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# Bethe's Formula for Coulomb-Nuclear Interference*

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Approximations involved in the deduction of Bethe's formula for Coulomb-nuclear interference are reexamined. Some corrections to the formula are obtained by improving the approximations. It is, however, expected that these corrections are small.

USING the semiclassical WKB approximation, Bethe[1] deduced a formula for the small-angle scattering of protons by a nucleus which takes into account the interference between Coulomb and nuclear amplitudes. In recent years, the Bethe formula has been extensively used by various experimental groups[2–5] to evaluate the real part of the nuclear amplitude in the forward direction. This real part of the amplitude is of considerable physical importance, since it provides a check on dispersion relations,[6] indicates the possible asymptotic behavior of the total cross section,[7] and serves as a test for various theoretical models.[8] Except for the work of Soloviev,[9] very little theoretical work has been done since Bethe's on the Coulomb-nuclear interference, as compared to the large amount of experimental effort.[10]

In this paper we reexamine the approximations involved in Bethe's formula and explore the possibility of any significant corrections to it. This work has partly been stimulated by the fact that the semiclassical WKB result has recently been found to be exactly the same as that of a completely relativistic impact-parameter approach.[11]

The spin-independent elastic-scattering amplitude in the relativistic impact-parameter description is given by

$$f(s,\Delta)=ik\int_0^\infty bdb\; J_0(b\Delta)[1-e^{2i\delta(s,b)}], \tag{1}$$

where $\Delta=2k\sin\frac{1}{2}\theta$, $k$ being the c.m. momentum and $\theta$ the c.m. scattering angle, and $s$ is the square of the c.m. energy. $\delta(s,b)$ is related to the complex, energy-dependent "potential" $V(s,r)$ by

$$\delta(s,b)=-\frac{1}{2k}\int_0^\infty \frac{V(s,r)rdr}{(r^2-b^2)^{1/2}}. \tag{2}$$

Since we are dealing with the scattering of protons by another charged particle, there are two potentials involved, namely, the Coulomb potential $V_C(s,r)$ and the nuclear (or strong-interaction) potential $V_N(s,r)$. Correspondingly, we have $\delta(s,b)=\delta_C(s,b)+\delta_N(s,b)$. The scattering amplitude (1) can now be written as

$$f(s,\Delta)=f_{CN}(s,\Delta)+f_C(s,\Delta), \tag{3}$$

where

$$f_{CN}(s,\Delta)=ik\int_0^\infty bdb\; J_0(b\Delta)e^{2i\delta_C(s,b)}[1-e^{2i\delta_N(s,b)}] \tag{4}$$

and

$$f_C(s,\Delta)=ik\int_0^\infty bdb\; J_0(b\Delta)[1-e^{2i\delta_C(s,b)}]; \tag{5}$$

$f_{CN}$ involves both Coulomb and nuclear interactions, while $f_C$ involves only the Coulomb interaction.

Bethe's formula is now based on the following two approximations:

(a) In the amplitude $f_{CN}(s,\Delta)$, Eq. (4), the Coulomb phase shift can be considered as independent of $b$. The reason is that because of the short range of the nuclear interaction, this amplitude gets its main contribution from small values of $b$, and for these values of $b$, the Coulomb phase shift $\delta_C(s,b)$ does not vary appreciably.

(b) In the Coulomb amplitude $f_C(s,\Delta)$, Eq. (5), the phase shift can be taken as that due to two point charges, neglecting the finite charge distributions of the particles. This is based on the argument that for small $\Delta$, the main contribution to Coulomb scattering comes from large values of $b$, for which the phase $\delta_C(s,b)$ is not very much affected by the finite sizes of the particles.

Using the above approximations, we get

$$f_{CN}(s,\Delta)=e^{2i\eta_N}f_N(s,\Delta), \tag{6}$$

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where $\eta_N$ is the average nuclear Coulomb phase shift, and $f_N(s,\Delta)$ is the pure nuclear amplitude given by

$$f_N(s,\Delta)=ik\int_0^\infty bdb\, J_0(b\Delta)[1-e^{2i\delta_N(s,b)}]. \quad (7)$$

Further,

$$f_C(s,\Delta)=-(2k\xi/\Delta^2)e^{2i\eta_C}, \quad (8)$$

where $\xi=Ze^2/v_L$ ($v_L$=lab velocity of the incident particle), and $\eta_C=\eta_0-\xi\ln\sin\frac{1}{2}\theta$ is the pure Coulomb phase shift; here $\eta_0=\arg\Gamma(1+i\xi)$ and $c=\hbar=1$. The differential cross section for small values of $\Delta$ now takes the form

$$\frac{d\sigma}{d\Omega}=|f(s,\Delta)|^2=\left|-\frac{2k\xi}{\Delta^2}e^{2i(\eta_C-\eta_N)}+f_N(s,\Delta)\right|^2 \quad (9)$$

which is Bethe's formula.

We now want to examine the approximations involved in the deduction of the above formula. Let us begin with the Coulomb amplitude first, both because of its familarity and because it brings out the usefulness of the impact-parameter description. To calculate $\delta_C(s,b)$, we have to know the Coulomb potential $V_C(s,r)$. Since the integral in (2) has to converge, we need a screened Coulomb field. Further, it has been pointed out[11] that in this formalism the potential has to be less singular than $1/r$ at $r=0$. We therefore have to smooth out the Coulomb potential at the origin. The above two considerations can be put together by taking the radial dependence of $V_C(s,r)$ as $\{\exp[-\mu(r^2+\beta^2)^{1/2}]\}/(r^2+\beta^2)^{1/2}$. In the limit $\beta\rightarrow 0$, we obtain the usual screened Coulomb field. The limit $\beta\rightarrow 0$ corresponds to treating the particles as point charges.[12] As regards the energy dependence of $V_C(s,r)$, we find that the relativistic impact-parameter description[11] does not indicate what it should be, and we need additional physical information for it. We assume it to be such that the corresponding $s$ dependence of $\delta_C(s,b)$ is $1/v_L$, that is, the same as Bethe's.[13] Thus, the potential finally takes the form[14]

$$V_C(s,r)=2k\xi\, e^{-\mu(r^2+\beta^2)^{1/2}}/(r^2+\beta^2)^{1/2}, \quad (10)$$

where $\xi=Ze^2/v_L$, as given before. Putting this into Eq. (2), we get

$$\delta_C(s,b)=-\xi K_0(\mu(b^2+\beta^2)^{1/2}); \quad (11)$$

$K_0(z)$ is the modified Bessel function of the second kind. The Coulomb scattering amplitude is now obtained from Eqs. (5) and (11). To do the integration involved we replace $J_0(b\Delta)$ by $\frac{1}{2}[H_0^{(1)}(b\Delta)+H_0^{(2)}(b\Delta)]$, and then take the integral containing $H_0^{(1)}(b\Delta)$ along the positive imaginary axis and that containing $H_0^{(2)}(b\Delta)$ along the negative imaginary axis in the complex $b$ plane.[15] The final result is

$$f_C(s,\Delta)=-\frac{2k}{\pi}\int_\beta^\infty K_0(\zeta\Delta)\times\sinh[\pi\xi J_0(z)]e^{i\pi\xi Y_0(z)}\zeta d\zeta, \quad (12)$$

where $z=\mu(\zeta^2-\beta^2)^{1/2}$; and $Y_0(z)$ is the Bessel function of the second kind. Because of the oscillations of the Bessel functions in (12), the main contribution should come from small values of $z$. Correspondingly, we can use the approximations $Y_0(z)\approx(2/\pi)[\ln\frac{1}{2}z+\gamma]$ and $J_0(z)\approx 1$; $\gamma$ is the Euler's constant ($\gamma=0.577$). Integration of (12) can then be done analytically[16] and the result is

$$f_C(s,\Delta)=-(k/\pi)(2\beta/\Delta)^{1+i\xi}\Gamma(1+i\xi)K_{1+i\xi}(\beta\Delta)\sinh(\pi\xi)\times\exp[2i\xi(\ln\tfrac{1}{2}\mu+\gamma)]. \quad (13)$$

For $\Delta\rightarrow 0$, $K_{1+i\xi}(\beta\Delta)=\frac{1}{2}\Gamma(1+i\xi)(2/\beta\Delta)^{1+i\xi}$, so that Eq. (13) becomes

$$f_C(s,\Delta)=-(2k\xi/\Delta^2)\exp 2i\{\xi[\ln(\mu/\Delta)+\gamma]+\eta_0\}. \quad (14)$$

We notice that apart from a phase, (14) is exactly the Coulomb amplitude from the Schrödinger equation.[17] Putting $\eta_0\approx-\xi\gamma$, we find that the phase in (14) is the same infrared phase factor obtained by Soloviev[18] for $\Delta$ small. In the nonrelativistic limit it goes to the well-known divergent phase of the Coulomb amplitude.[19] It is worth noticing that in arriving at (14), we used $\beta\Delta$ small but not $\beta\rightarrow 0$, i.e., we did not consider the particles as point charges. Thus, approximation (b) in the deduction of Bethe's formula has been bypassed.

Let us now examine the Coulomb-nuclear amplitude $f_{CN}(s,\Delta)$, Eq. (4), together with $\delta_C(s,b)$ given by (11). For this amplitude, only small values of $b$ are important, so that we may take

$$\delta_C(s,b)\approx\xi\{\ln[\tfrac{1}{2}\mu(b^2+\beta^2)^{1/2}]+\gamma\}. \quad (15)$$

[12] This can be seen in the following way. For details see Ref. 11. Let us assume that the above potential occurs because the two particles involved in the scattering have finite charge distributions and that these distributions interact with each other through a screened Coulomb potential. If $F_1(\Delta^2)$ and $F_2(\Delta^2)$ are the form factors of the particles, then the Born amplitude becomes $F_1(\Delta^2)\times F_2(\Delta^2)/(\Delta^2+\mu^2)$. On the other hand, the Born amplitude calculated directly from the potential is $\beta K_1[\beta(\Delta^2+\mu^2)^{1/2}]/(\Delta^2+\mu^2)^{1/2}$. Therefore, we have

$$F_1(\Delta^2)F_2(\Delta^2)=\beta(\Delta^2+\mu^2)^{1/2}K_1[\beta(\Delta^2+\mu^2)^{1/2}].$$

Now, limit $\beta\rightarrow 0$, $F_1(\Delta^2)F_2(\Delta^2)=1$; that is, the particles behave as point charges in this limit.

[13] As we shall see later, this energy dependence is also in agreement with the relativistic calculation of Soloviev on the Coulomb phase.

[14] In the nonrelativistic limit, the factor $2k\xi$ becomes $2mZe^2$ ($m$=reduced mass) as it should. We note that when the incident particle and the target particle have similar charges, $\xi$ should be positive, corresponding to the fact that the potential is repulsive. If they have opposite charges, $\xi$ should be replaced by $-\xi$.

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Using this in (4),

$$f_{CN}(s,\Delta)=ik\int_0^\infty bdb\, J_0(b\Delta)e^{i\xi\,\ln(1+b^2/\beta^2)}\times[1-e^{2i\delta_N(s,b)}]e^{2i\xi[\ln(\mu\beta/2)+\gamma]}. \quad (16)$$

If now approximation (a) is made, i.e., the $b$ dependence of the Coulomb phase shift in (16) is neglected, then we get

$$f_{CN}(s,\Delta)=f_N(s,\Delta)e^{2i\xi[\ln(\mu\beta/2)+\gamma]}, \quad (17)$$

where $f_N(s,\Delta)$ is given by (7). Combining (14) and (17) we have

$$\frac{d\sigma}{d\Omega}=\left|-\frac{2k\xi}{\Delta^2}e^{2i(\eta_C-\eta_N)}+f_N(s,\Delta)\right|^2, \quad (18)$$

where

$$\eta_C=\eta_0-\xi\ln\sin\tfrac{1}{2}\theta \quad\text{and}\quad \eta_N=\xi\ln k\beta. \quad (19)$$

Equation (18) is exactly Bethe's formula. However, (19) indicates that some small numerical difference with Bethe's calculation can occur. Bethe[1] gives $\eta_C-\eta_N=\xi\ln(1.06/ka\theta)$, where $a$ is a parameter, while from (19) we find $\eta_C-\eta_N=\xi\ln(1.123/k\beta\theta)$.[20]

Let us next see what correction occurs if the $b$ dependence of the Coulomb phase in (16) is not completely neglected. To this end, we expand the logarithm in (16) and keep the first term. In order to carry out the integration, some knowledge of the nuclear phase shift $\delta_N(s,b)$ is needed. For this purpose, we can safely assume[21]

$$1-e^{2i\delta_N(s,b)}=Ce^{-2b^2/R^2} \quad (C\text{ complex}), \quad (20)$$

when $s$ is large and we are interested in small $\Delta$. The amplitude $f_{CN}(s,\Delta)$ now takes the form

$$f_{CN}(s,\Delta)=\tfrac{1}{2}ikC\left[\frac{2}{R^2}-\frac{i\xi}{\beta^2}\right]^{-1}\exp\left\{-\tfrac{1}{4}\Delta^2\left(\frac{2}{R^2}-\frac{i\xi}{\beta^2}\right)^{-1}+2i\xi[\ln\tfrac{1}{2}\mu\beta+\gamma]\right\}. \quad (21)$$

On the other hand, the parametrization (20) gives the following expression for the pure nuclear amplitude:

$$f_N(s,\Delta)=\tfrac{1}{4}ikCR^2\,e^{-R^2\Delta^2/8}. \quad (22)$$

The parameter $R$ can be easily identified as the optical-model radius. Neglecting $\xi^2$ and $\Delta^2\xi$ terms, we obtain from (21) and (22),

$$f_{CN}(s,\Delta)=f_N(s,\Delta)\left(1+i\frac{R^2\xi}{2\beta^2}\right)e^{2i\xi[\ln\frac{1}{2}\mu\beta+\gamma]}, \quad (23)$$

so that the differential cross section in the near-forward direction in the c.m. system becomes

$$\frac{d\sigma}{d\Omega}=\left|-\frac{2k\xi}{\Delta^2}e^{2i(\eta_C-\eta_N)}+f_N(s,\Delta)\left[1+\frac{iR^2\xi}{2\beta^2}\right]\right|^2. \quad (24)$$

The quantity that occurs in the square bracket is the correction that we have obtained to Bethe's formula (18). This correction adds a term $-\mathrm{Im}f_N(s,0)R^2\xi/2\beta^2$ to the real part of the forward nuclear amplitude. However, since $\beta^2$ and $R^2$ should be of the same order of magnitude, and $\xi$ is small ($\xi\approx Z/137$), the correction to $p$-$p$ and $\pi$-$p$ scattering is not very significant, though for proton-nucleus scattering it may add up to a few percent in the measurement of Re $f(s,0)$/Im $f(s,0)$.

Summarizing, we find that Bethe's formula (18) is much more accurate than the original approximations involved would suggest. We have noticed that keeping finite charge distributions of the particles does not alter the Coulomb contribution. (The only requirement is that $\beta\Delta$ should be small.) Further, when the $b$ dependence of the Coulomb amplitude is not completely neglected in the Coulomb-nuclear part, the correction that comes in is also small. Finally, the nuclear amplitude that occurs in Bethe's formula (18) and in our formula (24) is the pure strong-interaction amplitude, in contrast to the nuclear amplitude occurring in Soloviev's formula,[9] which contains finite radiative corrections.

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[20] Bethe's parameter $a$ and our parameter $\beta$ are physically similar. They are connected with the finite sizes of the particles. To see how they compare with each other, let us examine the Born amplitude, which in Bethe's case is $-2k\xi\exp(-a^2\Delta^2/4)/\Delta^2$ and in our case is $-2k\xi\beta\Delta K_1(\beta\Delta)/\Delta^2$ (when the limit $\mu\to 0$ is taken). The form factors $\exp(-a^2\Delta^2/4)$ and $\beta\Delta K_1(\beta\Delta)$ are both equal to 1 at $\Delta=0$. If we consider that both the form factors fall to a value $1/e$ at the same value of $\Delta$, we get $\beta=0.83a$.

[21] For example, L. Van Hove, lectures given at CERN, 1964 (unpublished).