Broken Symmetry Sum Rules and the Algebra of Currents. II*

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An analysis of current-algebra methods of deriving broken $SU(3)$ sum rules for three-point coupling constants initiated in a previous paper is expanded and concluded. It is shown quite definitely that current algebra does not intrinsically provide more information than is present in conventional group-theoretical derivations. In so doing, we rederive broken $SU(3)$ sum rules for both strong- and weak-interaction coupling constants. In addition consistency conditions in the form of first order, broken $SU(3)$ sum rules for partially conserved axial-vector current and partially conserved tensor current proportionality constants are found.

I. INTRODUCTION

In a previous paper[1] [hereafter referred to as (I)], we derived a current-algebra method for obtaining broken $SU(3)$ sum rules for form factors. The method was illustrated by a derivation of the Muraskin-Glashow[2] sum rules for spin-$\frac{1}{2}^+$-baryon–pseudoscalar-meson coupling constants. This, of course, conflicted with the results of an independent current-algebra derivation[3] of broken $SU(3)$ coupling constant sum rules and we attempted to resolve and explain this difference.

In the present paper, we extend and conclude the analysis described in I. The explanation of the differences involved between our approach and that of Ref. 3 is given in detail in Sec. II. This is followed by a derivation in Sec. III of broken $SU(3)$ sum rules for couplings involving vector mesons. We use partial conservation of both axial-vector currents and tensor currents (PCAC and PCTC) in our analysis, and in so doing, find sum rules to first order in the symmetry breaking for the PCAC and PCTC proportionality constants. It is also found that the vector-meson coupling constants satisfy the same broken $SU(3)$ sum rules that were derived by conventional group-theory techniques.[2] Moreover, this latter result follows without having the meson source densities (currents) forming part or all of a closed algebra with the $SU(3)$ generators [as assumed in (I)] but from commutators[4] of the generators with axial-vector and tensor currents along with PCAC and PCTC.

In Sec. IV, we consider the coupling constants of the spin-$\frac{3}{2}^+$ baryon decuplet with the octets of spin-$\frac{1}{2}$ baryons and pseudoscalar mesons. Once again we obtain the same broken $SU(3)$ sum rules derived previously[5] by a group-theory method. One of these sum rules can be tested from experiment and the agreement is excellent.[5]

Finally, we consider weak-interaction coupling constants in Sec. V with the usual agreement with the group-theory derivation.[6] In Sec. IV and V very slight modifications of our general method are required and these are described in detail.

Our final conclusions and a summary of our results are presented in Sec. VI.

II. THREE-POINT FUNCTIONS AND BROKEN SYMMETRY

In the approach used by Bose and Hara[3] to derive coupling constant sum rules, it is assumed that all the first-order breaking effects are confined to the matrix element $\langle B_2,\pi^\beta|H_{\text{s.b.}}|B_1\rangle$, where $H_{\text{s.b.}}=\lambda\bar{q}\lambda_8 q$ is the symmetry-breaking part of the Hamiltonian. By citing a particular example, we showed in I that this approach is equivalent to saturating a commutator with octet single-particle states. The situation thus appears very similar to using a quark model and, in fact, Eberle[7] has recently shown that both the quark model and the method of Ref. 3 are equivalent, from a perturbation-theory point of view, to assuming that the breaking is due only to graphs such as the one[8] in Fig. 1. In a quark-model approach[9] this is, of course, unavoidable. However, we shall show, using the method derived in I, that the additional breaking terms present in a group theory derivation should also be present in the type of approach used in Ref. 3. We start by considering the matrix element

$$\langle B_2,\pi^\beta|[Q^{(\alpha)},S_0]|B_1\rangle=0,$$

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[1] L. A. Copley and D. Masson, Phys. Rev. **156**, 1564 (1967).

[2] M. Muraskin and S. L. Glashow, Phys. Rev. **132**, 482 (1963).

[3] S. K. Bose and Y. Hara, Phys. Rev. Letters **17**, 409 (1966). The Bose and Hara method has also been used to derive sum rules for vector-meson coupling constants. See R. Rockmore, Phys. Rev. **153**, 1490 (1967); C. S. Lai, *ibid.* **155**, 1562 (1967).

[4] These commutators are related to the $U(12)$ algebra of currents. See S. Fubini, G. Segrè, and J. D. Walecka, Ann. Phys. (N. Y.) **39**, 381 (1966); G. Segrè and J. D. Walecka, *ibid.* **40**, 337 (1966).

[5] V. Gupta and V. Singh, Phys. Rev. **135**, B1442 (1964).

[6] K. Kawarabayashi and W. W. Wada, Phys. Rev. **137**, B1002 (1965).

[7] E. Eberle, Nuovo Cimento **46**, 803 (1966).

[8] Note that only one {8} contributes since M and S_8 are combined symmetrically by an equal-time commutator.

[9] R. J. Rivers, Phys. Letters **22**, 514 (1966).

where $Q^{(\alpha)}$ is the αth $SU(3)$ generator and S_0 is the $SU(3)$ preserving part of the S matrix. Following the method of I, we find the dispersion sum rule

$$\sum_{(B_n,\pi^\nu);B_n'} \left\{ \sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_n & \alpha & B_2 \end{pmatrix} r_{2n}{}^{\alpha} g_{B_1B_n}{}^{\nu}(\Delta^2)\delta_{\nu\beta} + \sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ \pi^\nu & \alpha & \pi^\beta \end{pmatrix} \times r_{\beta n}{}^{\alpha} g_{B_1B_n}{}^{\nu}(\Delta^2)\delta_{n2} - \sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_1 & \alpha & B_n' \end{pmatrix} r_{n'1}{}^{\alpha} g_{B_2B_{n'}}{}^{\beta}(\Delta^2) \right\} + \frac{1}{\pi}\int \frac{\mathrm{Im}A_{B_1B_2}{}^{\alpha\beta}(\nu',\Delta^2)d\nu'}{\nu'} = 0, \quad (1)$$

where $g_{B_1B_2}{}^{\beta}$ is the contribution to the $B_1B_2\pi^\beta$ coupling constant from S_0 and $\mathrm{Im}A_{B_1B_2}{}^{\alpha\beta}$ is the coefficient of $\bar{u}_{B_2}\gamma_5 u_{B_1}$ in the expansion $[i(2\pi)^4\delta(p_f-p_i)T_0=1-S_0]$,

$$A(\nu',\Delta^2)=\tfrac{1}{2}i(2\pi)^4\sum\{\langle B_2,\pi^\beta|D^{(\alpha)}(0)|n\rangle \times\langle n|T_0|B_1\rangle\delta(p_f+k-p_n) -\langle B_2,\pi^\beta|T_0|n\rangle\langle n|D^{(\alpha)}(0)|B_1\rangle \times\delta(p_i-k-p_n)\}. \quad (2)$$

It should be noted that the integrand in (1) is at least of order λ, since all zeroth-order contributions have been explicitly extracted as pole terms and are contained in the sum of residues.

Again following the analysis and notation of (I), we see that $\mathrm{Im}A_{B_1B_2}{}^{\alpha\beta}$ is analogous to a sum of amplitudes for the "reactions"

$$B_1+\xi_0\to B_2+\pi^\beta+\bar{\zeta}_\alpha, \quad (\{8\}\to\{8\}\times\{8\}\times\{8\}),$$

and (3)

$$B_1+\zeta_\alpha\to B_2+\pi^\beta+\bar{\xi}_0, \quad (\{8\}\times\{8\}\to\{8\}\times\{8\}),$$

respectively, where ξ_0 is a "scalar particle" corresponding to T_0. Using this analogy by expanding the continuum contribution into the $SU(3)$ amplitudes indicated by (3), one finds all the breaking terms[7] which were previously overlooked through the assumption[3] that $g_{B_1B_2}{}^{\beta}$ is defined only by its $SU(3)$-symmetry-limit value. In particular, by virtue of the $\{8\}\times\{8\}\to\{8\}\times\{8\}$ amplitude one obtains an expansion of the continuum in terms of five parameters exactly analogous to that obtained in I in rederiving the Muraskin-Glashow sum rules.

In summary, then, we have found that the matrix elements of S_0 are subject to renormalization by the symmetry breaking interaction and that their contributions $g_{B_1B_2}{}^{\beta}$ to the strong-interaction coupling constants satisfy *broken* $SU(3)$ sum rules determined by the $SU(3)$ properties of the continuum contribution to (1).

We should note, however, that if we take single-particle matrix elements of any operator which is a linear combination of singlet and octet operators, $J_0+\lambda J_8$ say, then the matrix elements of the singlet part can be assumed to attain only their symmetry limit value. If we take the particular example of octet single-particle states then we can apply the same analysis as above to the quantities $\langle B_2|J_0|B_1\rangle$. In this case the amplitudes in the continuum have the form $\{8\}\to\{8\}\times\{8\}$ and $\{8\}\times\{8\}\to\{8\}$. Thus these matrix elements will satisfy the same sum rule as $\langle B_2|\lambda J_8|B_1\rangle$ and so can be ignored in sum-rule derivations such as the Gell-Mann–Okubo mass formulas[10] for which $J_0+\lambda J_8$ is the mass operator. Another such example will be given below in which the same sum rule as that derived by the Bose and Hara method is found for the singlet vector-meson coupling constants.

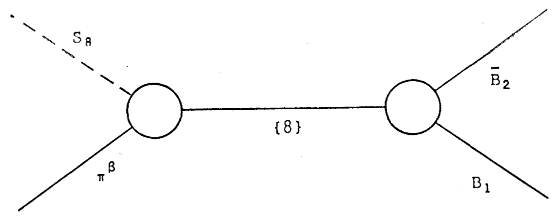

FIG. 1. The basic graph which contributes to the $SU(3)$ breaking in the quark, Bose and Hara (see Ref. 3) models.

III. VECTOR-MESON COUPLING CONSTANTS

In deriving broken $SU(3)$ sum rules for the baryon-pseudoscalar-meson coupling constants in I we considered both the case in which the meson source density $J_\pi{}^{(\alpha)}$ transformed like an 8-vector even when $SU(3)$ is broken[11] and the case involving not $J_\pi{}^{(\alpha)}$ but rather, the axial-vector current $A_\mu{}^{(\alpha)}$ with the PCAC hypothesis used to obtain the meson fields. However, *vector*-meson source currents would appear to admit only one such possibility since there exist arguments[12] opposing an assumption that they transform like an 8-vector in the presence of symmetry-breaking interactions. We are led then to make the hypothesis of a partially conserved tensor current[13] (PCTC)

$$\partial_\mu J_{\mu\nu}{}^{(\alpha)}(x)=K_\alpha\phi_\nu{}^{(\alpha)}(x), \quad (4)$$

where $\phi_\nu{}^{(\alpha)}(x)$ is the αth vector-meson field, K_α is a proportionality constant, and $J_{\mu\nu}{}^{(\alpha)}(x)$ is an antisymmetric tensor current defined in terms of quark

[10] This method of deriving the mass formulas can be found, for example, in Riazuddin and K. T. Mahanthappa, Phys. Rev. **147**, 972 (1966).

[11] $J_\pi{}^{(\alpha)}$ forms part or all of a closed algebra with the $SU(3)$ generators, $Q^{(\beta)}$.

[12] Experimentally, an immediate example of such an argument is $\omega-\phi$ mixing. Theoretically, W. Królikowski [Trieste International Center Report No. IC/66/7 (unpublished)] has shown that $J_\mu{}^{(\alpha)}$ (which is rigorously conserved) cannot be the source of generators of unitary transformations between particles of different masses, thus excluding the typical quark-model analogy $\bar{q}\gamma_\mu(\lambda^i/2)q\sim J_\mu{}^{(i)}$.

[13] W. Królikowski, Trieste International Center Report No. IC/66/15 (unpublished); S. Fubini *et al.*, Ref. (4).

fields by

$$J_{\mu\nu}^{(i)}=\bar{q}\sigma_{\mu\nu}\frac{\lambda^{(i)}}{2}q \quad (i=0,\cdots,8).$$

In analogy with the PCAC case of I, we shall require the commutator

$$[Q^{(\alpha)},J_{\mu\nu}^{(\beta)}(x)]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}J_{\mu\nu}^{(\gamma)}(x), \quad (5)$$

where the $Q^{(\alpha)}$ are the $SU(3)$ generators as defined in (I). Taking the divergence of this commutator, we obtain

$$[\dot{Q}^{(\alpha)},J_{0\nu}^{(\beta)}]+[Q^{(\alpha)},\partial_\mu J_{\mu\nu}^{(\beta)}(x)]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}\partial_\mu J_{\mu\nu}^{(\gamma)}(x). \quad (6)$$

Matrix elements of the first commutator on the left-hand side of (6) are of at least order λ and have the same $SU(3)$ properties as the continuum contribution to the corresponding matrix elements of the second commutator.[14] Thus, its contribution is removed at the same time as that of the continuum and so we can ignore it in all further considerations.

In the derivation of vector-meson–pseudoscalar-meson coupling constant (VPP and VVP) sum rules either PCTC or PCAC may be used. These sum rules should therefore test the consistency of the two hypotheses and so we shall consider them first.

For VPP couplings we require the matrix elements

$$\epsilon_\mu\langle\pi^\alpha;p_1|J_\mu^{(\beta)}|\pi^\gamma;p_2\rangle=\epsilon\cdot p_2 G_{\pi^\alpha\pi^\gamma}{}^{\beta}(\Delta^2),$$

$$\langle V^\alpha;p_1,\epsilon|J_\pi^{(\beta)}|\pi^\gamma;p_2\rangle=\epsilon\cdot p_2 G_{V^\alpha\pi^\gamma}{}^{\beta}(\Delta^2),$$

$$\langle\pi^\alpha,p_1|D^{(\beta)}|\pi^\gamma,p_2\rangle=\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \gamma & \beta & \alpha\end{pmatrix}ir^\beta(m_\alpha^2-m_\gamma^2),$$

and

$$\langle V^\alpha;p_1,\epsilon_\mu|D^{(\beta)}|V^\gamma;p_2,\rho_\nu\rangle=\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \gamma & \beta & \alpha\end{pmatrix}ir_V^\beta(m_{V^\alpha}^2-m_{V^\gamma}^2)\epsilon\cdot\rho,$$

where $D^{(\beta)}(x)$ is the divergence of the unitary spin vector current, $J_\mu^{(\beta)}(x)$ is the source current of the βth vector meson and $G_{\pi^\alpha\pi^\gamma}{}^{\beta}=G_{\pi^\alpha\beta^\alpha}$. We note that if a phenomenological Hamiltonian is written for the VPP interaction and charge conjugation invariance is invoked then one finds[2] that $G_{\pi^\alpha\pi^\gamma}{}^{\beta}(\Delta^2)$ must be completely antisymmetric [i.e., pure F coupling in the $SU(3)$ limit].

We now take matrix elements of (6) between pseudoscalar-meson states. Since the use of boson rather than fermion states introduces no additional complications, the method described in I is directly applicable and yields the sum rules[15]:

$$\sqrt{3}G_{\eta\bar{K}^0}{}^{K^{0*}}-2G_{\bar{K}^0\bar{K}^0}{}^{\rho^0}-3G_{K^0\pi^0}{}^{K^{0*}}+G_{\pi^+\pi^+}{}^{\rho^0}=0, \quad (7)$$

and

$$4\frac{K_{K^*}}{m_{K^*}^2}G_{\pi^0\bar{K}^0}{}^{K^{0*}}+\frac{K_\rho}{m_\rho^2}(G_{\pi^+\pi^+}{}^{\rho^0}-G_{\bar{K}^0\bar{K}^0}{}^{\rho^0})-\sqrt{3}\frac{K_{\omega_8}}{m_{\omega_8}^2}G_{\bar{K}^0\bar{K}^0}{}^{\omega_8}=0. \quad (8)$$

It is important to realize at this point that the proportionality constants K_α/m_α^2 attain a universal value independent of α in the $SU(3)$ limit. When the symmetry is broken, the constants deviate from this universal value to first order in λ. Thus, since $\sqrt{3}G_{\eta\bar{K}^0}{}^{K^{0*}}=3G_{K^0\pi}{}^{K^{0*}}$ and $G_{\pi^-\pi^+}{}^{\rho^0}=2G_{\bar{K}^0\bar{K}^0}{}^{\rho^0}$ in the $SU(3)$ limit, the K_α/m_α^2 were cancelled out in (7) to first order in λ. However, similar $SU(3)$ limit combinations do not occur in (8) and so, in this case, the proportionality constants cannot be removed. This procedure also applies to the PCAC constants obtained below.

If we now derive the same sum rules using PCAC by taking matrix elements of the equation[1]

$$[\dot{Q}^{(\alpha)},A_0^{(\beta)}(x)]+[Q^{(\alpha)},\partial_\mu A_\mu^{(\alpha)}(x)]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}\partial_\mu A_\mu^{(\gamma)}(x), \quad (9)$$

between a vector-meson state and a pseudoscalar-meson state we obtain relation (7) again and a simplified version of (8):

$$\sqrt{3}G_{\bar{K}^0\bar{K}^0}{}^{\omega_8}-4G_{\pi^0\bar{K}^0}{}^{K^{0*}}+G_{\bar{K}^0\bar{K}^0}{}^{\rho^0}-G_{\pi^+\pi^+}{}^{\rho^0}=0. \quad (10)$$

Since we must maintain consistency to first order in λ, we therefore obtain, in addition to the coupling-constant sum rules (10) and (7), a first-order sum rule for the PCTC constants which is of the same form as the Gell-Mann–Okubo mass formula:

$$3\frac{K_{\omega_8}}{m_{\omega_8}^2}+\frac{K_\rho}{m_\rho^2}-4\frac{K_{K^*}}{m_{K^*}^2}=0. \quad (11)$$

Finally, we note that (7) and (10) are the same as the Muraskin-Glashow sum rules for these coupling constants, although, because of the use of PCTC and PCAC, all the coupling constants are evaluated off the mass shell at $\Delta^2=0$.

[14] This is most easily seen if we assume $H_{\text{s.b.}}=\lambda S_8$ so that $\dot{Q}^{(i)}\propto -i\lambda S^{(i)}$ and $[\dot{Q}^{(i)},J_{0\nu}^{(j)}]\propto -i\lambda[S^{(i)},J_{0\nu}^{(j)}]=-2id_{ijk}V_\nu^{(k)}$, $V_\nu^{(k)}=\bar{q}\gamma_\nu(\lambda^{(k)}/2)q$.

[15] We should note at this point that if we assumed

$$[Q^{(\alpha)},J_\mu^{(\beta)}]=-\sqrt{3}\begin{pmatrix}8 & 8 & 8_a\\ \alpha & \beta & \gamma\end{pmatrix}J_\mu^{(\gamma)},$$

we would directly obtain the same sum rules as Muraskin and Glashow.

To obtain sum rules for VVP coupling constants, we take matrix elements of (9) between vector-meson states. Using the matrix element

$$\langle V^\alpha;\, p_1,\epsilon | J_\pi^{(\beta)} | V^\gamma;\, p_2,\rho\rangle = G_{V^\alpha V^\gamma}{}^{\beta}{}_{(\Delta^2)}\epsilon_{\mu\nu\lambda\sigma}\Delta_\mu\rho_\nu\epsilon_\lambda p_{2\sigma}\,,\quad \Delta_\mu = p_{1\mu} - p_{2\mu}\,,$$

and the same analysis as before, we obtain[16]

$$\sqrt{3}G_{\omega_8\bar{K}^{0*}}{}^{K^0} - \sqrt{3}G_{\rho^0\omega_8}{}^{\pi^0} - 2G_{\bar{K}K^{0*}}{}^{\pi^0} - G_{K^{0*}\rho^0}{}^{K^0} = 0\,, \quad (12)$$

$$4\sqrt{3}G_{K^{0*}K^{0*}}{}^{\eta} - 3\sqrt{3}G_{\omega_8\omega_8}{}^{\eta} - \sqrt{3}G_{\rho^0\rho^0}{}^{\eta} + 12G_{K^{0*}K^{0*}}{}^{\pi^0} + 6\sqrt{3}G_{\rho^0\omega_8}{}^{\pi^0} = 0\,, \quad (13)$$

$$\sqrt{3}\frac{C_\eta}{m_\eta^2}(G_{\rho^0\rho^0}{}^{\eta} - G_{K^{0*}K^{0*}}{}^{\eta}) - \frac{C_\pi}{m_\pi^2}G_{K^{0*}K^{0*}}{}^{\pi^0} + 4\frac{C_K}{m_K^2}G_{K^{0*}\rho^0}{}^{K^0} = 0\,, \quad (14)$$

and

$$3\frac{C_\eta}{m_\eta^2}G_{\omega_1\omega_8}{}^{\eta} - 4\frac{C_K}{m_K^2}G_{\omega_1\bar{K}^{0*}}{}^{K^0} + \frac{C_\pi}{m_\pi^2}G_{\omega_1\rho^-}{}^{\pi^+} = 0\,, \quad (15)$$

where C_α is the PCAC constant of proportionality. However, we can also obtain these sum rules by taking matrix elements of (6) between a vector-meson state and a pseudoscalar-meson state and using PCTC. This procedure leads to the following sum rules in place of (14) and (15):

$$\sqrt{3}G_{K^{0*}\eta}{}^{K^{0*}} - \sqrt{3}G_{\rho^0\eta}{}^{\rho^0} - 2G_{K^{0*}K^0}{}^{\rho^0} + G_{K^{0*}\pi^0}{}^{K^{0*}} - 2G_{\rho^0\bar{K}^0}{}^{K^{0*}} = 0\,,$$

or

$$\sqrt{3}G_{\bar{K}^{0*}\bar{K}^{0*}}{}^{\eta} - \sqrt{3}G_{\rho^0\rho^0}{}^{\eta} - 4G_{\rho^0\bar{K}^{0*}}{}^{K^0} + G_{\bar{K}^{0*}\bar{K}^{0*}}{}^{\pi^0} = 0\,, \quad (16)$$

and,

$$3G_{\omega_1\omega_8}{}^{\eta} - 4G_{\omega_1\bar{K}^{0*}}{}^{K^0} + G_{\omega_1\rho^-}{}^{\pi^+} = 0\,. \quad (17)$$

We therefore have the four coupling-constant sum rules (12), (13), (16), and (17) and, from consistency, the first-order "Gell-Mann–Okubo" sum rule for the PCAC constants

$$3(C_\eta/m_\eta^2) + (C_\pi/m_\pi^2) - 4(C_K/m_K^2) = 0\,. \quad (18)$$

We note that once again we have derived the same sum rules as those of Muraskin and Glashow. Further, we note that (17) is the same as the corresponding sum rule derived by the Bose and Hara method.[3] This confirms our analysis in Sec. II. In addition, relation (18) permits us to return to the analysis of the baryon-pseudoscalar-meson couplings in (I), extending our results derived under the assumption of PCAC to obtain just the five Muraskin-Glashow sum rules rather than only three as stated in I.

It is now a simple task to derive the baryon–vector-meson coupling-constant sum rules. We start by defining the matrix element

$$\langle B_2,p_2 | J_\mu{}^{(\alpha)} | B_1,p_1\rangle = \bar{u}(p_2)\left[G_{1B_2B_1}{}^{(\alpha)}(\Delta^2)\left(\gamma_\mu - \Delta_\mu\frac{(m_2-m_1)}{\Delta^2}\right) + G_{2B_2B_1}{}^{(\alpha)}(\Delta^2)\sigma_{\mu\nu}\Delta_\nu\right]u(p_2)\,.$$

We now take matrix elements of (6) between baryon states and with the aid of PCTC, relation (11) and our usual analysis we find[17]

$$\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\omega_8} - \sqrt{3}G_{NN}{}^{\omega_8} - G_{NN}{}^{\rho^0} + 4G_{N\Sigma^0}{}^{K^{0*}} - G_{\Sigma^+\Sigma^+}{}^{\rho^0} = 0\,, \quad (19)$$

$$\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\omega_8} - \sqrt{3}G_{\Xi^0\Xi^0}{}^{\omega_8} - G_{\Xi^0\Xi^0}{}^{\rho^0} + 4G_{\Sigma^0\Xi^0}{}^{K^{0*}} + G_{\Sigma^+\Sigma^+}{}^{\rho^0} = 0\,, \quad (20)$$

$$3\sqrt{3}G_{\Sigma^0\Sigma^0}{}^{\omega_8} - 3\sqrt{3}G_{\Lambda\Lambda}{}^{\omega_8} + 4G_{NN}{}^{\rho^0} + 4G_{\Xi^0\Xi^0}{}^{\rho^0} + 8G_{N\Sigma^0}{}^{K^{0*}} + 8G_{\Sigma^0\Xi^0}{}^{K^{0*}} + 6\sqrt{3}G_{\Sigma\Lambda}{}^{\rho^0} = 0\,, \quad (21)$$

$$\sqrt{3}G_{N\Lambda}{}^{K^{0*}} - \sqrt{3}G_{\Sigma^0\Lambda}{}^{\rho^0} - 2G_{NN}{}^{\rho^0} + G_{N\Sigma^0}{}^{K^{0*}} - 2G_{\Sigma^0\Xi^0}{}^{K^{0*}} - G_{\Sigma^+\Sigma^+}{}^{\rho^0} = 0\,, \quad (22)$$

$$\sqrt{3}G_{\Lambda\Xi^0}{}^{K^{0*}} - \sqrt{3}G_{\Sigma^0\Lambda}{}^{\rho^0} - 2G_{\Xi^0\Xi^0}{}^{\rho^0} - 2G_{N\Sigma^0}{}^{K^{0*}} + G_{\Sigma^0\Xi^0}{}^{K^{0*}} + G_{\Sigma^+\Sigma^+}{}^{\rho^0} = 0\,, \quad (23)$$

and

$$3G_{\Lambda\Lambda}{}^{\omega_1} - 2G_{NN}{}^{\omega_1} - 2G_{\Xi^0\Xi^0}{}^{\omega_1} + G_{\Sigma^+\Sigma^+}{}^{\omega_1} = 0\,. \quad (24)$$

Finally, we note that all the sum rules involving an ω_1 or ω_8 coupling constant can be rewritten in terms of the physical ω and ϕ mesons by taking[18]

$$\omega = (\sqrt{\tfrac{1}{3}})\omega_8 + (\sqrt{\tfrac{2}{3}})\omega_1 \quad\text{and}\quad \phi = -(\sqrt{\tfrac{2}{3}})\omega_8 + (\sqrt{\tfrac{1}{3}})\omega_1\,.$$

IV. SUM RULES FOR BARYON DECUPLET COUPLINGS TO THE BARYON AND MESON OCTETS

Our analysis is also easily applicable, with a few small modifications, to the coupling constants of the $J^P=\frac{3}{2}^+$ baryon decuplet to the $J^P=\frac{1}{2}^+$ baryon octet and pseudoscalar-meson octet.

We must take matrix elements of (9) between an octet B state and a decuplet B^* state. We thus require the matrix elements

$$\langle B_1^*,p_1 | J^{(\beta)} | B_2,p_2\rangle = G_{B_1^*B_2}{}^{\beta}(\Delta^2)\bar{u}_\mu(p_1)\Delta_\mu u(p_2)\,,$$

$$\langle B_1^*,p_1 | D^{(\alpha)} | B_2^*,p_2\rangle = (\sqrt{6})\begin{pmatrix}10 & 8 & 10\\ B_2^* & \alpha & B_1^*\end{pmatrix} i(m_1^* - m_2^*)r^{*\alpha}\bar{u}_\mu(p_1)u_\mu(p_2)\,,$$

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[16] These coupling constants are completely symmetric (D-type coupling) as can again be seen from a phenomenological Hamiltonian.

[17] We consider only $G_{1B_1B_2}{}^{(\alpha)}$. Sum rules for $G_{2B_1B_2}{}^{(\alpha)}$ follow analogously.

[18] S. Okubo, Phys. Letters 5, 165 (1963).

and

$$\langle B_1,p_1|D^{(\alpha)}|B_2,p_2\rangle = \sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \alpha & B_1 \end{pmatrix} i(m_1-m_2)r^\alpha \bar{u}(p_1)u(p_2).$$

Since the projection operator for the Rarita-Schwinger spin $\frac{3}{2}$ field is

$$\Lambda_{\mu\nu}=\frac{\gamma\cdot p+m}{2m}\left(g_{\mu\nu}-\tfrac{1}{3}\gamma_\mu\gamma_\nu-2\frac{p_\mu p_\nu}{3m^2}+\frac{p_\mu\gamma_\nu-p_\nu\gamma_\mu}{3m}\right),$$

it is easily determined that our initial dispersive sum rule is

$$\sum_{B_{n^*},B_{n'}}\left[(\sqrt{6})\begin{pmatrix} 10 & 8 & 10 \\ B_n^* & \alpha & B_1^* \end{pmatrix} r^{*\alpha}\frac{C_\beta}{m_\beta^2}G_{B_n^*B_2}{}^\beta - \sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \alpha & B_{n'} \end{pmatrix} r^\alpha \times \frac{C_\beta}{m_\beta^2}G_{B_1^*B_{n'}}{}^\beta\right] + \frac{1}{\pi}\int\frac{\mathrm{Im}A_{B_1^*B_2}{}^{\alpha\beta}(\nu')d\nu'}{\nu'} = -\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ \alpha & \beta & \gamma \end{pmatrix}\frac{C_\gamma}{m_\gamma^2}G_{B_1^*B_2}{}^\gamma, \quad (25)$$

where $\mathrm{Im}A_{B_1^*B_2}{}^{\alpha\beta}$ is the coefficient of $\bar{u}_\mu(p_1)\Delta_\mu\, u(p_2)$ in the expansion

$$A(\nu')=\tfrac{1}{2}i(2\pi)^4\sum\frac{C_\beta}{m_\beta^2}[\langle B_1^*|D^{(\alpha)}(0)|n\rangle \times\langle n|J_\pi{}^{(\beta)}(0)|B_2\rangle\delta(p_1+k-p_n) - \langle B_1^*|J_\pi{}^{(\beta)}(0)|n\rangle\langle n|D^{(\alpha)}(0)|B_2\rangle \times\delta(p_2-k-p_n)].$$

We can now apply the same analysis as used in I to get rid of the continuum contribution. We must note, however, that in this case there need not exist simple relations between the s- and u-type "amplitudes" but this does not add any additional complications. All the sum rules may be obtained, for example, from the commutators $[Q^{(K^0)},\ A_\mu{}^{(K^0)}]=\frac{1}{2}A_\mu{}^{(\pi^0)}-\frac{1}{2}\sqrt{3}A_\mu{}^{(\eta)}$,

$$[Q^{(K^0)},A_\mu{}^{(\eta)}]=\tfrac{1}{2}\sqrt{3}A_\mu{}^{(K^0)},$$

and

$$[Q^{(K^0)},A_\mu{}^{(\pi^0)}]=-\tfrac{1}{2}A_\mu{}^{(K^0)}.$$

We find the following relations

$$3G_{\Xi^*\Xi^0}{}^{\pi^0}+3\sqrt{3}G_{\Xi^*\Xi^0}{}^{\eta}-\sqrt{6}G_{\Omega^-\Xi^-}{}^{K^0}-6G_{Y_1^*\Xi^0}{}^{K^0}=0, \quad (26)$$

$$3G_{N^*N}{}^{\pi^0}-2\sqrt{3}G_{Y_1^*\Lambda}{}^{\pi^0}-6G_{Y_1^*N}{}^{K^0}+2\sqrt{3}G_{\Xi^{0*}\Lambda}{}^{K^0}=0, \quad (27)$$

$$\sqrt{6}G_{\Omega^-\Xi^-}{}^{K^0}+6G_{\Xi^*\Sigma^0}{}^{K^0}+3G_{N^*\Sigma}{}^{K^0}-6G_{Y_1^{*0}\Xi^0}{}^{K^0}=0, \quad (28)$$

$$\sqrt{6}G_{\Omega^-\Xi^-}{}^{K^0}-6G_{Y_1^*N}{}^{K^0}+3G_{\Xi^{0*}\Sigma^0}{}^{K^0}+3\sqrt{3}G_{\Xi^*\Lambda}{}^{K^0}=0, \quad (29)$$

$$G_{Y_1^{*-}\Sigma^-}{}^{\pi^0}+2G_{\Xi^*\Xi^0}{}^{\pi^0}-G_{N^*N}{}^{\pi^0}+\sqrt{3}G_{Y_1^*\Lambda}{}^{\pi^0}=0, \quad (30)$$

$$2\sqrt{3}G_{\Xi^*\Xi^0}{}^{\eta}-2\sqrt{3}G_{Y_1^{*-}\Sigma^-}{}^{\eta}+3G_{N^*\Sigma}{}^{K^0}-6G_{Y_1^*\Xi^0}{}^{K^0}=0, \quad (31)$$

$$2G_{\Xi^{0*}\Xi^0}{}^{\pi^0}+2G_{Y_1^{*-}\Sigma^-}{}^{\pi^0}+G_{N^*\Sigma}{}^{K^0}-2G_{Y_1^*\Xi^0}{}^{K^0}=0, \quad (32)$$

where use was made of (17) in relation (26). These sum rules are the same as those obtained by Gupta and Singh[5] by a conventional group theory approach. The one testable sum rule (30) agrees with experiment very well.

V. WEAK-INTERACTION COUPLING CONSTANTS

As a final application of our analysis, we shall consider the leptonic weak-interaction coupling constants. In Cabibbo's theory[19] of leptonic decays, the weak hadron current is written as

$$J_\mu=\cos\theta J_\mu{}^{(\pi^+)}+\sin\theta\, J_\mu{}^{(K^+)},$$

where $J_\mu{}^{(\alpha)}=V_\mu{}^{(\alpha)}+A_\mu{}^{(\alpha)}$, $V_\mu{}^i\sim\bar{q}\gamma_\mu(\lambda^i/2)q$, and $A_\mu{}^i\sim\bar{q}\gamma_\mu\gamma_5(\lambda^i/2)q$. The angle θ is assumed to have nothing to do with the strong interactions and hence all renormalization due to $SU(3)$ breaking will involve the current $V_\mu{}^{(\alpha)}$ and $A_\mu{}^{(\alpha)}$ only.

The derivation of the sum rules for the axial-vector coupling constants is quite trivial and has, in fact, been done implicitly in this paper and in I. We need only consider the commutators

$$[Q^{(K^0)},A_\mu{}^{(K^-)}]=0,$$

and

$$[Q^{(K^0)},A_\mu{}^{(\pi^+)}]=0,$$

from which we obtain via the usual analysis

$$\sqrt{3}g_{P\Lambda}{}^A-\sqrt{3}g_{\Sigma^+\Lambda}{}^A+\sqrt{2}g_{PN}{}^A-g_{P\Sigma^0}{}^A+2g_{\Sigma^0\Xi^-}{}^A+g_{\Sigma^+\Sigma^0}{}^A=0, \quad (33)$$

and

$$\sqrt{3}g_{\Lambda\Xi^-}{}^A-\sqrt{3}g_{\Sigma^+\Lambda}{}^A+\sqrt{2}g_{\Xi^0\Xi^-}{}^A+2g_{P\Sigma^0}{}^A-g_{\Sigma^0\Xi^-}{}^A-g_{\Sigma^+\Sigma^0}{}^A=0. \quad (34)$$

One should note the similarity between these relations and the relations (9) and (10) of paper (I).

In order to derive the *second*-order sum rules for the vector coupling constants we shall require some modifications to our usual analysis.

We consider the commutator

$$[Q^{(K^-)},V_\mu{}^{(K^+)}]=-\tfrac{1}{2}V_\mu{}^{(\pi^0)}-\tfrac{1}{2}\sqrt{3}V_\mu{}^{(\eta)}. \quad (35)$$

Since $V_\mu{}^{(\alpha)}$ is the current corresponding to $Q^{(\alpha)}$, the currents $V_\mu{}^{(\pi^0)}$ and $V_\mu{}^{(\eta)}$ carry conserved quantum

[19] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

numbers. Using the matrix element

$$\langle B_1,p_1|V_\mu^{(\alpha)}|B_2,p_2\rangle = i\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \alpha & B_1 \end{pmatrix}\bar{u}_1 \times\{G_{1B_1B_2}^{(\alpha)}(\Delta^2)\gamma_\mu + G_{2B_1B_2}^{(\alpha)}(\Delta^2)(p_1-p_2)_\mu + G_{3B_1B_2}^{(\alpha)}(\Delta^2)\sigma_{\mu\nu}(p_1-p_2)_\nu\}u_2,$$

where $G_{1B_1B_2}^{(\alpha)}(0)=g_{B_1B_2}^V$, $m_1\neq m_2$, we easily derive the relation

$$\sum_{n,n'}\left\{3\begin{pmatrix} 8 & 8 & 8_a \\ n & K^- & B_1 \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & K^+ & n \end{pmatrix} g_{B_1n}{}^V g_{nB_2}{}^V - 3\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & K^- & n' \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_a \\ n' & K^+ & B_1 \end{pmatrix} g_{n'B_2}{}^V g_{B_1n'}{}^V\right\} + \frac{1}{\pi}\int\frac{\mathrm{Im}A_{B_1B_2}{}^{K^-K^+}(\nu',0)d\nu'}{\nu'} = -3\begin{pmatrix} 8 & 8 & 8_a \\ K^- & K^+ & \gamma \end{pmatrix}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & \gamma & B_1 \end{pmatrix}, \quad (36)$$

where $A_{B_1B_2}{}^{K^-K^+}(\nu',\Delta^2)$ is the coefficient of $\bar{u}_1\,\gamma_\mu\,u_\alpha$ in the expansion

$$A(\nu',\Delta^2)=\tfrac{1}{2}i(2\pi)^4\sum_n\{\langle B_1|D^{(K^-)}|n\rangle \times\langle n|J_\mu{}^{(K^+)}|B_2\rangle\delta(p_1-p_n+k) - \langle B_1|J_\mu{}^{(K^+)}|n\rangle\langle n|D^{(K^-)}|B_2\rangle \times\delta(p_n-p_2+k)\}.$$

Once again we can draw the analogy between the continuum contribution and the sum of "*s*- and *u*-channel amplitudes". However, since $CJ_\mu{}^{(\alpha)}C^{-1}=-J_\mu{}^{(-\alpha)}$, where C is the charge conjugation operator,[20] we must replace the relations between $A_i{}^s$ and $A_i{}^u$ in (I) by $A_i{}^s=-A_i{}^u$ for $i=27$, 8_s, 8_a, 10, $\overline{10}$ and 1. Thus taking matrix elements of (35) between P states and Ξ^- states, we find

$$3g_{\Lambda\Xi^-}{}^{V2}+g_{\Sigma^0\Xi^-}{}^{V2}-3g_{P\Lambda}{}^{V2}-g_{P\Sigma^0}{}^{V2}=0, \quad (37)$$

which is the only independent sum rule. Since, however, $g_{B_1B_2}=1+g_{B_1B_2}'$ where $g_{B_1B_2}'=O(\lambda^2)$, we can write $g_{B_1B_2}{}^2=1+2g_{B_1B_2}'$ and so derive the second-order sum rule

$$3g_{\Lambda\Xi^-}{}^{V}+g_{\Sigma^0\Xi^-}{}^{V}-3g_{P\Lambda}{}^{V}-g_{P\Sigma^0}{}^{V}=0. \quad (38)$$

Finally, if we redefine our coupling constant to be

$$G_{B_1B_2}{}^V=\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ B_2 & K^+ & B_1 \end{pmatrix} g_{B_1B_2}{}^V,$$

we find

$$\sqrt{3}(G_{\Lambda\Xi^-}{}^V+G_{P\Lambda}{}^V)+G_{\Sigma^0\Xi^-}{}^V+G_{P\Sigma^0}{}^V=0. \quad (39)$$

We note once again that (39), (34), and (33) are the same as the corresponding sum rules derived by Kawarabayashi and Wada[6] by the conventional group-theory technique.

VI. CONCLUSION

We have performed a reasonably complete analysis of the derivation of broken $SU(3)$ sum rules for three-point vertex functions using current algebra. Our one over-all conclusion must be that the current-algebra method is completely consistent with the various group theory methods. We must then ask what are the added virtues associated with our method since in some cases it is actually less tedious to use the alternative methods of group theory. The answer to this question is twofold: (1) The current-algebra approach appears more physical in the sense that we do not require such entities as symmetry-breaking spurions nor do we require our meson fields to transform like 8-vectors even in the presence of symmetry breaking, and (2) It is possible, given experimental justification, to calculate second- or higher-order corrections[21] to the sum rules.

It is interesting to note our result that assuming that both PCAC and PCTC are valid hypotheses then we obtain the broken $SU(3)$ sum rules (11) and (18) for the proportionality constants. Writing $C_\alpha/m_\alpha{}^2=f_\alpha$, we can restate (18) in the form:

$$3(C_\eta/m_\eta{}^2)+f_\pi-4f_K=0.$$

However, we know from experiment[22] that $f_K/f_\pi=1.21$ and hence we find

$$C_\eta=1.28m_\eta{}^2f_\pi.$$

Since f_π can be determined from the pion decay width we thus have a measure of the η PCAC proportionality constant. This has important implications for calculations depending on PCAC in which the proportionality constants were previously assumed to be universal.[22]

One can test the sum rules in Sec. VI (in the absence of good experimental data) with the calculations of corrected weak-interaction coupling constants performed by Calucci *et al.*[23] In particular, relation (37) checks with their results to within 0.5% as we would expect. Further, their calculations present an interesting example to follow for determining higher-order corrections to these sum rules and coincidentally (in the case of the axial-vector sum rules) to the corresponding sum rules for baryon-meson coupling constants.

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[20] Note all our pseudoparticle sources were such that $CS^{(\alpha)}C^{-1}=S^{(-\alpha)}$, i.e., self-conjugate octets.

[21] See the comment on this point in I.

[22] For a particular example of such a calculation, see K. Raman, Phys. Rev. **149**, 1122 (1966).

[23] G. Calucci, G. Denardo, and C. Rebbi, University of Trieste Report, 1966 (unpublished).