Testing General Relativity with Laser Ranging to the Moon

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This paper assesses some possibilities inherent in precise laser ranging to the moon for testing Einstein's theory of gravitation. The anticipated accuracy of the determination of the light transit time for a laser pulse returned by an optical corner reflector on the lunar surface is about two parts in 10^{10}. Such high precision opens the possibility of detecting general relativistic effects in both the light propagation itself and in the lunar motion. The detailed analysis presented here indicates that, although the effects on light propagation are probably not detectable, there are general relativistic effects in the lunar motion which appear to be observationally accessible with the expected laser-ranging data. Observation of the dominant effect would provide a significant test of the correctness of the geodesic equation, to $O(1/c^2)$ beyond the Newtonian approximation, for describing the motion of bodies in a gravitational field.

I. INTRODUCTION

THE aim of this paper is an assessment of some possibilities inherent in precise laser ranging to the moon for testing Einstein's general theory of relativity. With present techniques, laser ranging from an earth-based station to an optical corner reflector located on the lunar surface is capable of determining a transit time to an accuracy of somewhat better than a nanosecond, or about 2 parts in 10^{10}. This would correspond to a precision of some 10 cm in distance if the speed of light were well enough known.[1] Although general relativistic effects are notoriously small, this unprecedented accuracy opens the possibility of detecting such effects both in the light propagation itself and in the lunar motion. To gain the full benefit of the anticipated two parts in 10^{10} precision one need only structure the confrontation between theory and observation in such a way that the uncertainty in the local speed of light is unimportant.

The idea of using the moon to test relativity theory is an old one. In a paper published in 1916, the same year in which Einstein's definitive paper appeared, de Sitter[2] calculated the expected general relativistic contributions to the secular motion of the lunar perigee and node. They are of the order of two seconds of arc per century. The difference between the observed values and the Newtonian theoretical values calculated by Brown, as well as the uncertainties in both these quantities, were of the same order of magnitude. No test was possible.

Following the theoretical discovery in 1918 of the Lense-Thirring effect, namely that the rotation of a massive body makes a characteristic contribution to the gravitational field. Einstein and Thirring apparently[3] considered the possibility of detecting this effect in the lunar motion. They concluded that the contributions to the secular motions were too small to be detected, at least over any reasonable time span.

The theoretical situation appears to have remained largely where de Sitter left it until 1958, when Brumberg[4] published a calculation of relativistic corrections to the lunar motion based on the Hill-Brown method. The relativistic corrections to the secular motion of the node and perigee were carried to a higher order of approximation than in de Sitter's work, but Brumberg concluded that the then-contemporary observational accuracy did not permit a test of the theory. More recently Eckert[5] has made a detailed comparison of theory and observation for the lunar perigee and node. An inference to be drawn from his work is that these secular motions are indeed unlikely to provide a test of general relativity in the near future.

The scope and content of this paper are most easily summarized in a section-by-section fashion. In Sec. II we outline the derivation of the approximate metric for the earth-moon-sun system as an expansion in powers of $1/c$. The calculation is done within the context of Einstein's theory (without the cosmological constant), but we assign labels (to be regarded as having the numerical value 1) to various terms in the final metric. These coefficients, introduced in the spirit of Eddington's analysis, serve as markers throughout the remaining calculations and indicate how the various parts of the metric contribute to the final results. Thus they give a partial answer to the necessary question: What part of the total theoretical package could a particular observation test?

This prepares the ground for Sec. III, in which we calculate the round-trip proper time for the laser pulse through relativistic corrections of order $1/c^2$. This expression is then analyzed for those portions which may be expected to show a periodic behavior dependent on the relative positions and velocities of the earth, moon, and sun. This is essential if one hopes to separate relativistic effects from uncertainties in parameters such

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[1] J. Faller (private communication); C. O. Alley, P. L. Bender, R. H. Dicke, J. E. Faller, P. A. Franken, H. H. Plotkin, and D. T. Wilkinson, J. Geophys. Res. 70, 2267 (1965).

[2] W. de Sitter, Monthly Notices Roy. Astron. Soc. 77, 155 (1916).

[3] V. L. Ginzburg, Sci. Am. 200, 149 (1959).

[4] V. A. Brumberg, Bull. Inst. Theoret. Astron. (U.S.S.R.) 6, 733 (1958).

[5] W. J. Eckert, Astron. J. 70, 787 (1965).

as the masses, the orbital elements, and the local speed of light in vacuum. We find that when the labeling coefficients are explicitly given their values from the Einstein theory, namely unity, the theoretical expression indicates that there is no realistic expectation of detecting gravitational effects on the light propagation itself with the laser-ranging data. At this point there remains open, however, the possibility of observationally significant relativistic effects in the lunar motion.

Hence we turn in Sec. IV to a detailed calculation of the earth-moon separation. In Sec. IV A we describe the procedure employed to set up the equations of motion for the relative earth-moon separation in a Lagrangian form with relativistic corrections through $O(1/c^2)$. Section IV B is concerned with the rotation of the three bodies involved. Rotation could affect the end result in two distinct ways. First, the rotation of a body makes a contribution to the metric. Second, the equations of motion for a spinning body in relativity theory are qualitatively different from those of the usual nonspinning point particle. However, we conclude that to our desired accuracy both of these effects unfortunately make negligible contributions. There is no hope of testing these aspects of the theory with the anticipated laser-ranging data and so they are neglected in the calculation. In Sec. IV C we outline the procedure involved in setting up a Hill-Brown calculation of the relative separation. Our calculation extends Brumberg's treatment by including the eccentricity of the earth's orbit (to first order). Thus we can assess the relativistic corrections associated with both the lunar and earth eccentricities. The final expression for the major relativistic corrections to the earth-moon separation is given in Sec. IV D, and there we discuss their observational significance. It appears that the general relativistic contribution to at least one of the periodic terms is observationally accessible with the laser-ranging data and that a significant test of the theory is possible.

Section V summarizes the conclusions.

In an appendix we discuss briefly the general relativistic contributions to precession effects, meaning by this both the secular motions of the lunar perigee and node and the general precession. The primary aim here is to clear up a misunderstanding about the distinction between them which has arisen in the literature.

II. THE APPROXIMATE METRIC

The calculation of an approximate metric as a power series in $1/c$ is by now rather standard.[6] We will merely outline the steps and indicate the points at which one must exercise some care.

The exact metric may be written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\,, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric of special relativity.[7] The requirement that the metric $g_{\mu\nu}$ reduce asymptotically to the flat-space Cartesian-coordinate expression $\eta_{\mu\nu}$ as one approaches spatial infinity supplies the essential boundary condition. For our purposes we will need the $1/c$ expansion of $h_{\mu\nu}$ to the following orders:

$$\begin{aligned} &h_{00} \quad \text{to} \quad O(1/c^4)\,,\\ &h_{0i} \quad \text{to} \quad O(1/c^3)\,,\\ &h_{ij} \quad \text{to} \quad O(1/c^2)\,. \end{aligned} \tag{2.2}$$

Before one can calculate explicit expressions for the $h_{\mu\nu}$ from the Einstein equations (here taken with zero cosmological constant), one must decide upon the form of the stress-energy tensor $T^{\mu\nu}$ for the extended bodies. This problem does not have an unambiguous solution. However, the stress-energy tensor for a perfect fluid does provide both a tractable solution and a physically reasonable one:

$$T^{\mu\nu}=c^{-2}(\epsilon+p)u^\mu u^\nu+pg^{\mu\nu}\,, \tag{2.3a}$$

$$\epsilon=\rho c^2(1+\Pi/c^2)\,, \tag{2.3b}$$

$$u^\mu=dx^\mu/d\tau \quad \text{with} \quad u^\mu u_\mu=-c^2\,. \tag{2.3c}$$

Here u^μ is the 4-velocity of the fluid with the indicated normalization, τ being proper time. The quantities ϵ, ρ, and $\rho\Pi$ represent the energy, mass, and internal energy density, respectively, of the fluid; p is the isotropic pressure.[8] For our case of several isolated bodies the mass density ρ is nonzero only within each of the extended bodies.

We anticipate taking the limit wherein the extended bodies shrink to points. For a set $\{A\}$ of point particles of nominal mass m_A the stress-energy tensor has the following thoroughly-covariant form:

$$T^{\mu\nu}=\sum_A \int_{-\infty}^{\infty} m_A c(-g)^{-1/2} \times\delta[x^\alpha-x_A{}^\alpha(\tau_A)]u_A{}^\mu u_A{}^\nu d\tau_A\,, \tag{2.4}$$

where x^α represents the space-time field point and $x_A{}^\alpha(\tau_A)$, the position of particle A as a function of its proper time τ_A. The δ function is really a product of four δ functions, one for each value of α, and the integral extends over the entire world line of each particle.

The integral is readily done, giving

$$T^{\mu\nu}(\mathbf{x},t)=\sum_A m_A(c/u_A{}^0)(-g)^{-1/2} \times\delta[\mathbf{x}-\mathbf{x}_A(\tau_A)]u_A{}^\mu u_A{}^\nu\,, \tag{2.5}$$

with τ_A such that $ct\equiv x^0=x_A{}^0(\tau_A)$. A comparison of this expression with the explicitly ρ-dependent part of the perfect fluid stress-energy tensor suggests that we can

[6] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965); V. Fock, *The Theory of Space, Time, and Gravitation* (The Macmillan Company, New York, 1964).

[7] Greek letters run from 0 to 3; Latin, from 1 to 3. Summation over repeated indices is to be understood. The coordinates have an essentially "spatial" meaning, with $x^0=ct$. The Minkowski metric has the diagonal form $(-1, 1, 1, 1)$. The determinant of $g_{\mu\nu}$ is written as g.

[8] More specifically, the mass density ρ is defined such that the covariant divergence of ρu^μ is zero.

facilitate the passage to the point-particle limit by introducing a function $f(\mathbf{x},t)$ defined in the perfect fluid context by the relation

$$\rho=(c/u^0)(-g)^{-1/2}f. \tag{2.6}$$

As defined, f differs from the mass density ρ only by terms of order $1/c^2$ and higher. Thus they are identical at the Newtonian level. The point-particle limit then corresponds to

$$f(\mathbf{x},t)\rightarrow\sum_A m_A\delta[\mathbf{x}-\mathbf{x}_A(t)], \tag{2.7}$$

whenever this introduces no disastrous singularities. Here $\mathbf{x}_A(t)$ is merely a more explicit manner of specifying the position of particle A at time t.

Additional constraints must be imposed before we can consider the coordinate system as specified. A particularly convenient choice of four coordinate conditions for this problem is the following set:

$$\frac{\partial h_0{}^l}{\partial x^l}-\frac{1}{2}\frac{\partial h_l{}^l}{\partial x^0}=0, \tag{2.8a}$$

$$\frac{\partial h_j{}^l}{\partial x^l}-\frac{1}{2}\frac{\partial h_\alpha{}^\alpha}{\partial x^j}=0. \tag{2.8b}$$

(The indices on $h_{\mu\nu}$ have been raised with $\eta_{\mu\nu}$.) These coordinate conditions have the great virtue that the Einstein field equations in the approximation we need reduce to coupled equations of the Poisson form rather than to inhomogeneous wave equations. Thus one can immediately infer that the approximate metric has only an implicit time dependence, this through the time dependence of the positions and velocities of the bodies. In the solution to be presented for the perfect fluid the three equations (2.8b) are identically satisfied to appropriate order in $1/c$. To satisfy the single equation (2.8a) we require nothing more than mass conservation at the Newtonian level, a property of the fluid which we must certainly impose.

To economize in the expressions for the perfect-fluid metric we introduce the standard auxiliary functions U, U_i, χ, and Φ, defined as follows[9]:

$$\begin{aligned}
U(\mathbf{x},t)&\equiv G\int d^3x'\,\frac{1}{\Delta}f(\mathbf{x}',t),\\
U_i(\mathbf{x},t)&\equiv G\int d^3x'\,\frac{1}{\Delta}f(\mathbf{x}',t)v^i(\mathbf{x}',t),\\
\chi(\mathbf{x},t)&\equiv\frac{1}{2\pi}G\int d^3x'\,\frac{1}{\Delta}U(\mathbf{x}',t),\\
\Phi(\mathbf{x},t)&\equiv\tfrac{1}{2}G\int d^3x'\,\frac{1}{\Delta}[(\tfrac{3}{2}v^2+\Pi-U)f+3p].
\end{aligned} \tag{2.9}$$

The symbol Δ is $|\mathbf{x}-\mathbf{x}'|$ and v^i is defined by $v^i\equiv u^i(c/u^0)$, corresponding to the ordinary coordinate velocity dx^i/dt. The symbol G is of course the Newtonian gravitational constant.

To desired order in $1/c$ the perfect-fluid metric is then simply this:

$$g_{ij}=\delta_{ij}(1+2\gamma U/c^2)+O(c^{-4}), \tag{2.10a}$$

$$g_{0i}=\frac{1}{c^3}\left(-4\eta U_i+\tfrac{1}{2}\eta'\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\chi\right)+O(c^{-5}), \tag{2.10b}$$

$$g_{00}=-1+\frac{2U}{c^2}-\beta'\frac{1}{c^4}(2U^2-4\Phi)+O(c^{-6}). \tag{2.10c}$$

We have introduced the labeling coefficients γ, β', η, and η' as a means of keeping track, in the calculations to come, of the influence of the respective parts of the total metric. This follows a tradition initiated by Eddington[10] and now enjoying considerable popularity. (We omit the customary coefficient α of the isolated U in g_{00}. Agreement with Newtonian physics requires it to be unity to high precision. A prime is appended to β' because the $1/c^4$ term it multiplies is quite different from the usual spherically-symmetric solar term of the planetary problem.) If the Einstein theory is correct, the metric for our physical situation (to appropriate order in $1/c$) is properly represented when the labeling coefficients all have the value 1. Since we are looking for effects predicted by the Einstein theory, we regard these coefficients merely as labels whose numerical value of unity we can explicitly invoke when desirable.

We now consider the passage to the point-particle limit. This requires some care and entails a definite loss of physical attributes. To avoid infinities we should pass to the limit only in the case of field points $(\mathbf{x},t)$ which do not lie on the world line of the center of mass (c.m.) of any of the extended bodies. In the process we also lose certain contributions which might arise in the extended-structure case. In particular, quadrupole and higher moments of the mass distribution disappear from the potential U. Any rotation of the extended body, which would give a contribution to the fluid velocity $\mathbf{v}(\mathbf{x},t)$, ceases to contribute in the limit.

If we are willing to accept these restrictions and losses, we must then face the question of how to handle the internal energy $f\Pi$ and pressure p in the Φ part of h_{00}. Within the perfect-fluid context we can eliminate the internal energy $f\Pi$ by means of the approximate thermodynamic relation holding within each nonrotating body A:

$$p_A=(U_A-\Pi_A)f_A, \tag{2.11}$$

where the subscript A means that the quantities refer

[9] These differ somewhat from Chandrasekhar's expressions because we use f rather than ρ. Because the difference between these densities is $O(1/c^2)$, this really affects only Φ, an $O(1/c^4)$ part of the metric.

[10] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1960), p. 105.

to body A. In particular, U_A is solely the potential produced by body A. This relation follows at the Newtonian level if we demand internal equilibrium and neglect the "tidal forces" produced by the gravitational fields of the other bodies.[11] This leaves us with only a single awkward term of $2p$ in the integrand of Φ. This, too, may be simplified. As we approach the point limit we may approximate the integral over a single body A as

$$\int d^3x' \frac{1}{\Delta} 2p_A \approx \frac{1}{\Delta_A} \int d^3x'\, 2p_A\,, \tag{2.12}$$

where $\Delta_A \equiv |\mathbf{x}-\mathbf{x}_A(t)|$, with $\mathbf{x}_A(t)$ being the position of the c.m. of body A. Within the assumed context of internal equilibrium, no rotation, and neglect of tidal forces we have

$$\frac{\partial p_A}{\partial x^i} = f_A \frac{\partial U_A}{\partial x^i}\,, \tag{2.13}$$

within the body A. Multiplication by x^i, followed by integration over the body A and use of the properties of U_A, imply

$$\int d^3x\, 2p_A = \tfrac{1}{3} \int d^3x\, f_A U_A. \tag{2.14}$$

The latter integral may, to our order of approximation, be regarded as a constant. When the point limit is taken, it contributes to g_{00} in eactly the same manner as $m_A \equiv \int d^3x\, f_A(\mathbf{x},t)$ in the isolated U of g_{00}, namely as a coefficient of $1/\Delta_A$. (The expression defining m_A is rigorously constant provided we require that the covariant divergence of ρu^μ be zero, a property which is part of the definition of ρ.) Hence we may regard it as a "mass renormalization" of the nominal mass m_A. We define the "gravitationally renormalized" mass m_A' as

$$m_A' \equiv m_A + \tfrac{1}{3}\beta' c^{-2} \int d^3x\, f_A U_A. \tag{2.15}$$

To our order in powers of $1/c$ this is actually the mass of body A which would be measured by a Newtonian experiment in the asymptotic region (if A is at rest). If we substitute the renormalized mass m_A' for the nominal mass m_A in the isolated U of g_{00}, we have taken care of the awkward parts of Φ. Moreover, we may now replace m_A wherever it appears by m_A' since the renormalization is formally of order $1/c^2$ and will not affect other results to our order in $1/c$.

Thus in the point limit we may write the following as the expressions which should be substituted into Eqs. (2.10) for the metric:

$$U = \sum_A \frac{Gm_A'}{\Delta_A}\,, \tag{2.16a}$$

$$U_i = \sum_A \frac{Gm_A'}{\Delta_A} v_A{}^i\,, \tag{2.16b}$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\chi = \sum_A Gm_A' v_A{}^l \left(\frac{\delta^{il}}{\Delta_A} - \frac{\Delta_A{}^i \Delta_A{}^l}{\Delta_A{}^3}\right), \tag{2.16c}$$

$$\Phi = \tfrac{1}{2} \sum_A \frac{Gm_A'}{\Delta_A} \left(\tfrac{3}{2} v_A{}^2 - \sum_{B\neq A} \frac{Gm_B'}{\Delta_{BA}}\right). \tag{2.16d}$$

The additional symbols have the obvious definitions $\Delta_A{}^i \equiv x^i - x_A{}^i$ and $\Delta_{BA} \equiv |\mathbf{x}_B - \mathbf{x}_A|$. Needless to say, the field points should not be on the world line of any of the point particles.

III. ROUND-TRIP PROPER TIME FOR THE LASER PULSE

For the purpose of determining the general relativistic effects in the round-trip proper time for the laser pulse we may neglect the motion of the earth during the brief time interval between emission and reception of the pulse. (A nonrelativistic correction must certainly be made for the change in station coordinates due to the earth's rotation and center-of-mass motion during this approximately 2.5-sec interval.) Thus we may take the earth-station coordinates as instantaneously fixed in space.

This level of approximation brings the added benefit that the metric may be regarded as unchanging (in time) during the pulse transit time despite its implicit dependence on time through the positions and velocities of the earth, moon, and sun.

We first compute the round-trip coordinate time X^0, that is, the coordinate time interval between the two events of pulse emission and reception. This quantity is distinctly dependent on the particular choice of coordinate system. Later we will apply the factor required to convert this to proper time for an earth-based observer and hence to the observer's atomic time.

We make the usual assumption that the light pulse travels along a null geodesic, a property actually derivable for electromagnetic wave fronts from the combined Einstein-Maxwell equations. The null property implies

$$0 = g_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu. \tag{3.1}$$

As a quadratic in dx^0, this has the approximate solution

$$dx^0 = \left(\frac{1+2\gamma U/c^2}{1-h_{00}}\right)^{1/2} (\delta_{ij} dx^i dx^j)^{1/2} + \frac{h_{0i} dx^i}{1-h_{00}} + O(c^{-6})\,, \tag{3.2}$$

[11] V. Fock, *The Theory of Space, Time, and Gravitation* (The Macmillan Company, New York, 1964).

where we have inserted the form for h_{ij} from Eq. (2.10a).

In principle this should be integrated along the spatial projection of one null geodesic from the earth to the moon and then back along that of another null geodesic. In practice, to $O(1/c^2)$ in the relativistic corrections, it is sufficient to integrate along a straight line from the earth station to the lunar station and then back along the same line. The spatial projection of each of the two distinct null geodesics deviates little from a straight line. For fixed end points the difference in integrated arc length is $O(1/c^4)$ and may be neglected. The coefficients $h_{\mu\nu}$ are already at least $O(1/c^2)$. Thus differences in the spatial location of the paths of integration could lead only to effects of order $(1/c^2)\times(1/c^2)$ and hence may be neglected.

The second term, proportional to $h_{0i}dx^i$, introduces an asymmetry in the out-versus-back pulse times. However, in the integration along a straight line to the lunar station and back the contributions cancel. In any case, this is an $O(1/c^3)$ term; as such, it would make a negligible contribution even in a more precise treatment in which the out and back integrals did not exactly cancel.

In consequence of these simplifications the round-trip coordinate time to $O(1/c^2)$ is given (after further expansion of the first term above) by the integration over a straight connecting line of the following simple expression:

$$X^0=\int_{\text{round trip}}[1+(1+\gamma)U/c^2](\delta_{ij}dx^idx^j)^{1/2}. \tag{3.3}$$

At this point a diagram (Fig. 1) is a helpful means of introducing the notation we will need. The centers are the respective centers of mass, with r being the earth-to-moon separation. The magnitudes R_e and R_m with their associated unit vectors, are the center-to-station distances for the earth and moon, respectively. The vector $\mathbf{R}'$ denotes the position of the Newtonian earth-moon c.m. relative to the sun. The masses $M_\odot$, $M_\oplus$, and $M_{☾}$ designate the masses of the sun, earth, and moon, respectively, and correspond in the theory to the "renormalized" masses of Eq. (2.15). Within our approximation the pulse travels out and back along $\mathbf{S}$, the station-to-station straight-line separation.

Since only the Newtonian potential U is involved and since we may safely neglect multipole moments, the integral may be done exactly. However, the result is anything but transparent. So we expand and retain terms which are significant at the 5-cm level.

The solar contribution then reduces to a term of some 1.6×10^3 cm,

$$2S(1+\gamma)GM_\odot/c^2R', \tag{3.4}$$

with a term in S^2 of order 2 cm being the largest of the discarded terms. (In more detail, the primary discarded term is given by $-\mathbf{S}\cdot\mathbf{R}'/2R'^2$ times the term retained.

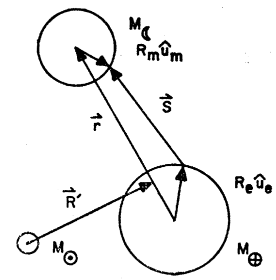

FIG. 1. Sketch to illustrate the notation used in describing the propagation of the laser pulse within the earth-moon-sun system.

The other discarded terms are orders of magnitude smaller than this one.)

The earth's contribution simplifies in part because of the experimental context, in which $|\hat{\mathbf{S}}\cdot\hat{u}_e-1|$ is small relative to unity. (One would observe when the moon is near the meridian in order to minimize effects in the earth's atmosphere.) Thus the earth contributes a constant term of some 7 cm,

$$2(1+\gamma)\frac{GM_\oplus}{c^2}\ln\left(\frac{r_0-R_m}{R_e}\right), \tag{3.5}$$

with r_0 being some mean earth-moon center-to-center distance. The discarded variable parts are of order 1 cm or less. (We have neglected the multipole structure of the earth's potential here. The quadrupole part would give a contribution considerably smaller than the earth's "gravitational radius" $GM_\oplus/c^2\approx0.4$ cm. Hence the neglect is amply justified.)

The lunar contribution has a structure similar to that of the earth's (with $|\hat{\mathbf{S}}\cdot\hat{u}_m+1|\ll1$, again for experimental reasons) but reduced by the ratio of the respective masses; hence it is utterly negligible.

Thus, to 5-cm accuracy, we get

$$X^0=2S+2S(1+\gamma)GM_\odot/c^2R'+2(1+\gamma)\times\frac{GM_\oplus}{c^2}\ln\left(\frac{r_0-R_m}{R_e}\right). \tag{3.6}$$

We must now convert X^0 to the elapsed proper time $\mathcal{T}$ of a fixed observer on the moving earth. This is given by an integral taken along the world line of the observer between the events of transmission and reception of the laser pulse:

$$\mathcal{T}=\int\frac{d\tau}{dx^0}dx^0\approx\frac{d\tau}{dx^0}X^0. \tag{3.7}$$

Here we have evaluated the integrand at some point along the small section of world line and extracted it from the integral. This involves nothing more than neglect of changes in the velocity and gravitational potential of the observer during the brief interval between emission and reception of the pulse. In this approximation, which is consistent with our calculation of the elapsed coordinate time, the conversion appears as an over-all factor.

With $v^i_{\text{obs}} \equiv (c\,dx^i/dx^0)_{\text{obs}}$ being the coordinate velocity of the earth-bound observer, we have

$$\frac{d\tau}{dx^0} = \frac{1}{c}\left(-g_{00} - 2g_{0i}\frac{v^i_{\text{obs}}}{c} - g_{ij}\frac{v^i_{\text{obs}}v^j_{\text{obs}}}{c^2}\right)^{1/2}. \quad (3.8)$$

To $O(1/c^2)$ in the expansion of the expression in parentheses we get

$$\frac{d\tau}{dx^0} = \frac{1}{c}(1 - U/c^2 - \tfrac{1}{2}v^2_{\text{obs}}/c^2). \quad (3.9)$$

The neglected parts, as $O(1/c^4)$ terms, are $O(10^{-16})$ relative to unity. Thus they are negligible in the short run. However, in a long term comparison of lunar motion with a detailed calculation done in coordinate time one may need to retain them.

From the experimental point of view it is highly desirable to split the result for the elapsed proper time into those terms which are constant and those which do vary with the periodic change in relative orientation and motion of the earth, moon, and sun. To this end we express some of the quantities in terms of the variables describing the relative motion of the earth-moon c.m. and the sun. Although we shall later need to describe a calculation of the lunar motion from the astronomer's conventional geocentric stance, the present computation is more easily visualized from the over-all barycentric frame (which corresponds more nearly to the heliocentric view). So we retain the latter view for the moment. Since we are working with terms which are already $O(1/c^2)$, we may regard the earth-moon c.m. as moving about the sun in a fixed Keplerian ellipse.

To required accuracy the solar gravitational potential at the earth station then enters $d\tau/dx^0$ as $GM_\odot/c^2R'$, with terms of $O(10^{-13})$ relative to unity dropped. The contribution of the earth's potential we write as $GM_\oplus/c^2R_e$ and we may drop the lunar contribution, for it is $O(10^{-13})$.

Any difference between the coordinate velocity v^i_{obs} of the observer and some more nearly "proper" or observational velocity must involve an additional factor of $1/c^2$ and hence is negligible. Introducing the velocity $\Delta\mathbf{v}$ of the observer relative to the earth's c.m., we write

$$\mathbf{v}_{\text{obs}} = \mathbf{V}' - \frac{M_☾}{M_\oplus + M_☾}\mathbf{v} + \Delta\mathbf{v}, \quad (3.10)$$

with $\mathbf{V}'$ being the motion of the earth-moon c.m. relative to the sun and $\mathbf{v}$ being the motion of the moon relative to the earth. (The motion of the sun in the over-all barycentric system is surely negligible in $1/c^2$ terms.) To adequate accuracy we get

$$\frac{v^2_{\text{obs}}}{c^2} = \frac{V'^2}{c^2} + \frac{2\mathbf{V}'\cdot\Delta\mathbf{v}}{c^2}, \quad (3.11)$$

with terms of order 5×10^{-12} dropped. The result of combining the contributions is then

$$\frac{d\tau}{dx^0} = \frac{1}{c}\left[1 - \left(\frac{GM_\odot}{c^2R'} + \frac{GM_\oplus}{c^2R_e}\right) - \tfrac{1}{2}\left(\frac{V'^2}{c^2} + \frac{2\mathbf{V}'\cdot\Delta\mathbf{v}}{c^2}\right)\right]. \quad (3.12)$$

We may now apply this as a factor to X^0. The result takes on a more useful form if we employ a relation valid for elliptic motion, namely,

$$\tfrac{1}{2}V'^2 = \frac{GM_\odot}{R'} - \frac{GM_\odot}{2a'}, \quad (3.13)$$

with a' being the semimajor axis for the motion of the earth-moon c.m. relative to the sun.

The final expression for the elapsed proper time is then

$$\mathcal{T} = \frac{2S}{c}\left[1 + (\gamma-1)\frac{GM_\odot}{c^2R'} + \frac{GM_\odot}{2c^2a'} - \frac{GM_\oplus}{c^2R_e} - \frac{\mathbf{V}'\cdot\Delta\mathbf{v}}{c^2}\right] + \frac{2(\gamma+1)}{c}\frac{GM_\oplus}{c^2}\ln\left(\frac{r_0 - R_m}{R_e}\right) \quad (3.14)$$

in seconds of proper time to an accuracy corresponding to 5 cm.

If the Einstein theory is correct, then γ is unity and the term in $1/R'$, which would vary primarily with a period of one year, vanishes. In the experimental context the term $(\mathbf{V}'\cdot\Delta\mathbf{v})/c^2$ has essentially the lunar period but is on the verge of insignificance, having a maximum magnitude of the order of 1.5×10^{-10}.

The logarithmic term is so small that we can consider multiplying it by S/r_0 without affecting the accuracy of Eq. (3.14). The entire right-hand side would then be proportional to S/c. With $\gamma - 1$ equal to zero and the velocity term virtually insignificant, the proportionality factor with the relativistic terms would be constant and hence experimentally indistinguishable from a correction to the speed of light. The value of c is known to no better than parts 10^7, whereas the factor would be $1+O(10^{-8})$.

We must conclude that within the framework of the Einstein theory there is no realistic expectation of detecting gravitational effects on the propagation of light with the laser-ranging data. The only hope for a relativistic correction which varies with time and has an experimentally significant amplitude lies in the lunar motion as it appears in the station-to-station coordinate separation S.

By virtue of its definition, S has the following form in terms of $\mathbf{r}$ and the earth- and lunar-station coordinates:

$$S^2 = r^2 + 2r(R_m\hat{r}\cdot\hat{u}_m - R_e\hat{r}\cdot\hat{u}_e) + (R_m^2 + R_e^2 - 2R_eR_m\hat{u}_e\cdot\hat{u}_m). \quad (3.15)$$

The determination of the variation of S due to variation of the scalar products is a quite difficult but tractable Newtonian problem. (There are also small variations in R_e and R_m arising from deformations of a tidal nature.) In looking for a test of the Einstein theory we must concentrate on the general relativistic effects in r, the earth-moon center-to-center separation. We turn now to an analysis of this problem.

IV. RELATIVISTIC CORRECTIONS TO THE EARTH-MOON SEPARATION

Lunar calculations are always approximate calculations. So at the outset we must establish the order of approximation necessary for our specific purpose. The mean earth-moon separation is some 4×10^{10} cm. The laser ranging really provides an observational time rather than a distance. Modulo the important uncertainty in the speed of light, the anticipated timing precision corresponds to some 10 cm in distance. So the experimental precision sets a goal of two parts in 10^{10} for the accuracy of the theoretical calculation of the earth-moon center-to-center separation r.

In the freshman physics view the equation one must solve approximately is simply:

$$\ddot{\mathbf{r}}=-\frac{G(M_\oplus+M_☾)}{r^3}\mathbf{r}+\begin{pmatrix}\text{Newtonian}\\ \text{solar}\\ \text{corrections}\end{pmatrix}+\begin{pmatrix}\text{Relativistic}\\ \text{corrections}\end{pmatrix}. \tag{4.1}$$

In the same spirit one would say that to determine r to two parts in 10^{10} one need not worry about correction terms which are smaller than $O(10^{-10})$ times the basic Newtonian term. This is certainly too naive. It totally neglects the resonant behavior of certain terms. Nonetheless, it gives an estimate of significance. A term of order 10^{-12}, say, should be negligible, and if dropping it helps to simplify this very complex problem, it is reasonable to do so. Certainly we see immediately that one needs the relativistic corrections through $O(1/c^2)$ only, for they are characterized by $GM_\odot/c^2R'\approx10^{-8}$.

A. Equations of Motion

Within the strict context of the Einstein theory one can derive the equations of motion for an extended body to $O(1/c^2)$ without additional assumptions of a characteristically relativistic nature (such as the geodesic postulate).[11] The derivation requires merely the statement that the covariant divergence of the stress-energy tensor is zero, and this follows directly from the field equations. The derivation goes through even for the relevant case of rigidly rotating bodies. No embarrassing infinities arise, and indeed the resulting point-limit equations for nonrotating bodies are exactly those which one would get by invoking the geodesic motion postulate, expanding to $O(1/c^2)$, and discarding the apparent infinities which arise from the action of a body on itself.

Our introduction of the coefficients γ, β', η, and η' makes the situation a bit awkward. However, if we regard the coefficients merely as labels, with each having the numerical value 1, then the metric is still a solution of the Einstein field equations. A 0 value for the covariant divergence of the stress-energy tensor follows immediately, and one can derive from this the equations of motion for fluid matter. When one applies this procedure with the labeled metric to the case of well-separated, nonrotating ideal fluid spheres in internal equilibrium and passes to the point limit, one derives the appropriate geodesic equation (without embarrassing infinities). Only in removing a single unwanted term need one explicitly invoke the numerical value of the labels. This term has the form of a mass renormalization term for the particular fluid sphere whose equation of motion is being determined. Thus, to $O(1/c^2)$ and with the specifically physical assumptions about the fluid, we may regard the resultant geodesic equation with labels in place as a deduction from the Einstein field equations (rather than as a separate postulate).

In restricting the derivation of the equations of motion with the labeled metric to the case of nonrotating spheres, we have explicitly dropped all effects which could arise from the known rotation of the three bodies involved and from multipole moments. In Sec. IV B we justify the neglect of general relativistic rotational effects. The multipole moments are certainly important at the Newtonian level but not at the $O(1/c^2)$ relativistic level. Since we are looking for only the major relativistic corrections to the Newtonian results, we may safely neglect the multipole moments. Thus we may reasonably reduce the problem to that of interacting point particles.

For a Hill-Brown calculation of the relative earth-moon separation one would like the equations of motion in a Lagrangian form. So one is confronted with the task of constructing a Lagrangian in the earth-moon relative position and velocity variables which gives the same equations of motion as does direct application of the geodesic equation to the moon and earth separately, with the difference then to be taken. Both of these are to go to $O(1/c^2)$ beyond the Newtonian equations.

For a body A which does contribute to the metric, one may get the geodesic equation from the Lagrangian integrand

$$m_Ac^2\frac{d\tau'_A}{dt}\equiv m_Ac^2[-g'_{00}-2g'_{0i}v_A{}^i/c-g'_{ij}v_A{}^iv_A{}^j/c^2]^{1/2}, \tag{4.2}$$

provided $g'_{\mu\nu}$ is $g_{\mu\nu}$ without the contributions of m_A except for the "interior" summation in the Φ of Eq. (2.16d) and provided the partial differentiations for the actual equations of motion are made with respect to the "field point" $\mathbf{x}$ of $g'_{\mu\nu}$, later set equal to $\mathbf{x}_A$. The square root may be expanded in powers of $1/c$ to our desired order of approximation, that is, $O(1/c^2)$ beyond that necessary for the Newtonian equations. If one chooses the particle A to be the moon, this provides a starting point for the desired Lagrangian. One must now add further terms (which come essentially from a similar Lagrangian for the earth) and then re-express the result in terms of the relative position $\mathbf{r}$ and relative velocity $\mathbf{v}$. We recapitulate the test which the proposed Lagrangian must pass: When the Lagrangian is treated as a function of $\mathbf{r}$ and $\mathbf{v}$, the equation for $d\mathbf{v}/dt$ which arises from the standard Lagrange procedure must coincide with the result obtained by taking the difference of the moon and earth equations of motion derived directly from the respective geodesic equations, all to $O(1/c^2)$ beyond the Newtonian equations.

The construction process is simplified by some additional approximations. In the over-all barycentric system the motion of the sun is very small. In the $O(1/c^2)$ terms one may set it to zero, although, of course, it remains relevant at the Newtonain level. The second additional approximation is the neglect of the lunar mass relative to the earth mass in the $O(1/c^2)$ terms. A careful look at these approximations shows that neither is likely to affect the result to the accuracy we desire. Where the neglect of $M_{☾}$ entains the loss of a particular form of term in the equation of motion, the term is typically of order 10^{-12} times the basic Newtonian term. When the term is retained but a factor like $1+M_{☾}/M_{\oplus}$ is set to unity, the error in the equations of motion may be as large as $O(10^{-10})$. However, we will see that the final relativistic corrections to r are so small that this is distinctly an acceptable approximation in the present context.

The resulting Lagrangian is long and awkward in appearance; we will refrain from writing it down here. Suffice it to say that, with the explicitly stated approximations given above, one can construct the appropriate Lagrangian with the labeling coefficients in place.

The next step in the conventional procedure involves a change of stance. One adopts the point of view that it is in fact the sun which moves around the earth-moon c.m., rather than vice versa. The vector $\mathbf{R}$ is defined to point from the Newtonian earth-moon c.m. to the position of the sun. It is the negative of the vector $\mathbf{R}'$ introduced in Sec. III. Likewise, $\mathbf{V}$ is defined as $(-\mathbf{V}')$.

One can then proceed to replace the solar mass $M_{\odot}$ by its equivalent in terms of the solar orbital parameters, with the orbit generally assumed to be a Keplerian ellipse. This serves also to introduce into the Lagrangian the mean solar motion n' as a parameter. The presence of relativistic effects complicates this process, but it can be carried through in a satisfactory order of approximation. By taking appropriate mass-weighted sums and differences of the geodesic equations for the earth, moon, and sun, one arrives at an equation for $d\mathbf{V}/dt$ which includes the $O(1/c^2)$ corrections. From the approximate solution of this equation we are to derive a substitute for $GM_{\odot}$ involving the solar mean motion. The Newtonian solar term in the equation for the relative earth-moon motion is $O(10^{-2})$ relative to the direct earth-moon interaction. So we need the $d\mathbf{V}/dt$ equation good only to two orders of magnitude less than the goal we set for the terms in $d\mathbf{v}/dt$.

We would like to approximate the solution of the $d\mathbf{V}/dt$ equation by a Keplerian ellipse. This implies the neglect of certain small Newtonian correction terms. In his lunar theory Brown did include some of these, but only as perturbations after solving the so-called main problem within the elliptic motion approximation. In any case, the deviations from elliptic motion due to the Newtonian corrections and the relativistic corrections to an assumed elliptic motion should be independent to first order. This is all we really need. Furthermore, inspection shows that we may safely neglect the eccentricity (which is approximately 0.017) in the relativistic terms. The result is a delightfully simple modification of the standard elliptic motion relation:

$$n'^2a'^3=G(M_{\odot}+M_{\oplus}+M_{☾})\times\left\{1-\frac{1}{c^2}\left[2(\gamma+\beta')\frac{GM_{\odot}}{a'}-\gamma(n'a')^2\right]\right\}, \quad (4.3)$$

where a' is the semimajor axis and n' is the mean solar motion. Introducing the ratio $\xi\equiv(M_{\oplus}+M_{☾})/(M_{\odot}+M_{\oplus}+M_{☾})\approx3\times10^{-6}$, we may solve for $GM_{\odot}$ to $O(1/c^2)$:

$$GM_{\odot}\approx(1-\xi)n'^2a'^3[1+(2\beta'+\gamma)(n'a')^2/c^2]. \quad (4.4)$$

General relativistic effects are present both in the obvious sense of the term in $1/c^2$ and in the more subtle sense that a' and n' refer to coordinate lengths and times, respectively. Their connection with observational quantities is by no means direct. In fact, the appearance of the over-all expression is distinctly dependent on the choice of coordinate conditions made earlier. To underline this, we point out that the corresponding planetary problem, when done with the coordinate condition choice which corresponds to the standard polar form of the Schwarzschild metric, contains no obvious relativistic corrections in the limit of a circular orbit. When the same problem is handled in a Cartesian frame with all axes treated equally in the metric, a correction term of the above form does arise.

B. Rotation Effects

Rotation effects arise in two distinct ways. First, the rotation of a body makes a direct contribution to the metric since the fluid velocity over which one integrates in Eqs. (2.9) is not merely the body's c.m. motion but a combination of that plus the rotational motion about the c.m. Moreover, there will be differences in the conditions for internal equilibrium, with the result that the mass distribution will not be spherically symmetric. This will lead to multipole moments in even the Newtonian potential. The multipole structure is certainly a major effect, but it is primarily a Newtonian effect. To estimate the relativistic corrections to the metric it suffices to look at those terms which involve the fluid velocity explicitly.

Second, within the general relativistic context the rotation of a body moving through a prescribed metric leads to deviations from purely geodesic motion. One might expect this, for a classical rotating body is necessarily an extended body; as such, it samples the gravitational field not merely along a world line but throughout a world tube. Hence even at the Newtonian level additional forces arise from a conjunction of the variation of the Newtonian gravitational field over the region together with the aspherical mass distribution produced by the rotation itself. However, the effect in question is really more subtle than this. The correction terms in the relativistic equations of motion depend on the angular-momentum four-vector of the spinning body and appear even in a derivation which explicitly drops the usual quadrupole moments as negligibly small.[12]

Within the context of our $1/c^2$ approximation for the equations of motion the influence of these two distinct rotational effects can be directly computed to an accuracy commensurate with the expected observational precision. However, dimensional arguments alone suffice to show that the effects are unfortunately negligible at our level of relevance.

We look first at the influences on the metric. The metric coefficients g_{0i} of Eq. (2.10b) involve the velocity linearly. (This is clear for the U_i term from its definition and is true for the χ term after the indicated differentiations have been performed.) For a mass B which rotates uniformly the first nonzero additional terms are of the dimensional form

$$\frac{1}{c^2}\frac{Gm_B}{\Delta_B}\frac{R_B}{\Delta_B}\frac{\omega_B R_B}{c}. \tag{4.5}$$

Here $\Delta_B \equiv |\mathbf{x}-\mathbf{x}_B|$ is the distance between the field point and the c.m. of body B; ω_B is the angular velocity about the c.m.; and R_B is a typical size dimension of the rotating body. The derivation of this involves nothing more than the standard expansion of the denominator in the integrals for field points far from the rotating body.

The geodesic equation specifies that g_{0i} contributes to the acceleration of a point particle through two types of terms:

$$c\times\begin{pmatrix}\text{velocity of the}\\ \text{point particle}\end{pmatrix}\times\begin{pmatrix}\text{spatial derivative}\\ \text{of } g_{0i}\end{pmatrix},$$

$$c\times(\text{time derivative of } g_{0i}). \tag{4.6}$$

[12] F. A. E. Pirani, Acta Phys. Polon. 15, 389 (1956); A. H. Taub, J. Math. Phys. 5, 112 (1964).

In both cases the resultant contribution to the acceleration has the form

$$\frac{Gm_B}{\Delta_B^2}\times\frac{R_B}{\Delta_B}\frac{\omega_B R_B}{c}\frac{(\text{some velocity})}{c}, \tag{4.7}$$

where "some velocity" is either the velocity of the point particle or the c.m. velocity of the rotating body. It is now simply a matter of inserting values of the parameters for earth, moon, and sun into the second bracket. Typically this has a value of $O(10^{-12})$ or less. Since the first bracket has the magnitude of the acceleration produced by the direct Newtonian interaction, the expression as a whole is negligible.

This disposes of rotational effects in g_{0i}, but there remains the v^2 term in Φ/c^4 of g_{00}. The contribution of v^2 to Φ/c^4 from a single rotating body is of this form

$$\frac{Gm_B}{c^2\Delta_B}\times\left[O\left(\frac{v_B^2}{c^2}\right)+O\left(\frac{R_B}{\Delta_B}\frac{\omega_B R_B}{c}\frac{v_B}{c}\right)+O\left(\frac{\omega_B^2 R_B^2}{c^2}\right)\right]. \tag{4.8}$$

Here v_B is the velocity of the c.m. of body B.

We must bear in mind that Φ/c^4 enters the geodesic equation with a factor of c^2 arising from a product of the time components of the point particle's four-velocity and with a spatial derivative applied. Hence we keep the first term in the bracket. The second is then of the same structure as the g_{0i} contributions and may be neglected in the geodesic equation for analogous reasons. The third term in the bracket is small, $O(10^{-11})$ or less, and could be neglected for that reason alone. However, there is an additional justification for dropping it: This term is indistinguishable from a change in the mass m_B. It acts like a further renormalization of the nominal mass. Since the masses must be determined empirically, there is no point in keeping this as a separate term. Thus we may conclude that the rotation effects arising from the metric itself may be neglected in our order of approximation.

Finally, we examine the extent to which a body fails to follow a geodesic because of its rotation. Once again the detailed equations are unnecessary for a demonstration that we may neglect corrections from this source. It suffices to say that the leading correction to the acceleration goes as the angular momentum per unit mass, involves the Riemann tensor, and is of course a $1/c^2$ effect. With the angular momentum per unit mass of the moving body A written as $\omega_A R_A^2$, the correction to the acceleration of body A must have the general form

$$\frac{Gm_B}{\Delta_B^2}\frac{R_A}{\Delta_B}\frac{\omega_A R_A}{c}\frac{(\text{some velocity})}{c} \tag{4.9}$$

for a mass m_B which contributes to the metric. This is the same type of expression we have met before and agreed to neglect. To our order the specifically general relativistic effects due to rotation are, unfortunately, negligible.

C. The Hill-Brown Calculation

Once one has arrived at the approximate Lagrangian for the earth-moon relative motion, the application of the Hill-Brown technique is a straightforward (though tedious) matter. The general method is carefully presented in Brown's *Treatise*[13] and the procedure with velocity-dependent terms is handled very nicely by Brumberg in his paper. However, a few words about the expansion procedures and further approximations are in order.

The elimination of the solar mass from the Lagrangian by the use of Eq. (4.4) introduces the parameters n' and a'. Since the sun is assumed in our approximation to follow an ellipse, one may regard $\mathbf{R}$ and $\mathbf{V}$ as known functions of n', a', the solar eccentricity e', the longitude of the solar perigee, and the solar mean anomaly. We expand the Lagrangian in powers of e', retaining the zeroth- and first-order terms, and in powers of $1/a'$.

The relativistic corrections contain the common factor $1/c^2$. Since the subsequent calculations are to be done to first order in $1/c^2$, it is convenient to follow Brumberg and to introduce a dimensionless parameter μ which characterizes the relativistic terms. We define

$$\mu \equiv (n'a')^2/c^2. \tag{4.10}$$

Thus μ is tantamount to $GM_\odot/c^2a'$, the difference being of order $1/c^4$ and hence negligible. The numerical value of μ is very closely 10^{-8}. After the introduction of μ the relativistic terms can be arranged according to powers of e' and $1/a'$.

It is now a matter of bringing the formidable apparatus of the Hill-Brown procedure to bear on the truncated Lagrangian. Brumberg has shown that an intermediate orbit in Hill's sense does exist even when the $1/c^2$ relativistic corrections are included in the problem. So one may compute the Hill-Brown series here with as much confidence as in the purely Newtonian case.

In general everything goes through as in the Newtonian problem. A single prominent exception arises in the computation of the secular motion of the lunar perigee. The presence of velocity-dependent terms in the relativistic part of the Lagrangian appears to prevent the reduction of the problem to a form of Hill's equation. Brumberg avoided this difficulty by a judicious change of variables at the start which eliminated velocities from the relevant part of the Lagrangian. With our desire to retain the coefficients γ, β', η, and η' as distinct labels, this stratagem is precluded for us. Consequently we have been forced to calculate the relativistic corrections to the perigee motion simultaneously with the determination of the corrections to the relative position which depend on the lunar eccentricity. This results in a rather awkward scheme of consistent successive approximations. So we have carried it through to second order only in the ratio $\tilde{m} \equiv n'/(n-n')$, where n is the lunar mean motion. In an appendix we discuss the relativistic corrections to the secular motion of both the lunar perigee and node and their connection with the general precession.

D. The Expression for the Relative Separation

Once the Hill-Brown coefficients have been determined in the auxiliary coordinate system rotating with angular frequency equal to the mean solar motion n', the determination of the magnitude of the relative separation is reasonably straightforward. It is really a matter of transforming from Cartesian coordinates in the rotating frame to polar coordinates in the geocentric frame.

In expressing r we use a notation involving the Delaunay angular variables and the Hill-Brown orbit parameters. The angular variables are differences in mean longitudes, as follows:

$$l = \text{moon minus lunar perigee},$$
$$l' = \text{sun minus solar perigee},$$
$$D = \text{moon minus sun},$$

with "mean longitude of" understood. To avoid ambiguity we give approximate numerical values with the listing of the Hill-Brown parameters[13a]:

$\tilde{e} \approx 0.1095$,	constant of lunar eccentricity
$e' \approx 0.0168$,	solar eccentricity
$\tilde{a} \approx 3.8 \times 10^{10}$ cm,	Hill's scale factor (of the order of the mean solar distance)
$\alpha \equiv \tilde{a}/a'$ ≈ 0.0025,	ratio of Hill's scale factor to the solar semimajor axis (Ref. 14)
$\tilde{m} \equiv n'/(n-n')$,	where n' is the mean solar motion and n the mean lunar motion.

With these definitions established, we can write down the theoretical expression for the $O(1/c^2)$ relativistic corrections to $r/\tilde{a}$.

[13] Ernest W. Brown, *An Introductory Treatise on the Lunar Theory* (Dover Publications, Inc., New York, 1960).

[13a] The quantities $\tilde{e}$, $\tilde{a}$, $\tilde{m}$ correspond to e, a, m in Brown's notation (Ref. 13).

[14] This differs from the usual definition in that the factor $(M_\oplus - M_☾)/(M_\oplus + M_☾)$ has been set to unity, which is appropriate since we are neglecting the lunar mass in the relativistic corrections.

$$
\begin{aligned}
\frac{r}{\tilde{a}} = & \left\{1+O(\tilde{m}^4)+\mu\left[-\tfrac{1}{12}(4\gamma+6\beta'+16\eta-\eta'+2)+\tfrac{1}{3}(2\gamma+1)\tilde{m}+(47/32)\tilde{m}^2-\frac{\alpha^2}{\tilde{m}^2}\right]\right\} \\
& +\left\{\tfrac{15}{16}\alpha\tilde{m}+\mu\left[(1+\gamma-2\eta)\frac{\alpha}{\tilde{m}^2}+\tfrac{1}{2}(9\gamma-4\beta'-16\eta-\eta'+12)\frac{\alpha}{\tilde{m}}-\tfrac{7}{2}\alpha\right]\right\}\cos D \\
& +\{-\tilde{m}^2+\mu[(\tfrac{1}{4}+\tfrac{1}{12}(1-\eta'))+(7/18)(1-\eta')\tilde{m}+(17/4)\tilde{m}^2]\}\cos 2D+\{O(\tilde{m}^4)+\mu[\tfrac{7}{32}\tilde{m}^2]\}\cos 4D \\
& +\tilde{e}\Big[\{-\tfrac{1}{2}+\mu[(1/24)(4\gamma+6\beta'+16\eta-\eta'+2)-(233/28)\tilde{m}]\}\cos l \\
& +\{-\tfrac{17}{32}\tilde{m}^2+\mu[\tfrac{3}{16}+(13/3\times 2^6)(1-\eta')]\}\cos(2D+l) \\
& +\left\{-\tfrac{15}{16}\tilde{m}+\mu\left[\tfrac{1}{32}(1-\eta')\frac{1}{\tilde{m}}+(-\tfrac{5}{16}+(85/3\times 2^7)(1-\eta'))+(165/64)\tilde{m}\right]\right\}\cos(2D-l) \\
& +\{-(255/2^8)\tilde{m}^3+\mu[(45/2^7)\tilde{m}]\}\cos(4D-l)\Big] \\
& +e'[\{\tfrac{3}{2}\tilde{m}^2+\mu[\tfrac{1}{2}(-8\gamma+16\eta-\eta'-10)]\}\cos l'+\{\tfrac{1}{2}\tilde{m}^2+\mu[-\tfrac{5}{4}\tilde{m}]\}\cos(2D+l') \\
& \qquad +\{-\tfrac{7}{2}\tilde{m}^2+\mu[(\tfrac{1}{2}+\tfrac{1}{6}(1-\eta'))+\tfrac{5}{4}\tilde{m}]\}\cos(2D-l')]. \qquad (4.11)
\end{aligned}
$$

For the distinct harmonics we have included the leading Newtonian term or indicated its magnitude. In writing out the relativistic terms we have at some point in the series in ascending powers of $\tilde{m}$ given the labeling coefficients γ, β', η, and η' the explicit numerical value of unity. The expressions would otherwise be even more cumbersome. For terms involving $\tilde{e}$ and e', the points were $O(\tilde{m})$; for α, $O(1)$; for the constant term, $O(\tilde{m}^2)$. In all cases higher-order terms were neglected, even where this meant the loss of a particular harmonic.

Before we inquire into the observational significance of this result a few points require discussion. First, in the process of transforming from the Hill-Brown representation to this polar form, contributions with the coefficients $\tilde{e}^2$, $\tilde{e}e'$, $(e')^2$, $\tilde{e}\alpha$, and $e'\alpha$ appear in r. They arise as the result of the multiplication and division of the original series. On the grounds of consistency we have not included them here. The reason is that terms with these coefficients would also appear in the original Hill-Brown series were we to carry the calculation through at a higher level of approximation. Since we have computed none of these terms directly, we cannot justify inclusion of merely some terms of their generic form in the expression for r. There is no reason to expect that terms of this form will make significant relativistic contributions. Certainly when those terms which arise through manipulation of the series are given the Einstein values of the labeling coefficients, the contributions from $\tilde{e}^2$ and $\tilde{e}e'$ are down at the 10-cm level, with the others considerably smaller. However, their general insignificance could be asserted positively only after a much more elaborate calculation, one which seemed unwarranted at this exploratory stage.

The same line of reasoning applies to contributions to r dependent on the inclination of the lunar orbit to the ecliptic. The contributions will appear in lowest order as proportional to the square of the constant of inclination. This means that one can expect the relativistic parts to be of order 5×10^{-11} or less relative to the mean lunar distance and hence negligible at the 10-cm level of relevance. Calculation of some of these contributions bears out this analysis.

Another point pertains to the relativistic corrections hidden in the angle l. This variable involves the secular motion of the lunar perigee, and the theoretical expression for the secular motion contains small relativistic corrections. Consequently, what appears to be a purely Newtonian term in $r/\tilde{a}$ may actually have a time dependence involving relativistic effects. However, this is largely a point of principle. In practice the relativistic effect is masked by the uncertainties in the Newtonian contributions to the secular motion, the primary difficulty arising from the unknown mass distribution within the moon.

A final point involves the definition of the scale parameter $\tilde{a}$. A Hill-Brown calculation with relativistic corrections is most conveniently done with its own characteristic linear scale parameter. The connection between Hill's scale parameter for the purely Newtonian problem and the convenient scale parameter for the relativistic problem can be established at the end by relating both to the parameter a defined by $n^2a^3 \equiv G(M_{\oplus}+M_{☾})$. The transformation from the relativistic linear scale parameter of the actual calculation to the Hill scale parameter has been performed for the $r/\tilde{a}$ expression given in Eq. (4.11). Thus the $r/\tilde{a}$ expression is couched, to appropriate order, in terms of the standard Hill scale factor of the Newtonian main problem of the lunar theory. Hence one can be sure that no spurious effects due to linear scale factors with different definitions have crept in.

To assess the magnitude of the corrections to $r/\tilde{a}$ arising from the Einstein theory we must give all the coefficients γ, β', η, and η' the value unity. This leads to a striking number of cancellations. For example, in the coefficient of $\cos D$ the leading relativistic terms, proportional to $\mu\alpha/\tilde{m}^2$ and $\mu\alpha/\tilde{m}$, vanish.

With the coefficients given their values from the Einstein theory, the dominant relativistic correction in a periodic term is in the $\cos 2D$ term:

$$\{-\tilde{m}^2+\cdots+\mu[\tfrac{1}{4}+(17/4)\tilde{m}^2]\}\cos 2D. \quad (4.12)$$

The contribution $\frac{1}{4}\mu$ amounts to a nominal amplitude of some 100 cm.

In potential significance the $\cos 2D$ term is followed by the $\cos l$ term. The $\cos(2D\pm l)$ and $\cos l'$ terms are next in line; these, however, are down at the 10-cm level.

Although there is a sizeable general relativistic contribution to the constant term in $r/\tilde{a}$, this is experimentally inaccessible because the lunar radius (as well as the earth radius) contributes a constant term to the round trip laser pulse time. One cannot with the laser experiment distinguish the relativistic part in the constant term of r from a very small correction to the distance of the corner reflector from the lunar c.m.

The relativistic contributions in the periodic terms cannot be brushed aside with the argument that they are artifacts arising from a particular choice of the *a priori* arbitrary coordinates used in the general relativistic calculation. In Sec. III we showed that the conversion from elapsed coordinate time for the laser pulse to proper time was, to sufficient accuracy, merely multiplication by a *constant* factor. So the periodic terms in r are tantamount to periodic terms in the elapsed proper time. The ratio of two such terms is both coordinate-invariant and independent of precise knowledge of the local speed of light. (The frequencies themselves are coordinate-dependent, but for the lunar case no ambiguity can arise at the observational level because the important frequencies are well separated.) Inspection of Eq. (4.11) is then sufficient to show that the ratios of the periodic terms are different from the corresponding ratios in the purely Newtonian case.

To give a particularly clear example, we look at the ratio of the $\cos(2D+l)$ amplitude to the $\cos(2D-l)$ amplitude. This ratio is independent of both the scale parameter $\tilde{a}$ and the lunar eccentricity constant $\tilde{e}$. The leading terms in the ratio go as follows:

$$(17/30)\tilde{m}+\cdots+\mu\left(-\frac{1}{5\tilde{m}}+\cdots\right)$$

$$\approx(17/30)\tilde{m}\left[1+\mu\left(-\frac{6}{17\tilde{m}^2}\right)\right]. \quad (4.13)$$

This ratio departs by a (relativistically speaking) considerable factor, of order 1.5×10^{-7}, from the corresponding ratio of the Newtonian theory. (The presence of $\tilde{m}$ cannot vitiate this conclusion, for $\tilde{m}$ is a ratio of observed frequencies and hence is insensitive to changes in the time coordinate used.)

To extricate the coefficient of the $\cos 2D$ term in the lunar motion from the round-trip proper time data for comparison with Eq. (4.12) requires a comprehensive analysis. In particular, improved values of the orbital parameters and of the parameters for the physical libration of the moon must be determined simultaneously. The coefficient must be extracted effectively as part of a ratio in which the speed of light cancels, as well as the linear scale factor $\tilde{a}$. Yet it appears to be possible with the laser-ranging experiment to determine the coefficient of the $\cos 2D$ term with sufficient accuracy to be of interest.[15]

In view of this possibility of measurement it is unfortunate that the non-Newtonian parts of the metric play so small a role in determining the relativistic part of the $\cos 2D$ coefficient. The bulk of the relativistic contribution is independent of those parts of the metric labeled by γ, β', η, and η'. This means that one is checking on the correctness of the geodesic equation, to $O(1/c^2)$, as the equation of motion for bodies in a gravitational field of simple Newtonian structure. This is more than another test of special relativistic dynamics, for special relativity theory fails to provide an unambiguous prescription for the incorporation of even a Newtonian potential $(-U)$ into the equations of motion.

V. CONCLUSION

The foregoing calculations provide an assessment of some possibilities opened by laser ranging for testing Einstein's gravitational theory. On the question of general relativistic effects in the light propagation itself we conclude that there is no significant hope of observing such effects with the round-trip proper time data. General relativistic effects in the lunar motion are quite a different matter. There is a realistic possibility of detecting the relativistic contribution to the $\cos 2D$ term in the earth-moon separation. Observation of this would provide a significant test of the correctness of the geodesic equation, to $O(1/c^2)$, for describing the motion of bodies in a gravitational field.

The promising results of this exploratory calculation indicate that a more extensive treatment is desirable. This should be commensurate with the increased precision of the Newtonian solution for the three-body problem provided by the work of Eckert and Smith.[16] In part this means a calculation to higher order in the orbital parameters. For comparison with a rival gravi-

[15] P. Bender (private communication).

[16] W. J. Eckert and H. F. Smith, in *International Astronomical Union Symposium No. 25*, edited by G. Contopoulos (Academic Press Inc., New York, 1966), p. 242.

tational theory such as the scalar-tensor theory of Brans and Dicke, the metric used in the orbit calculation should be generalized to include the terms from both theories with distinguishing coefficients. Work toward this end is in progress.

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APPENDIX

This Appendix deals with the connection between the relativistic corrections to the lunar secular motions and to the general precession. First we present the general relativistic contributions to the secular motion of the lunar node and perigee as they emerge from our Hill-Brown calculation.

$$\frac{d\Omega}{dt}=-\tfrac{3}{4}nm^2+\cdots+n\mu\{-\tfrac{1}{4}(1-\eta')+[\tfrac{1}{2}(2\gamma+1)-\tfrac{3}{16}(1-\eta')]m-(117/64)m^3+\cdots\},$$

$$\frac{d\pi}{dt}=+\tfrac{3}{4}nm^2+\cdots+n\mu[\tfrac{1}{2}(2\gamma+1)m+O(m^3)+(\alpha^2/m^2)(2+2\gamma-\beta')+\cdots].$$

For clarity we have inserted the leading Newtonian term. Here m is defined as $m\equiv n'/n$. In both expressions the coefficients labeling parts of the metric have been given their Einstein values of unity in the terms of order m^2 and higher. The term in α^2/m^2 in the perigee motion was calculated separately by a variation of orbital elements procedure. Because of the difficulties in our calculation of the perigee motion, alluded to earlier, we have not carried it through to $O(m^3)$.

If we give the remaining labels their Einstein values and drop the Newtonian terms, the expressions reduce to the following:

$$\frac{d\Omega}{dt}=n\mu[\tfrac{3}{2}m-(117/64)m^3+\cdots],$$

$$\frac{d\pi}{dt}=n\mu[\tfrac{3}{2}m-(1245/64)m^3+3\alpha^2/m^2+\cdots].$$

The $O(m^3)$ term in the perigee we have adopted from Brumberg's paper.[17]

We focus our attention on the second set of equations. In both the $\dot{\pi}$ and $\dot{\Omega}$ equations the (identical) leading terms with the factor $nm=n'$ are independent of n and thus of the details of the lunar motion. These strictly solar effects of some 1.91 in./century are often referred to as the geodesic precession effects. The terms in m^3 are combined luni-solar effects. The α^2 term in $\dot{\pi}$ is an effect due to the earth alone. (The over-all factor of $n\mu\alpha^2/m^2$ is independent of the solar parameters.) No comparable term appears in $\dot{\Omega}$ because even the relativistic corrections to the direct earth-moon interaction produce no acceleration perpendicular to the instantaneous orbital plane.

The combination of these small relativistic corrections with the manifold Newtonian effects then gives a *theoretical* secular motion relative to some *fixed inertial* system. This is to be compared with the observational results reduced to the corresponding inertial system. In practice this means that the *observations* of $\dot{\pi}$ and $\dot{\Omega}$ relative to the moving equinox must be corrected with the *observed* general precession of the equinox. As described here, the confrontation for either $\dot{\pi}$ or $\dot{\Omega}$ is between a single theoretical value and the appropriate difference of two observational values.

In the literature[18] a question has arisen over whether one should exclude the "geodesic precession" terms from the theoretical expressions for $\dot{\pi}$ and $\dot{\Omega}$. The confusion arises because relativity theory does predict a contribution to a *theoretical* calculation of the general precession with the same numerical value as the *leading* terms above. It bears the same name, geodesic precession, and the numerical agreement is by no means coincidental. Nonetheless, one is dealing with two distinct phenomena.

In the case of the general precession the Einstein theory predicts that the angular momentum of the earth undergoes a characteristically relativistic precession as the earth moves in its orbit about the sun. Mathematically, the angular-momentum four-vector of the earth is Fermi-Walker propagated along the earth's world line. The essential net result is that the component of angular momentum lying in the plane of the orbit precesses with a frequency given (to lowest order) by $\tfrac{3}{2}n'\mu$ rad/sec relative to some fixed inertial system. The appropriate inertial system is the asymptotically flat space at spatial infinity, here tantamount to the frame of the fixed stars. The sense of this precession is the same as that of the orbital motion of the earth about the sun. Thus, *if* one were concerned with a *theoretical* calculation of the general precession, one would have a relativistic contribution of 1.91 in./century to the direct motion of the zero of longitude along the ecliptic.

[17] The $O(m^3)$ term in the nodal motion agrees with Brumberg's when a transcription error in his paper is corrected, one involved in the passage from his Eqs. (48) to (48′).

[18] W. J. Eckert, Astron. J. **70**, 787 (1965).

The numerical agreement between the general precession contribution and the leading terms in $\dot{\pi}$ and $\dot{\Omega}$ should now not be surprising. If one wanted a quick result for the major relativistic solar contribution to the lunar secular motions, one would regard the earth-moon system as a composite spinning body whose center of mass moves in a circular orbit around the sun. The angular momentum of this composite system must precess just as does that of the spinning earth. Thus the leading terms in $\dot{\pi}$ and $\dot{\Omega}$ would appear directly.

This stratagem, however, would give only part of the relativistic contributions to the lunar secular motions. As the results quoted above show, application of the Hill-Brown procedure produces the primary terms plus additional (smaller) terms in both $\dot{\pi}$ and $\dot{\Omega}$. Moreover, these additional contributions are different for the two secular motions. At most, the solar parts of one of $\dot{\pi}$ or $\dot{\Omega}$ could be numerically equal to the geodesic precession result for the earth's axis. In point of fact, neither is. Within the context of the approximations made in our Hill-Brown calculations of the lunar secular motions, notably neglect of the lunar mass in $O(1/c^2)$ terms, the equations for calculating the relativistic contribution to the general precession are independent of the moon. This means that no term with a coefficient like $n\mu m^3$ can appear; so the final expressions must be different.

This discussion has been presented largely for the sake of clarifying a matter of principle. It remains true that the lunar secular motions are unlikely to provide a test of relativity theory. Certainly they will not until we know a good deal more about the mass distribution within the moon.

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Physical Consequences of Fock's Conformal Hypothesis

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It is shown that, when taken literally, Fock's method of reinterpreting the spatial metric in Einstein's general theory of relativity leads to the result that clocks do not measure proper time, and consequently to disagreement with gravitational red-shift experiments.

FOR static spaces, Fock has suggested[1] that in the metric[2]

$$ds^2=c^2V^2dt^2-g_{\mu\nu}dx^\mu dx^\nu\,,\quad \text{(Ref. 3)} \tag{1}$$

where $c^2V^2\equiv g_{00}$, one should interpret the spatial metric to be

$$d\sigma^2=h_{\mu\nu}dx^\mu dx^\nu\,, \tag{2}$$

where

$$h_{\mu\nu}=-V^{-2}g_{\mu\nu}$$

instead of the more usual[4–6]

$$dl^2=-g_{\mu\nu}dx^\mu dx^\nu. \tag{3}$$

He offers a generalization to nonstatic spaces that reduces to (2) whenever the metric is time-orthogonal. We shall therefore feel free to consider his assumption in all spaces admitting time-orthogonal metrics, although the results in the static case are startling enough. Note that

$$ds^2=c^2V^2dt^2-V^{-2}d\sigma^2 \tag{4}$$

and

$$d\sigma^2=V^2dl^2=g_{00}dl^2/c^2. \tag{5}$$

[Fock's Eqs. (2.03) and (2.08)].

At first glance, definition (2) might seem to lead to no new physical results. For example, since all the geodesics remain the same, planetary motions, the deflection of light by a gravitational field, and many other results come out the same as under definition (3). Indeed, Fock does not state that his theory leads to any results different from those of general relativity. It will here be shown that, taken at face value, Fock's theory has different physical consequences than the general theory of relativity, and that these consequences are in contradiction with experiment. It is possible, of course, to consider Fock's redefinition as a purely mathematical device, in which case it has no physical consequences. His later work (Ref. 3) has more of this flavor. To the author, Ref. 1 seems to make an explicitly physical redefinition. The most serious result in

[1] V. Fock, Zh. Eksperim. i Teor. Fiz. 38, 1476 (1960) [English transl.: Soviet Phys.—JETP 11, 1067 (1960)]; *Recent Developments in General Relativity* (Pergamon Press, Inc., New York, 1962).

[2] The term "a space" will denote a space-time manifold. The term "spatial" will be used to designate the splitting off of a three-dimensional metric.

[3] Greek indices are summed 1–3, Roman 0–3. We do *not* follow Fock's practice of writing dx_i for the *controvariant* vector dx^i. See, e.g., his book, *The Theory of Space, Time, and Gravitation* (The Macmillan Company, New York, 1964), p. 122.

[4] A. Einstein, in *The Principle of Relativity* (Dover Publications, Inc., New York), p. 129.

[5] C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), p. 238.

[6] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), 2nd ed., p. 273.