathbf{R}_{12}^0|^{-1}\left(\frac{d\mathbf{u}_1}{dz_1}\cdot\frac{d\mathbf{u}_2}{dz_2}\right)-\frac{\mathbf{r}_{12}\cdot(\mathbf{u}_1-\mathbf{u}_2)}{|\mathbf{R}_{12}^0|^3}-\frac{(\mathbf{u}_1-\mathbf{u}_2)^2}{2|\mathbf{R}_{12}^0|^3}+\frac{3}{2}\frac{[\mathbf{r}_{12}\cdot(\mathbf{u}_1-\mathbf{u}_2)]^2}{|\mathbf{R}_{12}^0|^5}\right\}.\qquad(6)$$

Equation (6) is a bilinear functional of the displacements $\mathbf{u}_1$ and $\mathbf{u}_2$, and a straightforward calculation shows that its variational derivative is equal to

$$\delta E_{12}/\delta\mathbf{u}_1(z)=\rho\kappa\hat{z}\times\dot{\mathbf{u}}_1(z).\qquad(7)$$

[16] See, for example, M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Blackie and Son, Ltd., London, 1954), 2nd ed., pp. 125–131 and 172–177.
[17] Reference 14, pp. 202–204, 211, and 217.

[18] This conclusion is inherent in the discussion of E. S. Raja Gopal, Ann. Phys. (N. Y.) 29, 350 (1964), Appendix.

This equation may be resolved into x and y components as

$$\delta E_{12}/\delta u_{x1}(z) = -\rho\kappa \dot{u}_{y1}(z)\,, \qquad (8)$$
$$\delta E_{12}/\delta u_{y1}(z) = \rho\kappa \dot{u}_{x1}(z)\,,$$

which explicitly exhibits the Hamiltonian nature of the system. In particular, the interaction energy E_{12} is identified with the Hamiltonian, while the deformations u_{xi} and u_{yi} are proportional to the conjugate variables. The zero-order term of Eq. (6) represents the interaction energy of the undeformed vortices and does not affect the equations of motion; the linear and quadratic terms in Eq. (6) correspond, respectively, to the zero-order and first-order terms in Eq. (4) for the velocity.

The above considerations may now be applied to a large vortex lattice confined to a circle of radius R. In equilibrium, the array is specified by the set of two-dimensional vectors $\{\mathbf{r}_i\}$, while the small horizontal displacement of the ith vortex at the height z is given by $\mathbf{u}_i(z)$. The total energy of the system is equal to

$$E = \sum_i E_i + \tfrac{1}{2} \sum_{ij}{}' E_{ij}\,, \qquad (9)$$

where E_i is the self-energy of the ith vortex and E_{ij} is the interaction energy between the ith and jth vortices. Here, the primed sum is over i and j separately, omitting the terms $i=j$. In the harmonic approximation, the explicit form of E_{ij} has already been given in Eq. (6). Furthermore, the self-energy may also be calculated from Eq. (6) by restricting $\mathbf{r}_i$ and $\mathbf{r}_j$ to the same circle of radius a, which represents the vortex core; the average value of E_{ij} over the circle then yields the self-energy E_i. It will be shown below (Sec. III) that this procedure correctly reproduces the self-induced motion originally calculated by Kelvin.[19]

Equation (6) contains terms linear in $\mathbf{u}_i$, which produce a uniform rotation of the vortex lattice about its center.[20] It is convenient to eliminate these linear terms by a transformation to a coordinate system rotating with angular velocity Ω. The transformed total energy of the system is given by

$$H = E - \Omega L\,, \qquad (10)$$

where L is the total angular momentum about the axis of rotation. This transformation does not alter the Hamiltonian properties of the system, so long as $\mathbf{u}_i$ is now interpreted as the displacement from equilibrium observed in the rotating frame.[21] The explicit form of the angular momentum is

$$L = -\rho\kappa \sum_i \int dz_i \{\mathbf{r}_i \cdot \mathbf{u}_i(z_i) + \tfrac{1}{2}[\mathbf{u}_i(z_i)]^2\}\,, \qquad (11)$$

apart from terms independent of $\mathbf{u}_i$ that do not affect the equations of motion.

Equation (10) has two distinct linear contributions,

$$\rho\kappa \sum_i \int dz_i \Omega \mathbf{r}_i \cdot \mathbf{u}_i(z_i) - (8\pi)^{-1}\rho\kappa^2 \sum_{ij}{}' \int\int dz_i dz_j \times \mathbf{r}_{ij} [\mathbf{u}_i(z_i) - \mathbf{u}_j(z_j)] |\mathbf{R}_{ij}{}^0|^{-3}\,, \qquad (12)$$

arising from L and E, respectively. We shall now show that these two terms cancel identically if Ω is determined self-consistently. The second term of Eq. (12) may be rewritten as

$$-(4\pi)^{-1}\rho\kappa^2 \sum_{ij}{}' \int dz_i \mathbf{r}_{ij} \cdot \mathbf{u}_i(z_i) \int dz_j |\mathbf{R}_{ij}{}^0|^{-3}$$
$$= -(2\pi)^{-1}\rho\kappa^2 \sum_{ij}{}' \int dz_i \mathbf{r}_{ij} \cdot \mathbf{u}_i(z_i) |\mathbf{r}_{ij}|^{-2}\,, \qquad (13)$$

where the symmetry in i and j has been used in the first line. Thus Eq. (12) reduces to

$$\rho\kappa \sum_i \int dz_i \mathbf{u}_i(z_i) \cdot \{\Omega \mathbf{r}_i - \sum_{j\neq i} (2\pi)^{-1} \kappa \mathbf{r}_{ij} |\mathbf{r}_{ij}|^{-2}\}\,. \qquad (14)$$

The summation over j does not converge absolutely, but a physically motivated procedure is to perform the sum in successive concentric circles about the origin of the lattice. Although the angular sum may be evaluated analytically, even for a finite lattice, the remaining radial sum requires numerical computation. For this reason, Eq. (14) will be calculated only in the continuum approximation, where the discrete lattice is replaced by a smoothed vortex density n. The summation over j then reduces to

$$\frac{\kappa}{2\pi} \sum_j{}' \frac{\mathbf{r}_{ij}}{r_{ij}{}^2} \approx \frac{n\kappa}{2\pi} \int d^2r' \frac{\mathbf{r}_i - \mathbf{r}'}{|\mathbf{r}_i - \mathbf{r}'|^2}$$
$$= \frac{n\kappa}{2\pi} \hat{r}_i \int_0^R r' dr' \int_0^{2\pi} d\varphi \frac{r_i - r' \cos\varphi}{r_i^2 - 2r_i r' \cos\varphi + r'^2}$$
$$= n\kappa \hat{r}_i r_i^{-1} \int_0^{r_i} r' dr' = \tfrac{1}{2} n\kappa \mathbf{r}_i\,. \qquad (15)$$

It is clear that the two terms in Eq. (14) cancel if the angular velocity satisfies the self-consistent relation[2] $\Omega = \frac{1}{2} n\kappa$. Notice that the angular integral in Eq. (15) vanishes if $r' > r_i$, which means that the continuum approximation to the sum is independent of the radius R. The corresponding exact analytical evaluation for a

[19] W. Thomson (Lord Kelvin), Phil. Mag. **10**, 155 (1880).

[20] In Ref. 10, the unphysical assumption of an infinite nonrotating array led to the erroneous conclusion that a vortex lattice in superfluid helium was unstable. As noted in Refs. 7 and 8, a rotating triangular lattice is stable if the angular velocity Ω is determined by the self-consistent condition $\Omega = \frac{1}{2} n\kappa$, where n is the density of vortex lines per unit area.

[21] See, for example, G. B. Hess, Phys. Rev. **161**, 189 (1967).

square or triangular array is possible only for an infinite lattice ($R\rightarrow\infty$); detailed calculations show that the angular velocity satisfies the same self-consistent relation.[7]

Equation (10) may now be expressed as a bilinear form in the displacements

$$H=\sum_i H_i+\tfrac{1}{2}\sum_{ij}{}' H_{ij}\,, \tag{16}$$

where

$$H_i=E_i+\tfrac{1}{2}\rho\kappa\Omega\int dz_i\mathbf{u}_i^2\,, \tag{17}$$

and

$$H_{ij}=\frac{\rho\kappa^2}{4\pi}\int\int dz_idz_j\bigg\{|\mathbf{R}_{ij}{}^0|^{-1}\bigg[\frac{d\mathbf{u}_i(z_i)}{dz_i}\cdot\frac{d\mathbf{u}_j(z_j)}{dz_j}\bigg]-\frac{1}{2}\frac{(\mathbf{u}_i-\mathbf{u}_j)^2}{|\mathbf{R}_{ij}{}^0|^3}+\frac{3}{2}\frac{[\mathbf{r}_{ij}\cdot(\mathbf{u}_i-\mathbf{u}_j)]^2}{|\mathbf{R}_{ij}{}^0|^5}\bigg\}. \tag{18}$$

Here, as mentioned previously, $\mathbf{u}_i$ represents the displacement from the equilibrium position $\mathbf{r}_i$, as measured in the rotating frame.

III. QUANTIZATION OF THE HAMILTONIAN

The equations of motion of the vortex lattice may be written as

$$\begin{aligned}\delta H/\delta u_{xi}(z)&=-\rho\kappa\dot{u}_{yi}(z)\,,\\ \delta H/\delta u_{yi}(z)&=\rho\kappa\dot{u}_{xi}(z)\,,\end{aligned} \tag{19}$$

where H is given in Eqs. (16)–(18). In order to bring Eq. (19) into conventional form, it is convenient to define new variables

$$\begin{aligned}q_i(z)&=(\rho\kappa)^{1/2}u_{xi}(z)\,,\\ p_i(z)&=(\rho\kappa)^{1/2}u_{yi}(z)\,,\end{aligned} \tag{20}$$

so that Eq. (19) becomes a set of strict Hamiltonian equations

$$\begin{aligned}\delta H/\delta q_i(z)&=-\dot{p}_i(z)\,,\\ \delta H/\delta p_i(z)&=\dot{q}_i(z)\,.\end{aligned} \tag{21}$$

We now assume that the vortex system may be quantized by interpreting q_i and p_i as quantum-mechanical operators that obey Heisenberg equations of motion

$$\begin{aligned}i\hbar\dot{q}_i(z)&=[q_i(z),H]\,,\\ i\hbar\dot{p}_i(z)&=[p_i(z),H]\,,\end{aligned} \tag{22}$$

where H is a symmetrized form of Eq. (16) written in terms of the q's and p's. Equation (22) reproduces the classical equations of motion (21) only if the operators q_i and p_i satisfy the canonical commutation relations

$$[q_i(z),p_j(z')]=i\hbar\delta_{ij}\delta(z-z')\,, \tag{23}$$

where the appearance of the two different δ's reflects the peculiar anisotropy of the vortex lattice.

The motion of the vortex lattice now assumes a standard quantum-mechanical form: a quadratic Hamiltonian expressed in canonical variables q and p. This Hamiltonian may be greatly simplified by the introduction of plane-wave states.[22] Since the translational invariance is continuous along the z axis and discrete in the xy plane, the plane-wave decomposition in the two directions requires separate treatments, and it is simplest to consider first the motion along the z axis. Suppose that the lattice extends a length L in the z direction; then the operators $p_i(z)$ and $q_i(z)$ may be expanded in a Fourier series

$$\begin{aligned}q_i(z)&=L^{-1/2}\sum_k e^{ikz}q_{ik}\,,\\ p_i(z)&=L^{-1/2}\sum_k e^{-ikz}p_{ik}\,,\end{aligned} \tag{24}$$

where $k=2\pi s/L$ ($s=0,\pm1,\pm2\cdots$) and periodic boundary conditions have been assumed. The Fourier coefficients are given by

$$\begin{aligned}q_{ik}&=L^{-1/2}\int dze^{-ikz}q_i(z)\,,\\ p_{ik}&=L^{-1/2}\int dze^{ikz}p_i(z)\,,\end{aligned} \tag{25}$$

and they obey the following commutation relations:

$$[q_{ik},p_{jk'}]=i\hbar\delta_{ij}\delta_{kk'}\,. \tag{26}$$

The Fourier-series representations of the canonical operators [Eq. (24)] may be substituted into the Hamiltonian Eq. (16). As an example, we shall consider the first term of Eq. (18) expressed in terms of the q's and p's:

$$\begin{aligned}&\frac{\kappa}{4\pi}\int\int dzdz'\bigg\{\frac{dq_i(z)}{dz}\frac{dq_j(z')}{dz'}+\frac{dp_i(z)}{dz}\frac{dp_j(z')}{dz'}\bigg\}[r_{ij}{}^2+(z-z')^2]^{-1/2}\\ &=-\kappa(4\pi L)^{-1}\int\int dzdz'\sum_{kk'}kk'\{q_{ik}q_{jk'}e^{i(kz+k'z')}+p_{ik}p_{jk'}e^{-i(kz+k'z')}\}[r_{ij}{}^2+(z-z')^2]^{-1/2}\\ &=(4\pi)^{-1}\kappa\sum_k k^2(q_{ik}q_{j-k}+p_{ik}p_{j-k})\int_0^\infty dz2\cos(kz)(r_{ij}{}^2+z^2)^{-1/2}=(4\pi)^{-1}\kappa\sum_k k^2(q_{ik}q_{j-k}+p_{ik}p_{j-k})2K_0(kr_{ij})\,,\end{aligned} \tag{27}$$

[22] See, for example, J. M. Ziman, *Electrons and Phonons* (Oxford University Press, Oxford, 1960), Chap. 1, which serves as a model for the present calculation.

where K_0 is the modified Bessel function of order zero, defined by the integral representation[23]

$$K_\nu(kr)=\frac{\Gamma(\nu+\frac{1}{2})(2r)^\nu}{k^\nu\Gamma(\frac{1}{2})}\int_0^\infty\frac{\cos(kz)dz}{(z^2+r^2)^{\nu+1/2}}. \tag{28}$$

The remaining terms of H_{ij} are evaluated in a similar manner, which yields[8,18]

$$\begin{aligned}H_{ij}=(4\pi)^{-1}\kappa\sum_k\{&[q_{ik}q_{i,-k}+q_{jk}q_{j,-k}-p_{ik}p_{i,-k}-p_{jk}p_{j,-k}]r_{ij}^{-4}(x_{ij}^2-y_{ij}^2)\\&+[q_{ik}p_{ik}+p_{ik}q_{ik}+q_{jk}p_{jk}+p_{jk}q_{jk}]r_{ij}^{-4}2x_{ij}y_{ij}+\tfrac{1}{2}k^2[q_{ik}q_{j,-k}+q_{jk}q_{i,-k}+p_{ik}p_{j,-k}+p_{jk}p_{i,-k}]K_0(kr_{ij})\\&-\tfrac{1}{2}k^2[q_{ik}q_{j,-k}+q_{jk}q_{i,-k}-p_{ik}p_{j,-k}-p_{jk}p_{i,-k}]r_{ij}^{-2}(x_{ij}^2-y_{ij}^2)K_2(kr_{ij})\\&-\tfrac{1}{2}k^2[q_{ik}p_{jk}+p_{jk}q_{ik}+q_{jk}p_{ik}+p_{ik}q_{jk}]r_{ij}^{-2}2x_{ij}y_{ij}K_2(kr_{ij})\}. \end{aligned}\tag{29}$$

Here, x_{ij} and y_{ij} denote the x and y components of the two-dimensional vector $\mathbf{r}_{ij}$.

In Sec. II, the detailed form of the self-energy was left unspecified, and it is now possible to complete the calculation. Although the vortex core will be treated as an assembly of parallel vortex filaments, we shall assume that a single quantum-mechanical operator suffices to describe the motion of each elementary length of core. The operator aspect of the self-energy is therefore obtained from H_{ij} merely be setting $i=j$ in the operators q and p. This procedure fails to treat the extended core with sufficient accuracy, however, and it is necessary to average the spatial variables $\mathbf{r}_i$ and $\mathbf{r}_j$ over a small circle of radius a. Such an average is equivalent to computing (twice) the total interaction energy between all pairs of elementary filaments. In addition, the restriction of small bending implies that $ka\ll 1$.

With these assumptions, the self-energy E_i becomes

$$\begin{aligned}E_i=(4\pi)^{-1}\kappa\sum_k(q_{ik}q_{i,-k}+p_{ik}p_{i,-k})\tfrac{1}{2}k^2[\ln 2-\gamma-\langle\ln(k|\mathbf{r}-\mathbf{r}'|)\rangle]\\+(4\pi)^{-1}\kappa\sum_k(q_{ik}q_{i,-k}-p_{ik}p_{i,-k})\tfrac{1}{4}k^2\langle|\mathbf{r}-\mathbf{r}'|^{-2}[(x-x')^2-(y-y')^2]\rangle\\+(4\pi)^{-1}\kappa\sum_k(q_{ik}p_{ik}+p_{ik}q_{ik})\tfrac{1}{2}k^2\langle|\mathbf{r}-\mathbf{r}'|^{-2}(x-x')(y-y')\rangle,\end{aligned}\tag{30}$$

where the expansions $K_0(x)\approx-\ln(\frac{1}{2}x)-\gamma$ and $K_2(x)\approx 2x^{-2}-\frac{1}{2}$ have been used, and $\gamma=0.5772\cdots$ is Euler's constant. An additional factor $\frac{1}{2}$ has been inserted into Eq. (30) to ensure that each pair of filaments is counted only once. The average implied by the angular brackets is defined as

$$\langle f(\mathbf{r},\mathbf{r}')\rangle=(\pi a^2)^{-2}\int\int d^2r d^2r' f(\mathbf{r},\mathbf{r}'), \tag{31}$$

where the integrals are confined to a circle of radius a. Detailed calculations yield

$$\begin{aligned}-\langle\ln(k|\mathbf{r}-\mathbf{r}'|)\rangle&=\ln(1/ka)+\tfrac{1}{4}\\\langle|\mathbf{r}-\mathbf{r}'|^{-2}[(x-x')^2-(y-y')^2]\rangle&=0,\\\langle|\mathbf{r}-\mathbf{r}'|^{-2}(x-x')(y-y')\rangle&=0,\end{aligned}\tag{32}$$

and the self-energy operator of the ith vortex reduces to

$$E_i=(4\pi)^{-1}\kappa\sum_k(q_{ik}q_{i,-k}+p_{ik}p_{i,-k})\times\tfrac{1}{2}k^2[\ln(2/ka)-\gamma+\tfrac{1}{4}]. \tag{33}$$

As a check on the averaging procedure, the additional term $\frac{1}{4}$ agrees precisely with that obtained in a different way by Kelvin[19] in his study of the vibrations of a classical vortex with a core in solid-body rotation.[18] The remaining term of Eq. (17) is easily evaluated, and the final expression becomes

$$H_i=\tfrac{1}{2}\sum_k(q_{ik}q_{i,-k}+p_{ik}p_{i,-k})\times\{\Omega+(4\pi)^{-1}\kappa k^2[\ln(2/ka)-\gamma+\tfrac{1}{4}]\}. \tag{34}$$

In the particular case of a single rectilinear vortex in an unbounded fluid, Ω vanishes and Eq. (33) represents the total Hamiltonian of the system.

Equations (16), (29), and (34) constitute the exact Hamiltonian operator of a system of rectilinear vortices uniformly filling the xy plane and executing small oscillations about their straight undeformed shapes. No further progress is possible without additional assumptions, and we shall therefore restrict the discussion to a square or triangular lattice of area A containing N vortices with a mean density $n=N/A$. The operators q_{ik} and p_{ik} may now be expanded in a second Fourier series

$$\begin{aligned}q_{ik}&=N^{-1/2}\sum_1 e^{i\mathbf{l}\cdot\mathbf{r}_i}q_{1k},\\p_{ik}&=N^{-1/2}\sum_1 e^{-i\mathbf{l}\cdot\mathbf{r}_i}p_{1k},\end{aligned}\tag{35}$$

where periodic boundary conditions have been used and $\{\mathbf{l}\}$ is the set of reciprocal lattice vectors associated with the two-dimensional array. The corresponding Fourier coefficients and commutation relations are easily found to be

$$\begin{aligned}q_{1k}&=N^{-1/2}\sum_j e^{-i\mathbf{l}\cdot\mathbf{r}_j}q_{jk},\\p_{1k}&=N^{-1/2}\sum_j e^{i\mathbf{l}\cdot\mathbf{r}_j}p_{jk},\end{aligned}\tag{36}$$

$$[q_{1k},p_{1'k'}]=i\hbar\delta_{11'}\delta_{kk'}. \tag{37}$$

[23] We follow the notation of G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1962), 2nd ed., p. 185.

Equation (35) must be substituted into the total Hamiltonian Eq. (16), and, as a typical example, we consider the first term of Eq. (29)

$$(8\pi)^{-1}\kappa \sum_k \sum_{ij}{}' (q_{ik}q_{i-k}+q_{jk}q_{j-k})r_{ij}^{-4}(x_{ij}^2-y_{ij}^2)$$
$$=(4\pi)^{-1}\kappa \sum_k \sum_{\mathbf{l}\mathbf{l}'} q_{\mathbf{l}k}q_{\mathbf{l}'-k}N^{-1} \times\sum_{ij}{}' e^{i(\mathbf{l}+\mathbf{l}')\cdot\mathbf{r}_i}r_{ij}^{-4}(x_{ij}^2-y_{ij}^2)$$
$$=(4\pi)^{-1}\kappa \sum_k \sum_{\mathbf{l}\mathbf{l}'} q_{\mathbf{l}k}q_{\mathbf{l}'-k}N^{-1} \sum_{ij}{}' e^{i(\mathbf{l}+\mathbf{l}')\cdot\mathbf{r}_j} \times e^{i(\mathbf{l}+\mathbf{l}')\cdot\mathbf{r}_{ij}}r_{ij}^{-4}(x_{ij}^2-y_{ij}^2).$$

In the limit of a large lattice ($lR\gg1$, $l'R\gg1$), the double sum over i and j may be evaluated approximately by shifting the origin in each term

$$N^{-1}\sum_j e^{i(\mathbf{l}+\mathbf{l}')\cdot\mathbf{r}_j}\sum_i{}' e^{i(\mathbf{l}+\mathbf{l}')\cdot\mathbf{r}_i}r_i^{-4}(x_i^2-y_i^2) =\delta_{\mathbf{l},-\mathbf{l}'}\sum_i{}' r_i^{-4}(x_i^2-y_i^2)=0,$$

where the final equality follows from the symmetry of the square or triangular lattice. The remaining terms of Eq. (16) are all evaluated in a similar manner, and the total Hamiltonian of the vortex lattice reduces to a sum of uncoupled terms

$$H=\sum_{\mathbf{l}k} H_{\mathbf{l}k}, \tag{38}$$

where

$$H_{\mathbf{l}k}=\tfrac{1}{2}(\Omega_G-\eta-\xi)q_{\mathbf{l}k}q_{\mathbf{l}k}^\dagger+\tfrac{1}{2}(\Omega_G-\eta+\xi)p_{\mathbf{l}k}p_{\mathbf{l}k}^\dagger +\tfrac{1}{2}\alpha(q_{\mathbf{l}k}p_{\mathbf{l}k}+p_{\mathbf{l}k}^\dagger q_{\mathbf{l}k}^\dagger). \tag{39}$$

Here, the adjoint operators are defined as[22]

$$q_{\mathbf{l}k}^\dagger=q_{-\mathbf{l},-k},\quad p_{\mathbf{l}k}^\dagger=p_{-\mathbf{l},-k}, \tag{40}$$

and the following abbreviations have been used[8,10]:

$$\Omega_G=\Omega+(4\pi)^{-1}\kappa k^2[\ln(2/ka)-\gamma+\tfrac{1}{4}] +(4\pi)^{-1}\kappa k^2\sum_j{}' K_0(kr_j), \tag{41}$$

$$\alpha=(4\pi)^{-1}\kappa k^2\sum_j{}'(1-e^{i\mathbf{l}\cdot\mathbf{r}_j})(2x_jy_j/r_j^2)K_2(kr_j), \tag{42a}$$

$$\xi=(4\pi)^{-1}\kappa k^2\sum_j{}'(1-e^{i\mathbf{l}\cdot\mathbf{r}_j})(y_j^2-x_j^2)r_j^{-2}K_2(kr_j), \tag{42b}$$

$$\eta=(4\pi)^{-1}\kappa k^2\sum_j{}'(1-e^{i\mathbf{l}\cdot\mathbf{r}_j})K_0(kr_j). \tag{42c}$$

The frequency Ω_G was first derived by Raja Gopal[18] in his study of the axial vibration modes of a vortex array; Eq. (41) differs from his result only in the explicit core correction $\frac{1}{4}$, and in the appearance of a lattice sum instead of an integral approximation. It is remarkable that these lattice sums describing the three-dimensional oscillations of a vortex lattice in superfluid helium are identical with those occurring in the two-dimensional oscillations of a vortex lattice in a bulk type-II superconductor,[9,10] when the vortices move without bending. This relation was first noted by Stauffer, who has carried out the classical analysis corresponding to that given above.[8]

IV. DIAGONALIZATION OF THE HAMILTONIAN

The Hamiltonian Eq. (38) may be diagonalized by a linear transformation of variables. Although this procedure is well known in the quantum theory of solids,[5,6,22] the present situation is slightly more complicated because Eq. (39) contains terms proportional to qp as well as the usual $qq^\dagger$ and $pp^\dagger$. Nevertheless, the proper linear combination of operators can still be determined from the corresponding classical equations. Define

$$q_{\mathbf{l}k}=[\hbar/2\omega(\Omega_G-\eta+\xi)]^{1/2}(\Omega_G-\eta+\xi)\times[a_{-\mathbf{l},-k}+a_{\mathbf{l}k}^\dagger],$$
$$p_{\mathbf{l}k}^\dagger=[\hbar/2\omega(\Omega_G-\eta+\xi)]^{1/2}\times[-(\alpha+i\omega)a_{-\mathbf{l},-k}-(\alpha-i\omega)a_{\mathbf{l}k}^\dagger], \tag{43}$$

where

$$\omega\equiv\omega(\mathbf{l},k)=[(\Omega_G-\eta)^2-(\alpha^2+\xi^2)]^{1/2} \tag{44}$$

represents the general dispersion relation as a function of the wave vector $(\mathbf{l},k)$ and was first obtained by Stauffer.[8] The inverse transformation is given by

$$a_{\mathbf{l}k}^\dagger=\frac{q_{\mathbf{l}k}(\alpha+i\omega)+p_{\mathbf{l}k}^\dagger(\Omega_G-\eta+\xi)}{i[2\hbar\omega(\Omega_G-\eta+\xi)]^{1/2}},$$
$$a_{-\mathbf{l}-k}=\frac{q_{\mathbf{l}k}(\alpha-i\omega)+p_{\mathbf{l}k}^\dagger(\Omega_G-\eta+\xi)}{-i[2\hbar\omega(\Omega_G-\eta+\xi)]^{1/2}}, \tag{45}$$

and it is not difficult to verify that the operators a and $a^\dagger$ obey the usual boson commutation relations

$$[a_{\mathbf{l}k},a_{\mathbf{l}'k'}^\dagger]=\delta_{\mathbf{l}\mathbf{l}'}\delta_{kk'},\quad [a_{\mathbf{l}k},a_{\mathbf{l}'k'}]=[a_{\mathbf{l}k}^\dagger,a_{\mathbf{l}'k'}^\dagger]=0. \tag{46}$$

Substitution of Eq. (43) into Eq. (38) eventually yields

$$H=\tfrac{1}{2}\sum_{\mathbf{l}k}\hbar\omega(a_{\mathbf{l}k}a_{\mathbf{l}k}^\dagger+a_{\mathbf{l}k}^\dagger a_{\mathbf{l}k}), \tag{47}$$

where the dependence of ω on $(\mathbf{l},k)$ has been suppressed for simplicity. Equation (47) shows that $a_{\mathbf{l}k}^\dagger$ may be interpreted as the creation operator for a single vibration quantum in the normal mode specified by the wave vector $(\mathbf{l},k)$ and frequency ω, determined by Eq. (44).

It has been shown[7] that an infinite square vortex lattice in He II is unstable with respect to small perturbations confined to the xy plane, and only the triangular lattice is expected to occur in physical situations. Unfortunately, the exact lattice sums have been evaluated[10,24] only in the long-wavelength limit ($kn^{-1/2}\ll1$, $ln^{-1/2}\ll1$), which is insufficient for the present calculation. In order to deal consistently with a single model, we shall therefore restrict this work to the continuum approximation, which allows a complete determination of the dispersion relation. In the long-wavelength limit, this approximation reproduces the exact results for the triangular lattice.[7,8,11] Furthermore, the specific heat and zero-point motion are rather insensitive to the detailed form of the frequency spectrum, assuming only that the lattice is stable, so that the continuum approximation is not expected to lead to an

[24] A. L. Fetter, Phys. Rev. 147, 153 (1966).

appreciable error. The continuum approximation to Eq. (42) has been evaluated in Appendix A of Ref. 10; the similar integral obtained from Eq. (41) was computed numerically in Ref. 18, but it can also be evaluated analytically as follows:

$$(4\pi)^{-1}\kappa k^2 \sum_j{}' K_0(kr_j) \approx (4\pi)^{-1}\kappa n k^2 \int d^2r K_0(kr)$$
$$= \tfrac{1}{2}n\kappa \int_{kb}^{\infty} x dx K_0(x)$$
$$= \tfrac{1}{2}n\kappa (kb) K_1(kb). \quad (48)$$

The lower cutoff $b=(n\pi)^{-1/2}$, which characterizes the intervortex spacing, excludes the area n^{-1} associated with the vortex at the origin.

The frequency spectrum must be considered separately in four distinct regions of the lk plane. It is clear that the wavelength for propagation perpendicular to the vortex axes cannot be shorter than the length b, so that l generally satisfies the restriction $lb \lesssim 1$. In contrast, no such restriction applies to the quantity kb.

(1). If $kb \gg 1 \gg lb$, then the wavelength for propagation along the vortex axis is much smaller than the intervortex spacing, and the vortices are effectively independent. The lattice sums α, ξ, η, and Eq. (48) all vanish exponentially in this case, and the vibration frequency is given as

$$\omega = \Omega_G = (4\pi)^{-1}\kappa k^2 [\ln(2/ka) - \gamma + \tfrac{1}{4} + O(k^{-2}b^{-2})], \quad (49)$$

which is just Kelvin's result for a vortex with a core in solid-body rotation.[19]

(2). If $1 \gg kb \gg lb$, then the wavelength for propagation along the vortex axis is much larger than the intervortex spacing b, and the vortices interact appreciably. In this region, the lattice sums α, ξ, and η are all small (of order $\Omega l^2 k^{-2}$), and we find

$$\Omega_G = 2\Omega[1 + \tfrac{1}{4}(kb)^2 \ln(b/a)]$$
$$= 2\Omega + (4\pi)^{-1}\kappa k^2 \ln(b/a), \quad (50)$$

$$\omega = 2\Omega, \quad (51)$$

where the relation $\Omega = \frac{1}{2}n\kappa = (2\pi b^2)^{-1}\kappa$ has been used. Equation (50) represents the dispersion relation for propagation along the vortex axis ($l=0$, $kb \ll 1$) and is implicit in the work of Raja Gopal.[18] The small correction of order $(kb)^2$ is given with only logarithmic accuracy because the additional constants depend on the detailed lattice structure; the precise values may be obtained from the sums in Ref. 24, but are not important here. It is interesting that this correction term does not contain a factor $\ln ka$, which arises from a cancellation between the logarithmic dependence on k due to the other vortices [Eq. (48)] and the self-induced logarithmic term in Eq. (41). The leading term in the dispersion relation [Eq. (51)] conflicts with an assertion of Vinen[25] that vortex waves cannot be propagated if $kb \ll 1$, but our result agrees with that of Raja Gopal[18] and Nozières.[26] Indeed, Eq. (51) has a simple physical interpretation as the natural oscillation frequency 2Ω of a fluid in solid-body rotation[27] with uniform vorticity $|\text{curl}\,\mathbf{v}| = 2\Omega$; for if $kb \ll 1$, then the propagating wave cannot resolve the discrete vortex structure, and the motion of the fluid is governed by the *mean* vorticity, given by $n\kappa = 2\Omega$.

(3). Finally, it is necessary to consider the case $1 \gg lb \gg kb$, where detailed calculation yields

$$\Omega_G - \eta = \Omega,$$

$$\omega = 2k\Omega/l, \quad (k \gg \tfrac{1}{4}l^2 b) \quad (52)$$

$$\omega = \tfrac{1}{2}lb\Omega, \quad (\tfrac{1}{4}l^2 b \gg k). \quad (53)$$

Equations (52) and (53) represent two nonoverlapping regions, as indicated in parentheses. In the limit $k=0$, Eq. (53) describes the motion of a lattice in which the vortices move without bending; this dispersion relation may also be written as $\omega = \frac{1}{2}\Omega l (n\pi)^{-1/2}$, first derived by Tkachenko.[7]

It is important to notice that the long-wavelength dispersion relation exhibits a very different structure for propagation parallel [Eq. (51)] and perpendicular [Eq. (53)] to the axis of rotation. Thus it is clear that the linear dependence of Eq. (53) on the wave number is unrelated to the usual phonon spectrum of a crystalline solid. In fact, the present results indicate a failure of elasticity theory, which arises from the long-range interaction between vortices in He II.[11] This feature represents one of the major differences between vortices in helium and in bulk type-II superconductors, where the long-wavelength dispersion relation agrees precisely with that predicted for an elastic continuum.[9–11]

V. SPECIFIC HEAT

The Hamiltonian [Eq. (47)] may be used to compute the thermal energy of a vortex lattice. It is simplest to consider first the case of a single vortex line in an infinite fluid, whose Hamiltonian is obtained from Eq. (34) in the limit $\Omega \to 0 (b \to \infty)$. The corresponding expression involving creation and destruction operators is

$$H = \tfrac{1}{2} \sum_k \hbar\omega (a_k a_k^\dagger + a_k^\dagger a_k); \quad (54)$$

here the only degrees of freedom are associated with waves propagating along the axis of the vortex, and ω is given in Eq. (49). At a temperature $T = (k_B \beta)^{-1}$, the

[25] Reference 4, p. 99.

[26] P. Nozières, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland Publishing Company, Amsterdam, 1966), p. 15.

[27] An especially lucid treatment may be found in S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, England, 1961), pp. 85–86.

mean thermal energy is

$$E=\langle H\rangle=\sum_k \tfrac{1}{2}\hbar\omega \coth(\tfrac{1}{2}\beta\hbar\omega) \\ =\sum_k \hbar\omega[(e^{\beta\hbar\omega}-1)^{-1}+\tfrac{1}{2}], \tag{55}$$

where the angular brackets denote an ensemble average. In Eq. (55), the second term represents the zero-point energy and will be now omitted because it does not affect the specific heat. The sum over k may be approximated by an integral

$$E=L\pi^{-1}\int_0^{k_m} dk\hbar\omega(e^{\beta\hbar\omega}-1)^{-1}, \tag{56}$$

where the additional factor of 2 arises from the two directions of propagation. The upper cutoff k_m is determined by the condition that the core radius a represent the shortest allowed wavelength

$$k_m=2\pi/\lambda_{\min}=2\pi/a, \tag{57}$$

which fixes the total number of states as

$$\sum_k=Lk_m\pi^{-1}=2L/a. \tag{58}$$

Equation (57) defines a "Debye" temperature Θ_D, given by[12]

$$\Theta_D=\hbar\omega(k_m)k_B^{-1} \\ \approx(4\pi k_B)^{-1}\hbar\kappa k_m^2 \\ =\hbar\kappa\pi/a^2k_B \\ \approx 2.4\times10^2\ {}^\circ\mathrm{K}, \tag{59}$$

where $\kappa\approx10^{-3}$ cm^2 sec^{-1} and $a\approx10^{-8}$ cm. Since $T\ll\Theta_D$ in all cases, Eq. (56) may be integrated to infinity with negligible error, and we find (with logarithmic accuracy)[12]

$$E\approx L\zeta(\tfrac{3}{2})(k_BT)^{3/2}(2\hbar\kappa)^{-1/2}[\ln(\hbar\kappa/a^2k_BT)]^{-1/2}, \tag{60}$$

where the slow logarithmic dependence on k has been neglected, and $\zeta(\tfrac{3}{2})=2.612\cdots$ is the Riemann zeta function. The heat capacity per unit length of vortex line is given by

$$L^{-1}dE/dT\approx\tfrac{3}{2}\zeta(\tfrac{3}{2})k_B(k_BT)^{1/2}(2\hbar\kappa)^{-1/2} \\ \times[\ln(\hbar\kappa/a^2k_BT)]^{-1/2}, \tag{61}$$

apart from corrections of relative order $[\ln(\hbar\kappa/a^2k_BT)]^{-1}$.

The specific heat of an array of vortices may be calculated similarly. Except for the zero-point contribution, the mean thermal energy is equal to

$$E=\langle H\rangle=\sum_{lk}\hbar\omega(e^{\beta\hbar\omega}-1)^{-1}, \tag{62}$$

where ω must now be taken from Eq. (44). As in Eq. (56), we shall approximate the sum in Eq. (62) by an integral, which now extends over a cylinder of height $2k_m$ and radius l_m

$$E=LA(2\pi^2)^{-1}\int_0^{k_m}dk\int_0^{l_m}ldl\hbar\omega(e^{\beta\hbar\omega}-1)^{-1}. \tag{63}$$

Here A is the area of the lattice containing N vortices, and the radial cutoff l_m is determined by the condition

$$N=\sum_l=(2\pi)^{-2}A\int d^2l \\ =Al_m^2(4\pi)^{-1},$$

or

$$l_m=2(\pi n)^{1/2}=2/b. \tag{64}$$

It is important to notice that $l_m\ll k_m$ for all physically attainable rotation speeds; in fact, the equality $l_m\approx k_m$ fixes the upper critical velocity $\Omega_{c2}\approx10^{12}$ rad sec^{-1} for the transition to the normal state.[4,26]

The thermal energy [Eq. (63)] may be evaluated approximately by dividing the lk plane into four separate regions, each characterized by the different dispersion relation Eqs. (49), (51), (52), or (53). If $\beta\hbar\Omega\ll1$, which is always satisfied in practice, it is not difficult to see that the dominant contribution arises from the region $kb\gg1$, and we find

$$E=LAn\zeta(\tfrac{3}{2})(k_BT)^{3/2}(2\hbar\kappa)^{-1/2}[\ln(\hbar\kappa/a^2k_BT)]^{-1/2}. \tag{65}$$

This expression shows that the thermal energy of the vortex lattice is N times that of a single vortex [Eq. (60)]; such a simple relation holds because of the small fraction of states with propagation vector perpendicular to the vortex axis ($l_m\ll k_m$). Differentiation of Eq. (65) yields the specific heat of the vortex lattice

$$C_{\mathrm{vor}}=(LA)^{-1}dE/dT \\ \approx\tfrac{3}{2}\zeta(\tfrac{3}{2})nk_B(k_BT)^{1/2}(2\hbar\kappa)^{-1/2}[\ln(\hbar\kappa/a^2k_BT)]^{-1/2}. \tag{66}$$

Notice that Eq. (66) is proportional to $T^{1/2}[\ln(T_0/T)]^{-1/2}$ and thus decreases quite slowly for small T. This functional form differs from a corresponding calculation for type-II superconductors,[28] which predicts $C_{\mathrm{vor}}\propto T^{3/2}$ as $T\rightarrow0$. It is not surprising that the two systems differ considerably, since the vortex density in the mixed state of a superconductor is usually much higher than in rotating He II.

Equation (66) may be compared with the phonon contribution to the specific heat of superfluid helium[29]

$$C_{\mathrm{ph}}=(2\pi^2/15)k_B(k_BT/\hbar v)^3, \tag{67}$$

where $v\approx238$ m sec^{-1} is the velocity of sound. Numerical evaluation yields

$$C_{\mathrm{vor}}\approx8.8\times10^{-6}\Omega T^{1/2}[\ln(76/T)]^{-1/2}, \\ C_{\mathrm{ph}}\approx3.0\times10^4T^3, \tag{68}$$

where C is measured in erg cm^{-3}(°K)$^{-1}$, T in °K, and Ω in rad sec^{-1}. Thus the vortex contribution to the specific heat of rotating He II is negligible for all but

[28] J. Matricon, in *Low Temperature Physics LT9*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, Inc., New York, 1965), p. 544.

[29] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 200.

the lowest temperatures and highest rotation speeds: at $T=10^{-3}$ °K and $\Omega=10^3$ rad sec^{-1}, Eq. (68) predicts $C_{\text{vor}}/C_{\text{ph}}\approx 2.8$.

VI. ZERO-POINT MOTION

The present Hamiltonian formalism allows a direct calculation of the zero-point motion associated with the vortex core. It is simplest to start with a single vortex line, in which case the creation and destruction operators have only the single label k. The mean-square displacement at the point z is given by

$$\begin{aligned}\langle \mathbf{u}^2(z)\rangle &= \langle u_x^2(z)+u_y^2(z)\rangle \\ &= (\rho\kappa)^{-1}\langle q^2(z)+p^2(z)\rangle ,\end{aligned} \tag{69}$$

where the angular brackets again denote a thermal average. Since the system is invariant under translations along the z axis, a second average may also be taken, defined as

$$\begin{aligned}\langle \mathbf{u}^2\rangle_{\text{av}} &\equiv L^{-1}\int dz\langle \mathbf{u}^2(z)\rangle \\ &= (\rho\kappa L)^{-1}\sum_k\langle q_kq_k{}^\dagger+p_kp_k{}^\dagger\rangle \\ &= \hbar(\rho\kappa L)^{-1}\sum_k\langle a_ka_k{}^\dagger+a_k{}^\dagger a_k\rangle \\ &= \hbar(\rho\kappa L)^{-1}\sum_k \coth(\tfrac{1}{2}\beta\hbar\omega) ,\end{aligned} \tag{70}$$

where Eqs. (24) and (43) have been used (note that α, ξ, and η all vanish as $b\to\infty$), and ω is given by Eq. (49). At zero temperature, the thermal factor $\coth(\frac{1}{2}\beta\hbar\omega)$ reduces to unity, and the mean-square displacement is simply

$$\langle \mathbf{u}^2\rangle_{\text{av}} = \hbar(\rho\kappa L)^{-1}\sum_k = 2\hbar/\rho\kappa a, \quad (T=0), \tag{71}$$

where Eq. (58) has been used. The quantity $\hbar(\rho\kappa)^{-1}$ is approximately equal to the volume v_0 per particle, so that $\langle \mathbf{u}^2\rangle_{\text{av}}=O(v_0a^{-1})$. Numerical evaluation ($\rho\approx 0.145$ g cm^{-3}, $\kappa\approx 10^{-3}$ cm^2 sec^{-1}, $a\approx 10^{-8}$ cm) yields

$$[\langle \mathbf{u}^2\rangle_{\text{av}}]^{1/2}=3.8\text{ Å} \quad (T=0) \tag{72}$$

as the root-mean-square displacement of the vortex core at $T=0$; this value is comparable with the original estimate of the vortex core radius.

At finite temperature, on the other hand, Eq. (70) formally diverges as $k\to 0$, and a more careful treatment is required. When the summation is replaced by an integral, it is necessary to cut off the integral at a lower limit $k_0\approx 2\pi/L$, which then gives

$$\langle \mathbf{u}^2\rangle_{\text{av}}\approx \hbar(\pi\rho\kappa)^{-1}\int_{k_0}^{k_m} dk \coth(\tfrac{1}{2}\beta\hbar\omega). \tag{73}$$

Since the dominant contribution arises from the lower range of integration, an approximate evaluation of Eq. (73) proceeds as follows[30]:

$$\begin{aligned}\langle \mathbf{u}^2\rangle_{\text{av}} &\approx 2(\pi\rho\kappa\beta)^{-1}\int_{k_0}^{\infty} dk\omega^{-1} \\ &= 8(\rho\kappa^2\beta)^{-1}\int_{k_0}^{\infty} dk k^{-2}[\ln(1/ka)]^{-1} \\ &= 8(\rho\kappa^2\beta)^{-1}k_0{}^{-1}[\ln(1/k_0a)]^{-1} \\ &= 4L(\rho\kappa^2\beta\pi)^{-1}[\ln(L/2\pi a)]^{-1}.\end{aligned} \tag{74}$$

Thus the root-mean-square displacement of a long vortex at finite temperature diverges like $L^{1/2}[\ln(L/a)]^{-1/2}$; if $T=1$°K, numerical evaluation of Eq. (74) yields $[\langle \mathbf{u}^2\rangle_{\text{av}}]^{1/2}\approx 8.5\times 10^2$ Å for $L=1$ cm and $[\langle \mathbf{u}^2\rangle_{\text{av}}]^{1/2}\approx 13$ Å for $L=10^{-4}$ cm. As the temperature is reduced, this divergent behavior persists until T reaches T_0, defined by the condition that k_BT_0 is comparable with the spacing of adjacent energy levels of the finite vortex. T_0 is given approximately by

$$T_0\approx \pi\hbar\kappa(k_BL^2)^{-1}\ln(L/2\pi a) , \tag{75}$$

which predicts that $T_0\approx 4.0\times 10^{-13}$ °K for $L=1$ cm and that $T_0\approx 1.8\times 10^{-5}$ °K for $L=10^{-4}$ cm. For $T\gtrsim T_0$, the finite-temperature expression [Eq. (74)] describes the mean-square displacement, while the zero-temperature expression [Eq. (71)] is correct only in the presently inaccessible range $T\lesssim T_0$.

It is interesting to consider how a finite density of vortices alters these results. The mean-square displacement of a vortex core must now be defined as

$$\langle \mathbf{u}^2\rangle_{\text{av}}\equiv (LN)^{-1}\sum_i \int dz\langle \mathbf{u}_i{}^2(z)\rangle , \tag{76}$$

where the sum is over all vortices; the evaluation is similar to Eq. (70), and we find

$$\langle \mathbf{u}^2\rangle_{\text{av}}=\hbar(LN\rho\kappa)^{-1}\sum_{1k}(\Omega_G-\eta)\omega^{-1}\coth(\tfrac{1}{2}\beta\hbar\omega). \tag{77}$$

At $T=0$, the thermal factor does not appear, and Eq. (77) reduces to

$$\langle \mathbf{u}^2\rangle_{\text{av}}\approx \hbar(2\pi^2n\rho\kappa)^{-1}\int_0^{k_m} dk\int_0^{l_m} ldl(\Omega_G-\eta)\omega^{-1}, \tag{78}$$

where the sum has been replaced by an integral, as in Eq. (63). The most important part of the integral is the rectangular region ($l_m>l>0$, $k_m>k>l_m$) in the lk plane, where $\Omega_G-\eta$ and ω are equal [Eq. (49)]; Eq. (78) may therefore be evaluated approximately as

$$\begin{aligned}\langle \mathbf{u}^2\rangle_{\text{av}} &= \hbar(4\pi^2n\rho\kappa)^{-1}k_ml_m{}^2 \\ &= 2\hbar/\rho\kappa a, \quad (T=0)\end{aligned} \tag{79}$$

[30] The integral in the second line may be expressed in terms of the logarithmic integral [E. Jahnke and F. Emde, *Tables of Functions with Formulae and Curves* (Dover Publications, Inc., New York, 1945), 4th ed., p. 3]; the asymptotic expansion for $k_0a\ll 1$ then yields the third line.

apart from corrections of order $a/b\ll 1$. This result is identical with that describing a single vortex [Eq. (71)], so that the mean-square displacement of each vortex core at zero temperature is unaffected by the presence of the other vortices. Equation (79) remains correct as long as the separation between vortices b is large compared to the core size a, which is the case for all feasible experiments with liquid He II ($\Omega\ll 10^{12}$ rad $\sec^{-1}$).

A different situation occurs at finite temperatures. As in Eq. (73), the integrals must be cut off at the lower limits

$$\langle \mathbf{u}^2\rangle_{\rm av}\approx \hbar(2\pi^2 n\rho\kappa)^{-1}\int_{k_0}^{k_m} dk\int_{l_0}^{l_m} l dl(\Omega_G-\eta)\omega^{-1}\coth(\tfrac{1}{2}\beta\hbar\omega)\,, \tag{80}$$

where $k_0=2\pi/L$ and $l_0=2\pi/A^{1/2}$. The most important part of the integral is the region $(l_m>l,\, l_m>k)$, and an approximate calculation gives

$$\langle \mathbf{u}^2\rangle_{\rm av}\approx 8b(\rho\kappa^2\beta)^{-1}\quad (T>0) \tag{81}$$

neglecting corrections of order $[\ln(b/a)]^{-1}$. In deriving Eq. (81), it is assumed that $l_m\gg k_0(L\gg b)$, so that the lattice spacing b acts as the cutoff in the mean-square displacement, instead of L [compare Eq. (74) for a single vortex]. For definiteness, consider a rotation speed $\Omega=1$ rad $\sec^{-1}$; the spacing between vortices is then given by $b=(\kappa/2\pi\Omega)^{1/2}\approx 1.3\times 10^{-2}$ cm, and Eq. (81) yields $[\langle \mathbf{u}^2\rangle_{\rm av}]^{1/2}\approx 9.8\times 10^2$ Å at 1°K.

It must be noted that $\langle \mathbf{u}^2\rangle_{\rm av}$ is proportional to b and therefore *decreases* as the vortex density increases: The presence of other vortices reduces the zero-point motion. In contrast, the dimensionless ratio

$$\langle \mathbf{u}^2\rangle_{\rm av} b^{-2}=8(b\rho\kappa^2\beta)^{-1}\,, \tag{82}$$

which characterizes the stability of the vortex lattice, increases as b decreases. When $\langle \mathbf{u}^2\rangle_{\rm av} b^{-2}\approx 1$, the lattice presumably "melts." If the melting temperature T_m is defined by the condition

$$T_m=(8k_B)^{-1}\rho\kappa^2 b\,, \tag{83}$$

then a rotation speed of 1 rad $\sec^{-1}$ produces a stable lattice up to a temperature $T_m\approx 2.3\times 10^6$ °K. Alternatively, at 2°K, the vortex lattice remains "solid" until $b\approx 1.5$ Å ($\Omega\approx 0.7\times 10^{12}$ rad $\sec^{-1}$). These numerical examples show that the root-mean-square displacement is always much less than the intervortex spacing, which clearly represents a necessary condition for stability.

The dimensionless parameter of Eq. (82) is also approximately equal to the Debye-Waller factor[5,6] W characterizing scattering of incident waves by the vortex lattice. More precisely, W is defined as

$$W=\tfrac{1}{2}\langle(\mathbf{K}\cdot\mathbf{u})^2\rangle_{\rm av}\,, \tag{84}$$

where $\hbar\mathbf{K}$ is the momentum transferred in the scattering process. The Bragg peaks in the differential cross section occur when $\mathbf{K}$ equals one of the reciprocal lattice vectors; hence $|\mathbf{K}|$ is of order b^{-1}, and Eq. (82) indeed provides an order-of-magnitude estimate of W. As noted above, this quantity is small for all reasonable physical situations, so that the quantum-mechanical or thermal motion of the vortices has negligible effect on the Bragg scattering by the vortex lattice. Whether such scattering can in fact be observed experimentally requires a detailed study, and has not been attempted here.

VII. DISCUSSION

The general formula for the mean-square displacement of the vortex core [Eq. (77)] may be compared with the corresponding mean-square displacement of a given atom in a crystal lattice, where $\langle \mathbf{u}^2\rangle_{\rm av}\propto\sum_{\mathbf{k}}\omega^{-1}\times\coth(\frac{1}{2}\beta\hbar\omega)$. The difference between the two expressions is due to the different structure of the Hamiltonians and represents a direct consequence of the distinction between Newtonian and vortex dynamics.[11] Thus the usual arguments[5] about the stability of a crystal lattice in 1, 2, or 3 dimensions are not directly relevant to vortex lattices, and it is necessary to reconsider the question from first principles.

Another important distinction between the dynamics of crystal lattices and vortex lattices is that the assumption of small deformations here plays an essential role· As a result, the present theory is meaningful only in the harmonic approximation, when the Hamiltonian is expressed as a quadratic form in the deformations. If anharmonic terms are included, the motion is no longer confined to the xy plane, and the displacements along the vortex axis must also be considered. Such terms destroy the symmetric relation between u_{xi} and u_{yi}, which is necessary for the Hamiltonian approach. Hence, we are unable to treat physical effects that depend on the anharmonic terms, such as thermal conduction due to vortex waves along the axis of the vortex. Such contributions to the total thermal conductivity are presumably negligible in superfluid helium, but might conceivably become important in the mixed state of a type-II superconductor.

A more important question concerns the rigorous basis for the quantization conditions [Eqs. (22) and (23)]. Although these relations are guaranteed to satisfy the correspondence principle, they have not been derived from the fundamental N-body Hamiltonian and must therefore be considered heuristic. Thus it is entirely possible that our theory omits additional terms of order $\hbar$ that also vanish in the classical limit ($\hbar\rightarrow 0$). The existence of such terms is suggested by the partial inconsistency between our initial treatment of the vortex core as a cylinder of well-defined radius ≈ 1 Å and the resulting root-mean-square displacement ≈ 3.8 Å associated with quantum-mechanical fluctuations.

It is interesting to observe that a decrease in the mass m of the background-fluid particles increases the quantum of circulation $\kappa(=h/m)$. Furthermore, simple quantum-mechanical estimates of the core size[31] indicate that a varies as $m^{-1/2}$. Equations (71) and (74) then imply the curious result that a reduced value of m leads to a *reduced* mean-square displacement of the vortex axis. This effect may be understood by noting that the circulation κ in vortex dynamics is analogous to the inertial mass in Newtonian dynamics: An increase in κ decreases the zero-point motion. Unfortunately, these qualitative arguments are not directly applicable to the motion of quantized flux lines in type-II superconductors, because the frequency spectrum is greatly altered from that studied here whenever the wavelength is larger than the penetration depth. The detailed theory of vortex dynamics in bulk type-II superconductors requires a separate treatment and will be presented in a subsequent paper.

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Lambda Curve of Liquid He^4†

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We have measured the pressure P_λ and the derivatives $(dP/dT)_\lambda$ and $(d\rho/dT)_\lambda$ of the lambda curve of He^4 as a function of temperature from the upper lambda point to the lower lambda point, using an apparatus of very high resolution. Empirical equations for P_λ and ρ_λ are presented which represent our data very well and agree generally with previous measurements. These equations define the position and slope of the lambda curve in the ρ, P, T space to a higher order of accuracy and detail than has been possible before.

I. INTRODUCTION

THE application of Pippard's[1] relations or the methods of Buckingham and Fairbank[2] to the lambda transformation of He^4, requires a knowledge of various thermodynamic derivatives along the lambda curve, such as $(dP/dT)_\lambda$ and $(d\rho/dP)_\lambda$. However, the values of these derivatives are not known with sufficient accuracy, since the pressure and density of the lambda transformation have not been measured at small enough temperature intervals to permit accurate differentiation.

The present experiment was designed to measure the derivatives $(dP/dT)_\lambda$ and $(d\rho/dP)_\lambda$ directly at many points along the lambda curve. The resolution of the apparatus was about 1 μdeg K in temperature, 10^{-5} atm in pressure, and better than 10^{-8} g/cm^3 in density. With this resolution it was no problem to make measurements over such small intervals that curvature corrections were negligible.

The slope $(dP/dT)_\lambda$ of the lambda curve at the point where it meets the vapor-pressure curve (the lower lambda point) is needed for the correlation of heat-capacity measurements with thermal-expansion and sound-velocity measurements at the lower lambda point. Many values have been quoted, ranging from -80 to -130 atm/deg. We have made a special effort to obtain a reliable value for this slope.

While this experiment was in progress, Elwell and Meyer[3] reported measurements of the lambda density and pressure at 22 temperatures and the slope $(dP/dT)_\lambda$ at 11 temperatures. Their results are in general agreement with ours, as will be seen later.

II. EXPERIMENTAL

The apparatus used in these experiments is similar to that used previously.[4–6] It is shown schematically in Fig. 1.

Helium gas was purified in a trap (not shown) immersed in liquid helium and was condensed into the sample compartment G through the low-temperature valve A, which has kept closed during measurements. G was isolated from the liquid helium bath by the

† Based on work performed under the auspices of the U. S. Atomic Energy Commission.

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