solution of (A5) is

$$f(-x)\sim e^{-x}/\sqrt{x},\quad x\gg 1. \tag{A6}$$

From (A1), (A3), (A4), and (A6), it follows that

$$f(x)\approx f(x)+f(-x)=2\sum_{n=0}^{\infty}\frac{[(\frac{1}{3}\sqrt[3]{4})x]^{6n}}{(2n)!(4n)!},\quad x\gg 1. \tag{A7}$$

Since the main contribution to the sum for large x comes from large n, we can use Stirling's formula

$$n!=(2\pi n)^{1/2}(n/e)^n \tag{A8}$$

to write

$$f(x)=\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{(x/2)^{6n}}{2\pi n(3n/e)^{6n}},\quad x\gg 1. \tag{A9}$$

In a similar manner, the zeroth-order Bessel function of imaginary argument can be written

$$I_0(x)=\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2}\approx\sum_{n=1}^{\infty}\frac{(x/2)^{2n}}{2\pi n(n/e)^{2n}},\quad x\gg 1 \tag{A10}$$

from which there follows

$$\begin{aligned}\tfrac{1}{3}[I_0(x)&+I_0(e^{2\pi i/3}x)+I_0(e^{4\pi i/3}x)]\\&=\tfrac{1}{3}\sum_{n=1}^{\infty}\frac{(x/2)^{2n}}{2\pi n(n/e)^{2n}}(1+e^{2\pi in/3}+e^{4\pi in/3}),\quad x\gg 1\\&=\sum_{n=1}^{\infty}\frac{(x/2)^{6n}}{6\pi n(3n/e)^{6n}},\quad x\gg 1.\end{aligned} \tag{A11}$$

A comparison of (A9) and (A11) yields

$$f(x)=(1/\sqrt{2})[I_0(x)+I_0(e^{2\pi i/3}x)+I_0(e^{4\pi i/3}x)],\quad x\gg 1. \tag{A12}$$

Making use of the asymptotic expression for $I_0(x)$,[17]

$$I_0(x)=e^x/(2\pi x)^{1/2},\quad x\gg 1 \tag{A13}$$

there follows

$$f(x)=e^x/(4\pi x)^{1/2},\quad x\gg 1. \tag{A14}$$

[17] B. O. Pierce and R. M. Foster, *A Short Table of Integrals* (Ginn and Company, Boston, 1956), p. 94.

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Current Algebras, Regge Poles, and the Isovector Anomalous Magnetic Moment of the Nucleon*

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We discuss the sum rule for the isovector anomalous magnetic moment of the nucleon $F_2{}^V(0)=1.85$, which is obtained from the current commutation relation $\delta(x_0)[A_0{}^+(x),A_\nu{}^-(0)]=2V_\nu{}^3(x)\delta^4(x)$ by use of the covariant method proposed by Fubini. We find that (1) the sum rule cannot be evaluated without explicit knowledge of one of the axial-vector–nucleon "scattering" amplitudes; (2) calculating the contributions from the $P_{33}(1236)$ and $D_{13}(1525)$ using a dispersion-pole model of the weak amplitude gives only $F_2{}^V(0)=0.37$, and (3) estimating the high-energy continuum contribution to the sum rule from Regge-pole fits to πp charge-exchange scattering increases the result to $F_2{}^V(0)\cong 1.0$. It seems that the sum rule is dominated by low- and high-energy continuum contributions, which must be more accurately known before the validity of the sum rule can be judged.

I. INTRODUCTION

THE success of the Adler-Weisberger[1] sum rule for $g_A{}^2$ has led to several attempts[2,3] to generalize the method and derive additional sum rules (or low-energy theorems) involving the parameters of πN scattering. We have independently made such an attempt which differs considerably from previous efforts, and leads us to a sum rule which can only be evaluated with the help of a model for the weak axial-vector amplitudes, and seems to be dominated by high-energy contributions. The results of this calculation have already been briefly described elsewhere.[4]

Our principal result may be stated as a sum rule for $F_2{}^V(0)=1.85$, the isovector anomalous magnetic moment of the nucleon

$$F_2{}^V(0)=-f_\pi{}^2M\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu}{\nu^2}\mathrm{Im}A^{(-)}(\nu)+2M\tilde{A}_{11}(0), \tag{1}$$

where M is the nucleon mass, $f_\pi=0.935m_\pi$ is the $\pi^\pm$

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[1] S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).

[2] N. Fuchs, Phys. Rev. 150, 1241 (1966).

[3] K. Raman, Phys. Rev. Letters, 17, 983 (1966); 18, 432E (1967).

[4] H. Goldberg and F. Gross, Cornell University Report (unpublished).

decay constant, $\nu = M\omega_L$, where ω_L is the total laboratory energy of the pion, and $A^{(-)}$ is half the difference between the A amplitudes for $\pi^- p$ forward scattering and $\pi^+ p$ forward scattering (extrapolated to zero pion mass). The amplitude $A_{11}(0)$ is one of the invariants in the decomposition of the axial-vector–nucleon "scattering" amplitude, and is defined in Sec. II. The tilde over A_{11} refers to the fact that the nucleon pole term has been extracted.

The derivation of Eq. (1) is described in Sec. II and III. In addition to (1), we obtain a second sum rule, which yields the original Adler-Weisberger sum rule when combined with (1). In this combination the model-dependent amplitude $\tilde{A}_{11}(0)$ cancels out. To obtain useful information from (1) requires a model. This is discussed in Sec. IV, and will be seen to present interesting difficulties.

High-energy contributions to the sum rule are estimated in Sec. V. For this purpose a Regge-pole model of π-N scattering is employed. A summary and conclusions can be found in Sec. VI.

II. DERIVATION OF THE SUM RULE

We consider the covariant axial-vector–nucleon scattering amplitude $T_{\mu\nu}^{\pm}$, defined as

$$T_{\mu\nu}^{\pm} = iN_1N_2 \int d^4x\, e^{iq_2\cdot x} \mathcal{K}_x \theta(x_0) \times \langle p_2 | [W_\mu^{\mp}(x), A_\nu^{\pm}(0)] | p_1 \rangle, \quad (2)$$

where N_1 and N_2 are nucleon normalization factors, W_μ is an axial-vector field which mediates the weak interactions, and A_μ is the axial-vector current. We have defined W_μ by

$$\mathcal{K}_x W_\mu^{\pm}(x) = A_\mu^{\pm}(x). \quad (3)$$

This scattering amplitude is shown schematically in Fig. 1. It corresponds to the scattering of (hypothetical) W particles by nucleons. The superscripts + and − refer to positive and negative charges, respectively.

The reason for starting with (2) is that we wish our analysis to be completely covariant, and the covariance of $T_{\mu\nu}$ is guaranteed by the covariance of the S matrix.

If we now operate on the integrand with $\mathcal{K}_x$, we obtain

$$T_{\mu\nu}^{\pm} = R_{\mu\nu}^{\pm} + \text{“seagull terms,”} \quad (4)$$

where

$$R_{\mu\nu}^{\pm} = iN_1N_2 \int d^4x\, e^{iq_2\cdot x} \theta(x_0) \times \langle p_2 | [A_\mu^{\mp}(x), A_\nu^{\pm}(0)] | p_1 \rangle, \quad (5)$$

and the seagull terms are the extra noncovariant polynomials in q_{20} resulting from the operation of $\mathcal{K}_x$ on the θ function. The existence of seagull terms implies the noncovariance of $R_{\mu\nu}^{\pm}$. This is directly related to the existence of Schwinger terms in the equal-time commutator of A_μ^+ and A_ν^-.[5]

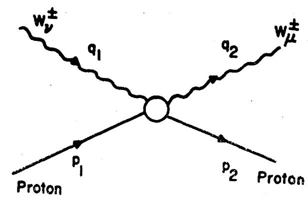

FIG. 1. Diagrammatic representation of the amplitude $T_{\mu\nu}$ discussed in the text.

We now take the dot product of Eq. (4) with q_2^μ, yielding

$$\begin{aligned} q_2^\mu T_{\mu\nu}^{\pm} &= q_2^\mu R_{\mu\nu}^{\pm} + q_2\cdot(\text{seagull}) \\ &= -N_1N_2 \int d^4x\, e^{iq_2\cdot x} \delta(x_0) \times \langle p_2 | [A_0^{\mp}(x), A_\nu^{\pm}(0)] | p_1 \rangle \\ &\quad - N_1N_2 \int d^4x\, e^{iq_2\cdot x} \theta(x_0) \times \langle p_2 | [\partial^\mu A_\mu^{\mp}(x), A_\nu^{\pm}(0)] | p_1 \rangle \\ &\quad + q_2\cdot(\text{seagull}). \end{aligned} \quad (6)$$

Using the equal-time commutation relation

$$\delta(x_0)[A_0^{\mp}(x), A_\nu^{\pm}(0)] = \mp 2V_\nu^3(x)\delta^4(x) + \text{Schwinger terms}, \quad (7)$$

Eq. (6) becomes

$$\begin{aligned} q_2^\mu T_{\mu\nu}^{\pm} &= -N_1N_2 \int d^4x\, e^{iq_2\cdot x} \theta(x_0) \times \langle p_2 | [\partial^\mu A_\mu^{\mp}(x), A_\nu^{\pm}(0)] | p_1 \rangle \\ &\quad \pm 2N_1N_2 \langle p_2 | V_\nu^3(0) | p_1 \rangle \\ &\quad - \text{Schwinger} + q_2\cdot(\text{seagull}). \end{aligned} \quad (8)$$

We now use a theorem recently proved by Brown,[6] which shows that the last two terms in (8) cancel if the equal-time commutator between the axial current and the W_μ field is no more singular than $\delta^3(x)$. Unlike Weisberger,[7] we must be careful with these terms since we shall later take derivatives with respect to ν (or, equivalently, q_{20}). Assuming this cancellation, we proceed to dot once more with q_1^ν, obtaining

$$L^{\pm} \equiv q_2^\mu T_{\mu\nu}^{\pm} q_1^\nu = C^{\pm} + D^{\pm} \pm 2q_1^\nu V_\nu^3, \quad (9)$$

[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[6] S. G. Brown, Phys. Rev. **158**, 1444 (1967). See also D. G. Boulware and L. S. Brown, Phys. Rev. **156**, 1724 (1967).
[7] W. I. Weisberger, Phys. Rev. **143**, B1302 (1965).

TABLE I. The tensor basis used in Eq. 14 of the text.

1	$P_\mu P_\nu$	$\bar{1}$	$P_\mu P_\nu \mathbf{Q}$
2	$P_\mu Q_\nu + P_\nu Q_\mu$	$\bar{2}$	$(P_\mu Q_\nu + P_\nu Q_\mu)\mathbf{Q}$
3	$P_\mu \Delta_\nu + P_\nu \Delta_\mu$	$\bar{3}$	$(P_\mu \Delta_\nu + P_\nu \Delta_\mu)\mathbf{Q}$
4	$P_\mu \gamma_\nu + P_\nu \gamma_\mu$	$\bar{4}$	$[\gamma_\nu,\mathbf{Q}]P_\mu + [\gamma_\mu,\mathbf{Q}]P_\nu$
5	$Q_\mu Q_\nu$	$\bar{5}$	$Q_\mu Q_\nu \mathbf{Q}$
6	$Q_\mu \Delta_\nu + Q_\nu \Delta_\mu$	$\bar{6}$	$(Q_\mu \Delta_\nu + Q_\nu \Delta_\mu)\mathbf{Q}$
7	$Q_\mu \gamma_\nu + Q_\nu \gamma_\mu$	$\bar{7}$	$[\gamma_\nu,\mathbf{Q}]Q_\mu + [\gamma_\mu,\mathbf{Q}]Q_\nu$
8	$\Delta_\mu \Delta_\nu$	$\bar{8}$	$\Delta_\mu \Delta_\nu \mathbf{Q}$
9	$\Delta_\mu \gamma_\nu + \Delta_\nu \gamma_\mu$	$\bar{9}$	$[\gamma_\nu,\mathbf{Q}]\Delta_\mu + [\gamma_\mu,\mathbf{Q}]\Delta_\nu$
10	$g_{\mu\nu}$	$\overline{10}$	$g_{\mu\nu}\mathbf{Q}$
11	$[\gamma_\mu,\gamma_\nu]$	$\overline{11}$	$[\gamma_\mu,\gamma_\nu]\mathbf{Q}$
12	$P_\mu Q_\nu - P_\nu Q_\mu$	$\overline{12}$	$(P_\mu Q_\nu - P_\nu Q_\mu)\mathbf{Q}$
13	$P_\mu \Delta_\nu - P_\nu \Delta_\mu$	$\overline{13}$	$\begin{cases}(P_\mu \Delta_\nu - P_\nu \Delta_\mu)\mathbf{Q} \\ [\gamma_\nu,\mathbf{Q}]P_\mu - [\gamma_\mu,\mathbf{Q}]P_\nu\end{cases}$
14	$P_\mu \gamma_\nu - P_\nu \gamma_\mu$		
15	$Q_\mu \Delta_\nu - Q_\nu \Delta_\mu$	$\overline{14}$	$[\gamma_\nu,\mathbf{Q}]Q_\mu - [\gamma_\mu,\mathbf{Q}]Q_\nu$
16	$Q_\mu \gamma_\nu - Q_\nu \gamma_\mu$	$\overline{15}$	$[\gamma_\nu,\mathbf{Q}]\Delta_\mu - [\gamma_\mu,\mathbf{Q}]\Delta_\nu$
17	$\Delta_\mu \gamma_\nu - \Delta_\nu \gamma_\mu$		

where

$$C^\pm = -iN_1N_2 \int d^4x\, e^{iq_2\cdot x}\delta(x_0) \times \langle p_2|[\partial^\mu A_\mu^\mp(x), A_0^\pm(0)]|p_1\rangle,$$

$$D^\pm = \quad iN_1N_2 \int d^4x\, e^{iq_2\cdot x}\theta(x_0) \times \langle p_2|[\partial^\mu A_\mu^\mp(x), \partial^\nu A_\nu^\pm(0)]|p_1\rangle,$$

$$V_\nu^3 = N_1N_2\langle p_2|V_\nu^3(0)|p_1\rangle. \tag{10}$$

Equation (9) is the basic equation from which the sum rules are derived. If we restricted ourselves to forward scattering, we would obtain only one relation, the Adler-Weisberger result. However, if we consider nonforward scattering, there are two independent invariants, and hence two relations can be derived.

We decompose L, C, and D as follows:

$$L = \bar{u}(p_2)[L_1(\nu,t,q^2,q^2) + \gamma\cdot Q L_2(\nu,t,q^2,q^2)]u(p_1), \tag{11}$$

and similarly for C and D. Here $Q^\mu = \frac{1}{2}(q_1+q_2)^\mu$, $\Delta^\mu = \frac{1}{2}(q_2-q_1)^\mu$, $P^\mu = \frac{1}{2}(p_1+p_2)^\mu$ and $\nu = P\cdot Q$, $t = 4\Delta^2$, $q^2 = q_1^2 = q_2^2$.[8] Defining the isovector-nucleon form factors

$$V_\nu^3 = [F_1^V(t) + F_2^V(t)]\gamma_\nu - \frac{F_2^V(t)}{M_p}P_\nu, \tag{12}$$

we obtain the two relations

$$L_1^\pm(\nu,t,q^2,q^2) = C_1^\pm(t) + D_1^\pm(\nu,t,q^2,q^2) \mp 2\nu F_2^V(t)/M_p, \tag{13a}$$

$$L_2^\pm(\nu,t,q^2,q^2) = C_2^\pm(t) + D_2^\pm(\nu,t,q^2,q^2) \pm 2(F_1^V(t) + F_2^V(t)). \tag{13b}$$

[8] We use the metric and γ matrices of Bjorken and Drell. Also $\mathbf{Q} \equiv \gamma\cdot Q$.

The next step is to examine the functions L_1 and L_2. We expand $T_{\mu\nu}$ in a complete tensor basis, $\vartheta_{\mu\nu}$:

$$\begin{aligned} T_{\mu\nu}^\pm &= \sum_{i=1}^{17} A_i^\pm(\nu,t,q^2,q^2)\bar{u}(p_2)\vartheta_{\mu\nu}{}^i u(p_1) \\ &\quad + \sum_{i=1}^{15} B_i^\pm(\nu,t,q^2,q^2)\bar{u}(p_2)\bar{\vartheta}_{\mu\nu}{}^i u(p_1) \quad \text{(Basis I)} \\ &= \sum_{i=1}^{17} A'_i{}^\pm(\nu,t,q^2,q^2)\bar{u}(p_2)\vartheta_{\mu\nu}{}^i u(p_1) \\ &\quad + \sum_{i=1}^{15} B'_i{}^\pm(\nu,t,q^2,q^2)\bar{u}(p_2)\bar{\vartheta}_{\mu\nu}{}^i u(p_1). \quad \text{(Basis II)} \end{aligned} \tag{14}$$

The two choices of basis $\vartheta_{\mu\nu}{}^i$ are given in Table I. They differ only in the choice of the tensor $\bar{\vartheta}_{\mu\nu}{}^{13}$. Basis I refers to the choice of $\bar{\vartheta}_{\mu\nu}{}^{13} = (P_\mu\Delta_\nu - P_\nu\Delta_\mu)\mathbf{Q}$ and is the one used in Ref. 4. Basis II (the primed basis) will refer to the choice of $\bar{\vartheta}_{\mu\nu}{}^{13} = [\gamma_\nu,\mathbf{Q}]P_\mu - [\gamma_\mu,\mathbf{Q}]P_\nu$. In our subsequent discussion we will present equations for both the A_i, B_i, and the A_i', B_i'.

The choice of tensor basis is important, and misleading results can be easily obtained with an inconvenient choice. This question is discussed thoroughly in Appendix A. We have chosen bases I and II above because they are the only two bases which are entirely free of kinematic singularities.

In terms of these bases, the L's become:

$$\begin{aligned} L_1 \equiv L_1(\nu,0,0,0) &= \nu^2 A_1(\nu,0,0,0) + 4\nu A_{11}(\nu,0\,0,0) \quad \text{(Basis I)} \\ &= \nu^2 A_1'(\nu,0,0,0) + 4\nu A_{11}'(\nu,0,0,0) \\ &\quad - 4\nu^2 B_{13}'(\nu,0,0,0), \quad \text{(Basis II)} \end{aligned} \tag{15a}$$

$$\begin{aligned} L_2 \equiv L_2(\nu,0,0,0) &= \nu^2 B_1(\nu,0,0,0) - 4M_p A_{11}(\nu,0,0,0) \\ &\quad + 2\nu A_4(\nu,0,0,0) \quad \text{(Basis I)} \\ &= \nu^2 B_1'(\nu,0,0,0) - 4M_p A_{11}'(\nu,0,0,0) \\ &\quad + 2\nu A_4'(\nu,0,0,0) + 4M_p\nu B_{13}'(\nu,0,0,0), \quad \text{(Basis II)} \end{aligned} \tag{15b}$$

and hence (13) becomes (suppressing the ν and $t = q_1^2 = q_2^2 = 0$ dependence and the $\pm$ superscript)

$$\nu^2 A_1 + 4\nu A_{11} = C_1 + D_1 \mp (2\nu/M_p)F_2, \quad \text{(Basis I)} \tag{16a}$$

$$\nu^2 A_1' + 4\nu A_{11}' + 4\nu^2 B_{13}' = C_1 + D_1 \mp (2\nu/M_p)F_2, \quad \text{(Basis II)}$$

$$\nu^2 B_1 - 4M_p A_{11} + 2\nu A_4 = C_2 + D_2 \pm 2(F_1+F_2), \quad \text{(Basis I)} \tag{16b}$$

$$\nu^2 B_1' - 4M_p A_{11}' + 2\nu A_4' + 4M_p B_{13}' = C_2 + D_2 \pm 2(F_1+F_2). \quad \text{(Basis II)}$$

Following Weisberger,[7] a consideration of Eq. (16a) to zeroth order in ν yields the Adler self-consistency condition if $C_1 = 0$. Since we shall take a derivative of

Eq. (16a), we need only C_1=constant (i.e. independent of ν). We neglect C_2.[9]

Keeping the neutron and proton masses unequal $M_n \neq M_p$ it is reasonable to assume no poles in the A's and B's at $\nu=0$. We then obtain the low-energy theorems

$$4A_{11}^{\pm}(0)=\frac{\partial D_1^{\pm}}{\partial\nu}\bigg|_{\nu=0}\mp\frac{2F_2^V}{M_p}, \qquad (17a)$$

$$-4A_{11}^{\pm}(0)=\frac{D_2^{\pm}(0)}{M_p}\pm\frac{2(F_1^V+F_2^V)}{M_p}, \quad \text{(Basis V)} \qquad (17b)$$

and two similar equations, with $A_{11}(0)\rightarrow A_{11}'(0)$, for Basis II. We have suppressed the arguments $t=q^2=0$.

Finally, we state our results in a form independent of the basis

$$\frac{\partial L_1^{\pm}}{\partial\nu}=\frac{\partial D_1^{\pm}}{\partial\nu}\mp\frac{2F_2^V}{M_p}, \qquad (18a)$$

$$L_2^{\pm}=D_2^{\pm}\pm 2(F_1^V+F_2^V). \qquad (18b)$$

In the next section we will restrict our discussion to the minus amplitudes (L_1^- and L_2^-), extract the neutron pole terms, and make the identification between the D's and the πp scattering amplitudes.

III. THE POLE TERMS

Let us continue to leave $M_n\neq M_p$.[7] This will facilitate taking the limit $q_1^2=q_2^2=q^2=t=\nu=0$.

The one-neutron pole terms of D_1^-, D_2^-, and A_{11}^-, as defined in (10) and (14), are obtained by noting that

$$N_pN_n\langle p|A_\mu^+(x)|n\rangle=g_A\bar{u}_p\gamma_\mu\gamma_5u_ne^{i(p-n)\cdot x}$$
$$N_pN_n\langle p|\partial^\mu A_\mu^+(x)|n\rangle=ig_A(M_p+M_n)\bar{u}_p\gamma_5u_ne^{i(p-n)\cdot x}. \qquad (19)$$

We obtain

$$D_1^{-\text{Born}}(\nu,0,0,0)=\frac{g^2{}_A(M_n+M_p)^2(M_n-M_p)}{M_p^2-M_n^2+2\nu},$$

$$D_2^{-\text{Born}}(\nu,0,0,0)=-\frac{g_A^2(M_n^2+M_p)}{M_p^2-M_n^2+2\nu},$$

$$4A_{11}^{-\text{Born}}(\nu,0,0,0)=\frac{2g_A^2(M_n+M_p)}{M_p^2-M_n^2+2\nu}. \qquad (20)$$

It is then easy to see that the Born parts of the left- and right-hand sides of Eq. (17a) cancel, leaving

$$4\tilde{A}_{11}^-(0)=f_\pi^2\frac{\partial\tilde{A}}{\partial\nu}\bigg|_{\nu=0}+\frac{2F_2^V(0)}{M_p}, \qquad (21a)$$

where the tilde denotes the continuum contribution, and we have used PCAC (hypothesis of partially conserved axial-vector current) in the form

$$\partial^\mu A_\mu^{\pm}=f_\pi m_\pi^2\phi^{\pm};\; f_\pi\cong\frac{\sqrt{2}Mg_A}{g_{NN\pi}}, \qquad (22)$$

giving the identification

$$D^-=\frac{f_\pi^2m_\pi^4}{(m_\pi^2-q^2)^2}M^-\underset{q^2=0}{\longrightarrow}f_\pi^2M^-. \qquad (23)$$

Here M^- is the usual π^--p scattering amplitude $[M=\bar{u}(A+BQ\cdot\gamma)u]$. In Eq. (17b), the cancellation is incomplete, and the equation becomes

$$-4\tilde{A}_{11}^-(0)=\frac{g_A^2}{M_p}+f_\pi^2\frac{\tilde{B}^-(0)}{M_p}-\frac{2(F_1^V(0)+F_2^V(0))}{M_p}. \qquad (21b)$$

Similar results are obtained in the other basis.

If we add the two Eqs. (21) together, $\tilde{A}_{11}(0)$ cancels and we obtain the Adler-Weisberger relation

$$1=g_A^2+2f_\pi^2M_p\left(\frac{\partial\tilde{A}^-}{\partial\nu}+\frac{1}{M_p}\tilde{B}^-\right)_{\nu=0}. \qquad (24)$$

Assuming that $\partial\tilde{A}^-/\partial\nu$ and $\tilde{B}^-$ satisfy unsubtracted dispersion relations, and using the πp crossing relations gives

$$1=g_A^2+2f_\pi^2M_p\times\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu\frac{\text{Im}A^{(-)}(\nu,0,0,0)+(\nu/M_p)\text{Im}B^{(-)}(\nu,0,0,0)}{\nu^2}, \qquad (25)$$

where $A^{(-)}=\frac{1}{2}(A^--A^+)$.

Using the crossing relations converts Eq. (21a) into the sum rule quoted in the Introduction:

$$F_2^V(0)=-f_\pi^2M_p\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu\frac{\text{Im}A^{(-)}(\nu,0,0,0)}{\nu^2}+2M_p\tilde{A}_{11}(0,0,0,0). \qquad (1)$$

IV. RESONANCE CONTRIBUTIONS TO THE SUM RULE

If $\tilde{A}_{11}(0)=0$, Eq. (21a) becomes the low energy theorem proposed by Raman[3] and Eq. (21b) is related to the consistency condition proposed by Fuchs.[2] Raman's equation was obtained by using a set of invariants which essentially substituted $L_\mu=\epsilon_{\mu\nu\rho\sigma}P^\nu Q^\rho\Delta^\sigma$ for γ_μ as an independent vector. We show in Appendix A that the set proposed by Raman has kinematic singularities at $t=0$, so as to make his limits at $t=0$ invalid. In fact, any basis that eliminates $\vartheta_{\mu\nu}$[11] produces kinematic singularities in the associated invariant amplitudes, and hence it is not possible to eliminate the explicit appearance of the weak amplitude in Eq. (1) by a different choice of basis.

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[9] One argument for this is that C seems to have the structure of a scalar form factor, and hence contains no **Q** term.

To test Eq. (1) we need some model to evaluate $\tilde{A}_{11}(0)$. We have two methods at our disposal. We could treat the known π-N resonances as elementary particles, and calculate their contributions using perturbation theoretic Feynman rules. This approach leads to numerous difficulties, not the least of which is the ambiguity in the definition of the off-mass-shell spin $\frac{3}{2}$ propagator which is arbitrary up to factors proportional to $M^{*2}-s$. These factors are not negligible.

We therefore follow the second procedure, which is to assume that $\tilde{A}_{11}$ satisfies an unsubtracted dispersion relation. We could not have assumed that $\partial\tilde{L}_1/\partial\nu$ satisfies an unsubtracted dispersion relation, since examination of the sum rule (18a) shows that

$$\frac{\partial\tilde{L}_1}{\partial\nu}\xrightarrow[\nu\to\infty]{} 2F_2/M_p\neq 0. \qquad (26)$$

(We assume that $\partial\tilde{D}_1/\partial\nu=f_\pi^2\ \partial\tilde{A}/\partial\nu\to 0$ as $\nu\to\infty$.) However, since

$$\partial\tilde{L}_1/\partial\nu=4\tilde{A}_{11}+2\nu\tilde{A}_1, \qquad (27)$$

it is possible to assume that the $\tilde{A}_i$ (and in particular $\tilde{A}_{11}$ or $\tilde{A}_{11}'$) are unsubtracted. From this viewpoint, the decomposition of $T_{\mu\nu}$ into a covariant basis is a device for obtaining unsubtracted amplitudes.

An evaluation of $2M\tilde{A}_{11}(0)$ [or $2M\tilde{A}_{11}'(0)$] via a pole model dispersion approach shows immediately that this term is comparable to $\frac{1}{2}f_\pi^2\ M(\partial\tilde{A}/\partial\nu)$. Numerical integration of the Roper $P_{33}(1236)$ phase shift to $\sim$700 MeV gives

$$-\tfrac{1}{2}f_\pi^2 M\frac{\partial\tilde{A}^-}{\partial\nu}\bigg|_{\nu=0}=+2.05.$$

Calculation of $2\tilde{A}_{11}(0)$ using the dispersion pole model with PCAC described below gives for the same resonance

$$2M\tilde{A}_{11}(0)=-1.39.$$

The contributions from the $P_{11}(1400)$, $D_{13}(1525)$, $F_{15}(1688)$, and $F_{37}(1920)$ are small. Because of this large cancellation in Eq. (1), there is little merit in retaining the separation of the model-dependent term $2M\tilde{A}_{11}(0)$ and the experimental term $\frac{1}{2}f_\pi^2\ M\partial\tilde{A}^-/\partial\nu$. Instead, we cast our sum rule into the original Fubini[10] form by the following procedure:

Eqs. (16) hold for all ν. In the limit $\nu\to\infty$, Eq. (16a) becomes

$$\lim_{\nu\to\infty}\nu A_1(\nu,0,0,0)=2F_2^V(0)/M, \qquad (28)$$

with the assumption $A_{11}\leqslant C\nu^{-1}$ as $\nu\to\infty$. Similarly, if $A_{11}'\leq C'\nu^{-1}$ as $\nu\to\infty$, then

$$\lim_{\nu\to\infty}\nu(A_1'+B_{13}')=2F_2^V(0)/M. \qquad (29)$$

[10] S. Fubini, Nuovo Cimento 43A, 475 (1966).

However, the $\mu\nu$-symmetric covariants are common to both bases. Hence, $A_1=A_1'$, and Eqs. (28) and (29) can both be true only if $\lim \nu B_{13}'=0$, i.e., B_{13}' is superconvergent.[11] If this is so, both equations reduce to

$$-\frac{M}{\pi}\int_{\nu_0}^{\infty} a_1^{(-)}(\nu,0,0,0)d\nu=F_2^V(0). \qquad (30)$$

We have made use of the crossing relation

$$a_1^-(\nu)=-a_1^+(-\nu) \qquad (31)$$

and the definition

$$a_1^{(-)}(\nu)=\tfrac{1}{2}[a_1^-(\nu)-a_1^+(\nu)]. \qquad (32)$$

Lower case letters denote the absorptive parts of the corresponding upper case amplitudes.

The form (30) of the sum rule for F_2^V suffers from the disadvantage that direct reference to experimental π-p data has been eliminated, so that evaluation of the sum rule is completely model-dependent. We wish to emphasize that this appears to be the essential nature of the sum rule for F_2^V; the model-dependent term $\tilde{A}_{11}(0)$ is so large, that a model must inevitably play a central role in any examination of the sum rule. The advantage of (30) is that the two terms $2M\tilde{A}_{11}(0)$ and $\frac{1}{2}f_\pi^2 M\partial\tilde{A}^-/\partial\nu$ are treated in the same fashion, and hence inconsistencies are avoided. To see this, observe that (30) can be obtained if one calculates the absorptive part of A from Eq. (16a)

$$f_\pi^2\tilde{a}(\nu,0,0,0)=\nu^2\tilde{a}_1(\nu,0,0,0)+4\nu\tilde{a}_{11}(\nu,0,0,0) \qquad (33)$$

assumes unsubtracted dispersion relations for $\tilde{A}_{11}$ and $\partial\tilde{A}/\partial\nu$, and substitutes (33) into (21a).

We have evaluated the contributions to $a_1^{(-)}$ for the $\Delta=P_{33}(1236)$ and $D=D_{13}(1525)$. We use the spin-$\frac{3}{2}$ projection operator

$$P_{\mu\nu}(K)=\frac{(\mathbf{K}+M^*)}{2M^*}\bigg[-g_{\mu\nu}+\tfrac{1}{3}\gamma_\mu\gamma_\nu+\frac{1}{3M^*}(\gamma_\mu K_\nu-\gamma_\nu K_\mu)+\frac{2}{3M^{*2}}K_\mu K_\nu\bigg], \qquad (34)$$

where $K=p_1+q_1=P+Q$ is the 4 momentum of the Δ, and the axial-vector–nucleon–Δ coupling

$$N_\Delta N_p\langle\Delta^0|A_\mu^-|p\rangle=(\sqrt{\tfrac{1}{3}})g_{33}\bar{u}_\mu u_p. \qquad (35)$$

The result for Δ is

$$a_1^{(-)}(\nu,0,0,0)=-\frac{\pi}{9}g_{33}^2\frac{M_{33}+M}{M_{33}^2}\delta\left(\frac{M_{33}^2-M^2}{2}-\nu\right), \qquad (36)$$

[11] The superconvergence of B_{13}' may be obtained in another way by integrating $q_2^\mu T_{\mu\nu}$ by parts and making the unsubtracted assumption for the invariant amplitude which multiplies $[\gamma_\mu,Q]$ in the tensor decomposition of the amplitude $W_\mu^-+p\to\pi^-+p$.

which contributes

$$-\frac{M}{\pi}\int_{\nu_0}^{\infty} a_1^{(-)}(\nu,0,0,0)d\nu = \frac{M}{9}\frac{(M_{33}+M)}{M_{33}^2}g_{33}^2. \quad (37)$$

For the D_{13} we take the interaction to be

$$N_D N_p \langle D|A_\mu^-|p\rangle = ig_{13}\bar{u}_\mu\gamma^5 u, \quad (38)$$

and obtain

$$a_1^{(-)}(\nu,0,0,0) = -\tfrac{1}{6}\pi g_{13}^2 \frac{M_{13}-M}{M_{13}^2}\delta\left(\frac{M_{13}^2-M^2}{2}-\nu\right). \quad (39)$$

This contributes

$$-\frac{M}{\pi}\int_{\nu_0}^{\infty} a_1^{(-)}(\nu,0,0,0)d\nu = \tfrac{1}{6}M\frac{(M_{13}-M)}{M_{13}^2}g_{13}^2. \quad (40)$$

The coupling constant g_{33} is estimated by two different methods in Appendix B. The first involves applying PCAC to the coupling (36), which is then related to the width of the N^*. Using this method we obtain

$$g_{33}^2 \cong 2.0 \quad (\text{B.6})$$

The second method requires that a dispersion relation pole model of the N^* reproduce the P_{33} contribution to $A^{(-)}$ which is obtained by direct integration over the phase shifts. This gives

$$g_{33}^2 \cong 2.69 \quad (\text{B.9})$$

This uncertainty in g_{33} is probably primarily a reflection of the unreliability of the simple pole model. The values of g_{33}^2 obtained here are similar to those obtained by Schnitzer,[12] who obtained g_{33} by fitting a pole to the Adler-Weisberger relation. In a similar manner, g_{13}^2 can be estimated

$$g_{13}^2 = 3.2. \quad (\text{B.6}')$$

Combining these results (using $g_{33}^2=2.0$) gives us finally

$$F_2^V(0) \cong 0.24+0.13 = 0.37. \quad (41)$$

In addition to this, there will be contributions from the $F_{15}(1688)$ and the $F_{37}(1920)$. It turns out that the $P_{11}(1400)$ will *not* contribute to $a_1^{(-)}$. Although the final results should be somewhat larger than (41), it does not appear that the sum rule is anywhere near satisfied (recall that $F_2^V=1.85$!).

Before we conclude this section, we examine the validity of the superconvergence condition on $\tilde{B}_{13}'$ mentioned above. The contributions to $\tilde{B}_{13}'$ come from the Δ and D. We have

$$b'_{13}{}^{(-)}(\nu,0,0,0) = +\frac{\pi g_{33}^2}{36M_{33}}\delta\left(\frac{M_{33}^2-M^2}{2}-\nu\right) \quad \text{for } \Delta,$$

$$b'_{13}(\nu,0,0,0) = -\frac{\pi g_{13}^2}{24M_{13}}\delta\left(\frac{M_{13}^2-M^2}{2}-\nu\right) \quad \text{for } D_{13}. \quad (42)$$

[12] H. J. Schnitzer, Phys. Rev. **158**, 1471 (1967).

Hence we obtain

$$\frac{M}{\pi}\int_{\nu_0}^{\infty} b'_{13}{}^{(-)}(\nu,0,0,0)d\nu = \frac{M}{24}\left(\tfrac{2}{3}\frac{g_{33}^2}{M_{33}}-\frac{g_{13}^2}{M_{13}}\right)$$

$$\cong (1/24)(1.01-1.94) = -0.04 \quad (43)$$

The model does not quite satisfy the superconvergence condition, but the numerical significance of the extra terms contributed by b_{13}' to Eq. (28) is negligible.

V. HIGH-ENERGY CONTRIBUTIONS TO THE SUM RULE

From the analysis of the last section we are forced to one of two conclusions. (i) Either the sum rule (1) or (30) is invalid or (ii) there exist large continuum contributions to $a_1^{(-)}$ in Eq. (30). In this section we examine the latter possibility.

To estimate the high-ν behavior of $a_1^{(-)}$, we rewrite Eq. (33)

$$a_1^{(-)}(\nu,0,0,0) = f_\pi^2\frac{a^{(-)}(\nu,0,0,0)}{\nu^2} - 4\frac{a_{11}^{(-)}(\nu,0,0,0)}{\nu}. \quad (44)$$

If we assume (i) that $A_{11}^{(-)}(\nu,0,0,0)\to 0$ at least as fast as ν^{-1} as $\nu\to\infty$ and (as we did in the last section) (ii) that the asymptotic behavior of $A^{(-)}(\nu,0,0,0)$ is dominated by a ρ Regge pole, then the first term on the right-hand side of Eq. (44) will dominate as $\nu\to\infty$. The Regge form for $A^{(-)}(\nu,t,0,0)$ is[13]

$$A^{(-)}(\nu,t,0,0) \approx \beta(t)\frac{1-e^{-i\pi\alpha_\rho(t)}}{\sin\pi\alpha_\rho(t)}\left(\frac{\nu}{\nu_0}\right)^{\alpha_\rho(t)}, \quad (45)$$

where $\alpha_\rho(t)$ is the ρ Regge trajectory, ν_0 is a scale factor chosen to be $M\times(1\text{ GeV})$, and $\beta(t)$ is a residue function determined by fitting π^-p charge exchange scattering.

With these two assumptions, Eq. (44) becomes

$$a_1^{(-)}(\nu,0,0,0) \underset{\nu\to\infty}{\approx} f_\pi^2\frac{\beta(0)}{\nu_0^2}\left(\frac{\nu}{\nu_0}\right)^{\alpha_\rho(0)-2}. \quad (46)$$

Substituting into (30), we obtain

$$-\frac{M}{\pi}\int_{\nu_l}^{\infty} a_1^{(-)}(\nu,0,0,0)d\nu = -\frac{Mf_\pi^2\beta(0)}{\pi\nu_0(1-\alpha_\rho(0))}\left(\frac{\nu_0}{\nu_l}\right)^{1-\alpha_\rho(0)}, \quad (47)$$

where ν_l is some lower bound on the integration. The recent fit of Arbab and Chiu[14] to π^-p charge exchange

[13] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962). We neglect any effects of the extrapolation of the pion mass to zero.

[14] F. Arbab and C. B. Chiu, Phys. Rev. **147**, 1045 (1966). We have neglected the negligible exponential t dependence of $\beta(t)$ used by these authors.

scattering gives $\alpha_\rho(0)=0.56$ and

$$\beta(t)=-\frac{1}{\sqrt{2}}[5.9(\alpha_\rho(t)+1)+99.2\alpha_\rho(t)(\alpha_\rho(t)+1)] \quad \text{GeV}^{-1}, \qquad (48)$$

so that $\beta(0)=-54.7$ GeV^{-1}. Making the most optimistic choice $\nu_l=\nu_0$, the high-energy continuum contribution to $F_2{}^V(0)$ becomes 0.72. The sum rule so far then reads

$$F_2{}^V=0.37+0.72=1.09. \qquad (49)$$

The sum rule (30) is still far from satisfied. However, the size of the high-energy continuum contribution indicates that detailed knowledge of the higher-resonance contributions and low- and high-energy continuum will be necessary before the validity of the sum rule can be judged.

The sign of the residue β used in Eq. (47) was not determined by the fit given Arbab and Chiu.[14] We have so far chosen $\beta<0$ so as to give a positive contribution to $F_2{}^V(0)$. We now justify this choice.

The expression (45) for $A^{(-)}$ may be expanded near $t=m_\rho{}^2$ as follows:

$$A^{(-)}(\nu,t,0,0) \underset{t\to m_\rho^2,\nu\to\infty}{\approx} -\frac{2\beta(m_\rho{}^2)}{\pi\alpha_\rho'(m_\rho{}^2)\nu_0}\frac{\nu}{t-m_\rho{}^2}, \qquad (50)$$

where we have used the fact $\alpha_\rho(m_\rho{}^2)=1$.

However, the usual expression for the ρ pole, supplemented by universality[15] gives

$$A^{(-)}(\nu,t,0,0) \underset{t\to m_\rho^2,\nu\to\infty}{\approx} \frac{2F_2'(0)f_\rho{}^2}{M}\frac{\nu}{t-m_\rho{}^2}, \qquad (51)$$

where f_ρ is the universal constant describing the coupling of the ρ to the isospin current. Since $\alpha_\rho'(m_\rho{}^2)>0$, we see that $\beta(m_\rho{}^2)<0$. This agrees with the assignment of signs made in Eq. (48).

It is interesting that there is also good numerical agreement between $\beta(m_\rho{}^2)$ obtained from (48) and the value obtained by equating Eqs. (50) and (51). With the choice

$$\alpha_\rho'(m_\rho{}^2)=\frac{\alpha_\rho(m_\rho{}^2)-\alpha_\rho(0)}{m_\rho{}^2}=\frac{0.44}{m_\rho{}^2}, \qquad (52)$$

and $f_\rho{}^2/4\pi\simeq2.4$,[16] we obtain from (50) and (51)

$$\beta(m_\rho{}^2)=-F_2{}^V(0)f_\rho{}^2\pi\alpha_\rho'(m_\rho{}^2)(\nu_0/M)=-132 \text{ GeV}^{-1}. \qquad (53)$$

From (48)

$$\beta(m_\rho{}^2)=-149 \text{ GeV}^{-1}. \qquad (54)$$

[15] J. J. Sakurai, Ann. Phys. (N.Y.) **11**, 1 (1960); M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

[16] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).

VI. SUMMARY AND CONCLUSIONS

Our paper may be summarized as follows:

(i) When the derivation of the Adler-Weisberger relation is extended to the nonforward direction (by means of a tensor decomposition), Eqs. (13a and 13b) involving $F_2{}^V(t)$ and $F_1{}^V(t)+F_2{}^V(t)$ are obtained. We have examined one of these equations in the limit $t=q_1{}^2=q_2{}^2=0$ (Eq. 1). A separate evaluation of these two sum rules requires a knowledge of one of the weak amplitudes A_{11} at $\nu=0$. This observation is contrary to the results of previous work by other authors.[2,3] When $\tilde{A}_{11}(0)$ is eliminated from the two sum rules, the Adler-Weisberger relation is obtained.

(ii) One's first inclination on obtaining sum rule (1) is to hope that $2M\tilde{A}_{11}(0)$ is a small correction to a sum rule relating $F_2{}^V(0)$ to a πN scattering amplitude. However, this is not the case, as we showed in Sec. IV. Because of the near equality of $2M\tilde{A}_{11}(0)$ and the πN term, it is important then to make use of the constraint in Eq. (33), and reexpress the right-hand side of Eq. (1) in terms of the absorptive part $a_1{}^{(-)}$. This gave us Eq. (30).

(iii) An evaluation of the integral in Eq. (30) by means of a dispersion pole model for the resonance contributions gave the result $F_2{}^V(0)=0.37$ (Eq. 41). The smallness of this result led to a consideration of the high-energy contributions.

(iv) Using a ρ Regge-pole fit for $\pi^-p\to\pi^0n$ scattering and assuming that $A_{11}\leq O(\nu^{-1})$ as $\nu\to\infty$, we could estimate the high-energy contribution to $a_1{}^{(-)}$. This, combined with the resonance contributions, gave $F_2{}^V(0)\simeq1$.

We make two final remarks.

(i) Gilman and Schnitzer[17] have emphasized recently that neglect of low-energy continuum contributions will often lead to large errors in the evaluation of sum rules; it is known that the P_{33} contribution to the Cabibbo-Radicati sum rule is even of the "wrong" sign, and the sum rule is satisfied only when continuum contributions are included. This investigation suggests that, in certain cases, high-energy continuum contributions may also be important.

(ii) Schnitzer[12] has recently determined π-N scattering lengths using a related approach. He obtains the four p-wave scattering lengths (see his Tables I and II). Two of these do not depend on the current commutation relations, and the good agreement he obtains is a reflection (in our language) of the near equality of $2M\tilde{A}_{11}$ and $\frac{1}{2}f_\pi{}^2M\partial\tilde{A}/\partial\nu$. Of the other two, only one depends significantly on the current algebras, and its agreement is less convincing .This difference in approach partially reconciles the contrary conclusions we obtain.

Note added in proof. We also learned after this work was completed that K. Raman has published a corrected

[17] F. Gilman and H. J. Schnitzer, Phys. Rev. **150**, 1362 (1966).

and expanded version of his Ref. 3 [Phys. Rev. **159**, 1501 (1967)].

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APPENDIX A. CONSTRUCTION OF A TENSOR BASIS FREE OF KINEMATIC SINGULARITIES

In this Appendix we sketch the arguments that show that the tensor bases presented in Table I are the only ones free of kinematic singularities. To show this it is sufficient to prove that *any* nonsingular second-rank tensor $\mathfrak{A}_{\mu\nu}$ can be expanded

$$\mathfrak{A}_{\mu\nu}=\sum_{i=1}^{17} a_i\vartheta_{\mu\nu}{}^i+\sum_{i=1}^{15} \bar{a}_i\bar{\vartheta}_{\mu\nu}{}^i, \tag{A.1}$$

where none of the a_i or $\bar{a}_i$ have singularities in ν, t, or q^2 for any values of ν, t or q^2. We must show this for both bases.

To facilitate the discussion we define $L_\mu=\epsilon_{\mu\nu\lambda\sigma}P^\nu Q^\lambda\Delta^\sigma$, and construct the 15 tensors

$$\begin{aligned}&(P_\mu L_\nu\pm P_\nu L_\mu)i\gamma^5, && (P_\mu L_\nu\pm P_\nu L_\mu)i\mathbf{Q}\gamma^5,\\ &(Q_\mu L_\nu\pm Q_\nu L_\mu)i\gamma^5, && (Q_\mu L_\nu\pm Q_\nu L_\mu)i\mathbf{Q}\gamma^5,\\ &(\Delta_\mu L_\nu\pm \Delta_\nu L_\mu)i\gamma^5, && (\Delta_\mu L_\nu\pm \Delta_\nu L_\mu)i\mathbf{Q}\gamma^5,\\ &L_\mu L_\nu,\quad L_\mu L_\nu\mathbf{Q},\quad \text{and} && (Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)\mathbf{Q}.\end{aligned} \tag{A.2}$$

It is understood that these are to be sandwiched between nucleon spinors $\bar{u}(p_2)$ and $u(p_1)$.

It seems clear that the basis containing all of the 33 ϑ's in Table I, plus the 15 tensors introduced above is sufficiently large to be free of kinematic singularities. This basis contains 48 tensors.

There are of course many more nonsingular tensors which can be constructed, but by successive use of the Dirac equation these can be reduced to one of the 48 given above without introducing extra singularities. As an example[18]

$$\bar{u}(p_2)\mathbf{Q}\mathbf{L}u(p_1)=\bar{u}(p_2)i\gamma^5[-\nu^2+Q^2P^2]u(p_1), \tag{A.3}$$

where we have used the identity

$$\gamma^\mu\gamma^\nu\gamma^\lambda=g^{\mu\nu}\gamma^\lambda+g^{\nu\lambda}\gamma^\mu-g^{\mu\lambda}\gamma^\nu-i\gamma^5\epsilon^{\mu\nu\lambda\rho}\gamma_\rho, \tag{A.4}$$

with $\epsilon^{0123}=+1$.

[18] We use the Lorentz metric (1, −1, −1, −1) and the Dirac matrices with γ^0 Hermitian, $\gamma^i(i=1, 2, 3)$ anti-Hermitian and $\gamma^5(\equiv i\gamma^0\gamma^1\gamma^2\gamma^3)$ Hermitian.

The basis we have considered so far is over complete. There are only 32 independent tensors, so the next step is to eliminate 16 of the 48 in such a way that no poles are introduced.

The tensors involving L_μ can be eliminated by means of equations like the following (with $d=P^2Q^2-\nu^2$):

$$\begin{aligned}4L_\mu L_\nu&=(td)g_{\mu\nu}-(tQ^2)P_\mu P_\nu-(tP^2)Q_\mu Q_\nu\\&\quad+(t\nu)(P_\mu Q_\nu+Q_\mu P_\nu)-(tQ^2)\Delta_\mu\Delta_\nu,\\(P_\mu L_\nu-P_\nu L_\mu)i\gamma^5&=(\tfrac{1}{8}t)\{[\gamma_\nu,\mathbf{Q}]P_\mu-[\gamma_\mu,\mathbf{Q}]P_\nu\}\\&\quad-(\nu)(P_\mu\Delta_\nu-P_\nu\Delta_\mu)+M(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q},\\(P_\mu L_\nu-P_\nu L_\mu)i\mathbf{Q}\gamma^5&=d(P_\mu\gamma_\nu-P_\nu\gamma_\mu)\\&\quad-(P^2)(P_\mu Q_\nu-P_\nu Q_\mu)\mathbf{Q}\\&\quad+(M\nu)(P_\mu Q_\nu-P_\nu Q_\mu).\end{aligned} \tag{A.5}$$

Similarly, none of the other relations which eliminate the L_μ tensors will introduce extra poles.

A basis which eliminates γ_μ in favor of L_μ is *not* free of kinematic poles. As a relevant example, we express the tensor $\vartheta_{\mu\nu}{}^{11}\equiv[\gamma_\mu,\gamma_\nu]$ in terms of the "L_μ" basis.

$$\begin{aligned}[\gamma_\mu,\gamma_\nu]&=A(P_\mu L_\nu-P_\nu L_\mu)i\gamma^5+B(P_\mu L_\nu-P_\nu L_\mu)i\mathbf{Q}\gamma^5\\&\quad+C(Q_\mu L_\nu-Q_\nu L_\mu)i\gamma^5+D(Q_\mu L_\nu-Q_\nu L_\mu)i\mathbf{Q}\gamma^5\\&\quad+E(\Delta_\mu L_\nu-\Delta_\nu L_\mu)i\mathbf{Q}\gamma^5+F(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\\&\quad+G(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q}+H(Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)\\&\quad+I(Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)\mathbf{Q},\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}&A=8\nu/td, && F=8(\nu^2+2t)/td,\\&B=-2MQ^2/d^2, && G=-8M\nu/td,\\&C=-8P^2/td, && H=-(8\nu P^2+2t\nu)/td,\\&D=2M\nu/d^2, && I=8MP^2/td,\\&E=-8M/td, && d=P^2Q^2-\nu^2.\end{aligned} \tag{A.7}$$

This tensor $[\gamma_\mu,\gamma_\nu]$ appears explicitly in the decomposition of the nucleon, $P_{33}(1236)$ and the other baryon pole contributions to $T_{\mu\nu}$. Furthermore, these contributions do not vanish at $t=0$, so that their reexpression in terms of the "L_μ" basis via Eq. (A.6) will introduce poles at $t=0$. These poles give a finite contribution in the limit $t=0$ to

$$q_2{}^\mu[F(P_\mu\Delta_\nu-P_\nu\Delta_\mu)+G(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q}]q_1{}^\nu, \tag{A.8}$$

which is precisely equal to $q_2{}^\mu[\gamma_\mu,\gamma_\nu]q_1{}^\nu=4\nu-4M\mathbf{Q}$. (The equalities hold between nucleon spinors $\bar{u}(p_2)$, $u(p_1)$.)

If one had neglected the singularities in F and G, the expression (A.8) would vanish at $t=0$.

After the 14 terms involving L_μ have been eliminated by expressions like those given in (A.5), two conditions can be found which relate the remaining 34 invariants. These are

$$\begin{aligned}(Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)\mathbf{Q}&=Z_1[\gamma_\mu,\gamma_\nu]+Z_2[\gamma_\mu,\gamma_\nu]\mathbf{Q}\\&\quad+Z_3(P_\mu\Delta_\nu-P_\nu\Delta_\mu)+Z_4(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q}\\&\quad+Z_5([\gamma_\nu,\mathbf{Q}]Q_\mu-[\gamma_\mu,\mathbf{Q}]Q_\nu)+Z_6(Q_\mu\Delta_\nu-Q_\nu\Delta_\mu)\\&\quad+Z_7(Q_\mu\gamma_\nu-Q_\nu\gamma_\mu)+Z_8(\Delta_\mu\gamma_\nu-\Delta_\nu\gamma_\mu),\end{aligned} \tag{A.9}$$

and

$$[\gamma_\nu,\mathbf{Q}]P_\mu-[\gamma_\mu,\mathbf{Q}]P_\nu=Z[\gamma_\mu,\gamma_\nu]+Z_{10}[\gamma_\mu,\gamma_\nu]\mathbf{Q}+Z_{11}(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q}+Z_{12}(Q_\mu\gamma_\nu-Q_\nu\gamma_\mu)+Z_{13}(\Delta_\mu\gamma_\nu-\Delta_\nu\gamma_\mu),\quad(\text{A.10})$$

where

$$\begin{aligned}
&Z_1=-tQ^2/8M, && Z_7=\nu t/4M^2,\\
&Z_2=\tfrac{1}{2}\nu\left(1+\frac{\nu}{M^2}\right), && Z_8=-(Q^2-\nu^2/M^2),\\
& && Z_9=-\nu,\\
&Z_3=-Q^2/M, && Z_{10}=P^2/M^2,\\
&Z_4=\nu/M^2, && Z_{11}=-2/M,\\
&Z_5=-t/8M, && Z_{12}=2P^2/M,\\
&Z_6=\nu/M, && Z_{13}=-2\nu/M.\quad(\text{A.11})
\end{aligned}$$

Both of these expansions are free of kinematic poles, and the only other possible choice is to invert the second equation into an expansion for $(P_\mu\Delta_\nu-P_\nu\Delta_\mu)\mathbf{Q}$ by dividing by Z_{11}.

APPENDIX B. EVALUATION OF g_{33}^2 AND g_{13}^2

Applying PCAC to the matrix element

$$N_\Delta N_p\langle\Delta^{++}|A_\mu^+|p\rangle=g_{33}\bar{u}_\mu u,\quad(\text{B.1})$$

we obtain

$$g_{33}=f_\pi f_{33}/m_\pi,\quad(\text{B.2})$$

where f_π is the charged pion decay constant $(=0.935m_\pi)$ and f_{33} is defined through the effective Lagrangian

$$\mathcal{L}_{\text{eff}}=\frac{f_{33}}{m_\pi}\bar{\psi}_\mu{}^{++}\psi_p\partial^\mu\phi^++\text{H.c.}\quad(\text{B.3})$$

describing the decay of the doubly charged P_{33}.

By standard methods (see, e.g., Brudnoy[19]), the full width of the P_{33} is given in terms of f_{33} by

$$\Gamma_{\text{el}}=\frac{f_{33}^2}{4\pi}\frac{2}{3}\frac{E_N{}^*+M}{m_\pi{}^2M_{33}}k^{*3},\quad(\text{B.4})$$

where $E_N{}^*$ = energy of decay proton in the c.m. system, k^* = decay momentum in the c.m. system, M_{33} = mass of $P_{33}(1236)$, and M = mass of proton.

Using $\Gamma=120$ MeV, we obtain from (B.4)

$$f_{33}^2=2.3,\quad(\text{B.5})$$

and

yielding

$$g_{33}^2=2.0.\quad(\text{B.6})$$

Owing to the negative parity, the analog of Eqs. (B.1), (B.2), (B.3), and (B.4) are

$$N_DN_p\langle D_{13}|A_\mu^-|p\rangle=g_{13}\bar{u}_\mu\gamma_5u,\quad(\text{B.1}')$$

$$g_{13}=f_\pi f_{13}/m_\pi,\quad(\text{B.2}')$$

$$\mathcal{L}_{\text{eff}}=\frac{f_{13}}{m_\pi}\bar{\psi}_\mu\gamma_5\psi_p\partial^\mu\phi^++\text{H.c.},\quad(\text{B.3}')$$

$$\Gamma_{\text{el}}=\frac{f_{13}^2}{4\pi}\frac{2}{3}\frac{E_N{}^*-M}{m_\pi{}^2M_{13}}k^{*3}.\quad(\text{B.4}')$$

Taking $\Gamma_{\text{el}}=0.65\,\Gamma_{\text{tot}}=68$ MeV[20] we find

$$f_{13}^2=3.64,\quad(\text{B.5}')$$

and from (B.2′)

$$g_{13}^2=3.2.\quad(\text{B.6}')$$

We can also evaluate f_{33}^2 by requiring that the N_{33}^* pole model give the experimental P_{33} contribution to $\frac{1}{2}\partial\bar{A}^{(-)}/\partial\nu|_{\nu=0}$

$$\frac{1}{2}\frac{\partial\bar{A}^{(-)}}{\partial\nu}\bigg|_{\nu=0}=\frac{1}{\pi}\int_{\nu_0}^\infty\frac{\bar{a}^{(-)}(\nu,0,0,0)d\nu}{\nu^2}=-0.35m_\pi{}^{-3}.\quad(\text{B.7})$$

Using Eq. (B.3),

$$\bar{a}^{(-)}=-\frac{f_{33}^2}{m_\pi{}^2M}\frac{\pi}{9\nu}\left\{(M+M_{33})\left(1+\frac{\nu}{M_{33}{}^2}\right)-\frac{\nu}{M_{33}}\right\}\times\delta\left(\nu-\frac{M_{33}{}^2-M^2}{2}\right).$$

Hence

$$\frac{1}{\pi}\int_{\nu_0}^\infty\frac{\bar{a}^{(-)}(\nu,0,0,0)d\nu}{\nu^2}=-\frac{2}{9}\frac{f_{33}^2}{m_\pi{}^2M}\left\{\frac{M}{M_{33}-M}+\frac{M^2}{2M_{33}{}^2}\right\}=-0.113f_{33}{}^2m_\pi{}^{-3}.\quad(\text{B.8})$$

Then $f_{33}^2=3.1$ and, from (B.2)

$$g_{33}^2=2.69.\quad(\text{B.9})$$

[19] D. Brudnoy, Phys. Rev. 145, 1229 (1966).

[20] Parameters for the resonances were obtained from A. H. Rosenfeld *et al.*, Rev. Mod. Phys. 39, 1 (1967).