form (2.8) which these vertices possess, the uniform orientation guarantees that at least one of the vertices in each infrared loop is proportional to an infrared momentum.

We conclude this section by repeating Weinberg's calculation of B in the nonrelativistic limit and correcting a minor mistake in his result. The quantity v_{nm}^2 is first expanded in the form

$$v_{nm}^2=(\mathbf{v}_n-\mathbf{v}_m)^2-\mathbf{v}_n^2\mathbf{v}_m^2+2(\mathbf{v}_n^2+\mathbf{v}_m^2)\mathbf{v}_n\cdot\mathbf{v}_m-3(\mathbf{v}_n\cdot\mathbf{v}_m)^2+\cdots, \quad (8.9)$$

where $\mathbf{v}_n=\mathbf{p}_n/E_n$. This expansion is then inserted into

$$\frac{1+v_{nm}^2}{(1-v_{nm}^2)^{1/2}}\frac{\eta_n\eta_m m_n m_m}{v_{nm}}\ln\frac{1+v_{nm}}{1-v_{nm}}=2\eta_n\eta_m m_n m_m\left(1+\frac{11}{6}v_{nm}^2+\frac{63}{40}v_{nm}^4+\cdots\right) \quad (8.10)$$

to obtain a lengthy expression for B correct to the fourth order in the velocities. This expression can be greatly simplified with the aid of the energy-momentum conservation laws

$$\sum_n \eta_n m_n(1+\tfrac{1}{2}\mathbf{v}_n^2+\tfrac{3}{8}\mathbf{v}_n^4+\cdots)=0,$$

$$\sum_n \eta_n m_n \mathbf{v}_n(1+\tfrac{1}{2}\mathbf{v}_n^2+\cdots)=0,$$

and one finally obtains the compact formula

$$B=(4G/5\pi)\,\mathrm{tr}(\Delta d^2\mathbf{Q}/dt^2)^2, \quad (8.11)$$

where $\Delta d^2\mathbf{Q}/dt^2$ is the dyadic previously defined by Eqs. (4.11) and (4.12), having the explicit traceless form[39]

$$\Delta d^2\mathbf{Q}/dt^2=\sum_n \eta_n m_n(\mathbf{v}_n\mathbf{v}_n-\tfrac{1}{3}\mathbf{1}\mathbf{v}_n^2). \quad (8.12)$$

[39] By inadvertently dropping a term Weinberg obtains a dyadic which is not traceless.

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Quantum Statistics of Coupled Oscillator Systems

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The statistical properties of systems of coupled quantum-mechanical harmonic oscillators are analyzed. The Hamiltonian for the system is assumed to be an inhomogeneous quadratic form in the creation and annihilation operators, and is allowed to have an explicit time dependence. The relationship to classical theory is emphasized by expressing pure states in terms of the coherent-state vectors, and density operators by means of the P representation and an analogous representation involving the Wigner function. The state which evolves from an initially coherent state of the system is found, and equations governing the time evolution of the Wigner function and the weight function for the P representation are derived, in differential and integral form, for arbitrary initial states of the system. The results remain valid for couplings which do not preserve the vacuum state, and for cases in which the time dependence of the coupling parameters gives rise to large-scale amplification of the initial field intensities. The analysis is performed by first treating general linear inhomogeneous canonical transformations on the oscillator variables, and then specializing to the case in which these transformations represent the solutions for the Heisenberg operators in terms of their initial values. The results are illustrated within the context of a model of parametric amplification.

I. INTRODUCTION

In a wide variety of physical processes, all of the dynamical elements which enter into the description of the state of the system may be treated formally as quantum-mechanical harmonic-oscillator modes. The coupling between the modes typically takes the form of a quadratic expression in the annihilation and creation operators $a_j(t)$ and $a_j^\dagger(t)$, in which the coupling parameters are time-dependent in the general case. In addition, driving terms linear in the oscillator variables may be present. The operators $a_j(t)$ and $a_j^\dagger(t)$ then obey linear inhomogeneous equations of motion, and the solutions to these equations take the same form as the solutions for the c-number complex amplitudes in the analogous classical system. The time-dependent expectation values of dynamical operators for a given initial state of the system may be evaluated straightforwardly with the aid of the solutions to the Heisenberg equations of motion and the commutation relations for a_j and $a_j^\dagger$, and some indication is thereby provided of the way in which quantum fluctuations influence the time development of the oscillator system.[1–5]

* National Science Foundation postdoctoral fellow.

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It is convenient for many purposes to be able to find the probability that a physical measurement will yield a particular value, rather than to be able merely to evaluate the moments of dynamical operators. In such cases it is desirable to describe the state of the system in the Schrödinger picture, i.e., in terms of a time-dependent state vector or density operator. The most common basis for the description of the quantum states of harmonic oscillators is the set of occupation-number eigenstates or n-quantum states, which are the eigenstates of the free Hamiltonian. In cases in which the effect of the coupling is small during the time interval in question, a perturbation-theory calculation may be used to evaluate the transition probabilities between states of the unperturbed system, which must be taken to be stationary, i.e., diagonal in the n-quantum representation. The n-quantum states do not, however, provide a very convenient basis for describing the state of coupled oscillator systems in cases in which the coupling gives rise to large-scale coherent changes in the oscillator amplitudes. The equations governing the time evolution of the matrix elements $\rho_{nm}(t)$ of the density operator are rather difficult to solve exactly, especially when the coupling parameters are allowed to have an arbitrary time dependence. A more serious drawback is that the n-quantum states are not appropriate for describing states in which one has some information about the phase of oscillation of the complex amplitudes, and are therefore not suitable for making comparisons between quantum and classical theory.

A formulation of the quantum mechanics of harmonic oscillators has been constructed[6] in terms which allow quantum-mechanical calculations to be made along lines analogous to classical ones. It is based on the coherent states, which are eigenstates of the annihilation operators a_j. The coherent states form a complete though nonorthogonal, set in terms of which arbitrary state vectors and operators may be expressed. It is found that a broad class of density operators may be written in the P representation,[6,7] which is a diagonal mixture of pure coherent states. The weight function $P(\alpha)$ for the P representation plays a role in the calculation of certain statistical averages analogous to that played by the phase-space distribution function in classical theory. In this respect, the P function is similar to the Wigner function $W(\alpha)$, a quantum-mechanical distribution function which may be shown to be the expansion function for the density operator in terms of a certain complete set of Hermitian operators.[8–10] The coherent states and the quasiprobability functions $P(\alpha)$ and $W(\alpha)$ provide a natural basis for the description of the time-dependent state of coupled oscillator systems. We find that the solution to Schrödinger's equation may be simply expressed within this framework, in terms of the c-number functions of time which define the solutions to the Heisenberg equations of motion for the oscillator variables, and which therefore define the solutions for the oscillator amplitudes in the analogous classical system. The time-dependent state of the oscillator system may thus be expressed in terms of functions which bear the strongest possible resemblance to the functions which arise in classical theory.[11,12]

The simplest case we treat is the one in which the quadratic part of the Hamiltonian has the form $\sum \omega_{jk}(t)a_j^\dagger(t)a_k(t)$, where the coupling parameters $\omega_{jk}(t)$ may be allowed to have an arbitrary time dependence. If driving terms linear in $a_j(t)$ and $a_j^\dagger(t)$ are also present, the solution to the Heisenberg equations of motion takes the form

$$a_j(t)=\sum_k u_{jk}(t)a_k+\mathcal{A}_j'(t)\,, \tag{1.1}$$

where $u_{jk}(t)$ and $\mathcal{A}_j'(t)$ are c-number functions of time, and $a_j\equiv a_j(0)$. Solutions of this form correspond to a particularly simple behavior of the state of the system in the Schrödinger picture: An initially coherent state remains coherent at all times,[13,14,9] and the time-dependent functions $P(\{\alpha_j\},t)$ and $W(\{\alpha_j\},t)$ corresponding to an arbitrary Schrödinger density operator $\rho(t)$, obey Liouville's equation.

The solution to Schrödinger's equation is somewhat more complicated when the Hamiltonian contains terms of the form $\sum \sigma_{jk}(t)a_j^\dagger(t)a_k^\dagger(t)+\text{H.c.}$, which are necessary for the description of such phenomena as Raman and Brillouin scattering, and which appear in a simple linear model of the parametric amplifier.[2] When couplings of this kind are present, the Heisenberg operators may be expressed in terms of their initial values in the form

$$a_j(t)=\sum_k [u_{jk}(t)a_k+v_{jk}(t)a_k^\dagger]+\mathcal{A}_j'(t)\,. \tag{1.2}$$

We find that when the solutions to the equations of motion take this form, an initially coherent Schrödinger state vector does not remain coherent at later times. The Wigner function $W(\{\alpha_j\},t)$, however, may be shown to obey Liouville's equation even in this more general case. The function $P(\{\alpha_j\},t)$, on the other hand, obeys an equation containing, in addition to the usual Liouville terms, certain second derivatives with respect to the variables α_j and α_j^*.

The unitary time translation operator $U(t)$ which defines the solution to Schrödinger's equation may also be

[6] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[7] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[8] U. Fano, Rev. Mod. Phys. **29**, 74 (1957), Sec. 9.
[9] B. R. Mollow, thesis, Harvard University, 1966 (unpublished).
[10] K. Cahill and R. J. Glauber (to be published).

[11] D. Holliday and A. E. Glassgold [Phys. Rev. **139**, A1717 (1965)] have employed methods similar to those used in this paper to solve for the reduced density operator which describes a single mode of the radiation field coupled to a system of two-level molecules.
[12] The time evolution of the density operator for the two-mode parametric amplifier has been found by the author and R. J. Glauber [Phys. Rev. **160**, 1076 (1967); **160**, 1097 (1967)].
[13] R. J. Glauber, Phys. Letters **21**, 650 (1966).
[14] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. Letters **22**, 574 (1966).

used to generate the solution to the Heisenberg equations of motion, by means of the relation $a_j(t)=U^{-1}(t)\times a_jU(t)$. The operator $U(t)$ is in fact determined to within a (time-dependent) phase factor by the solution (1.2) for $a_j(t)$. The behavior of the state of the system in the Schrödinger picture may therefore be found by investigating the behavior of state vectors and density operators under canonical transformations defined by a unitary operator U which generates the linear inhomogeneous transformation

$$U^{-1}a_jU=\sum_k (u_{jk}a_k+v_{jk}a_k^\dagger)+\mathcal{A}_j', \tag{1.3}$$

and then identifying the quantities u_{jk}, v_{jk}, and $\mathcal{A}_j'$ with the functions $u_{jk}(t)$, $v_{jk}(t)$, and $\mathcal{A}_j'(t)$, evaluated at some time t. When the analysis is performed in this way, the results may also be used to carry out transformations of variables at a given time, which may be useful for the purpose of simplifying the Hamiltonian in systems with more general couplings.[15]

The second section of this paper contains an outline of the basic properties of the coherent states and the quasiprobability functions $W(\alpha)$ and $P(\alpha)$. It is shown how the function $W(\alpha)$ may be used to expand the density operator in terms of an appropriate set of Hermitian operators. In Sec. III canonical transformations of the form (1.3) are introduced, and the behavior of state vectors and density operators under such transformations is discussed in Secs. IV and V, respectively. These results are then used in Sec. VI to solve Schrödinger's equation for systems of coupled oscillators. Differential equations governing the time evolution of the functions $W(\{\alpha_j\},t)$ and $P(\{\alpha_j\},t)$ for such systems are derived. The behavior of these functions when the Hamiltonian is taken to have the form characteristic of parametric interactions is discussed in Sec. VII.

II. QUANTUM STATES FOR HARMONIC OSCILLATORS

The comparison of the quantum and classical mechanics of harmonic oscillators has been greatly facilitated by the introduction of the coherent states[6] for the oscillator system. A coherent state with complex eigenvalue α is defined to be an eigenstate of the annihilation operator a

$$a|\alpha\rangle=\alpha|\alpha\rangle, \tag{2.1}$$

and is given by the expression

$$|\alpha\rangle=e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}(n!)^{-1}(\alpha a^\dagger)^n|0\rangle, \tag{2.2}$$

where $|0\rangle$ is the ground state or vacuum state. The coherent states form a complete set; arbitrary state vectors and operators may be expressed in terms of the coherent-state vectors by means of the completeness relation[16,6]

$$\pi^{-1}\int d^2\alpha|\alpha\rangle\langle\alpha|=1, \tag{2.3}$$

where $d^2\alpha=d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha)$. The coherent states are not orthogonal, however. The inner product of two coherent states with complex eigenvalues α and β is

$$\langle\beta|\alpha\rangle=e^{-\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha|^2+\beta^*\alpha}. \tag{2.4}$$

The coherent states may be expressed in terms of the unitary displacement operator[6]

$$D(\alpha)\equiv\exp(a^\dagger\alpha-\alpha^*a) \tag{2.5}$$

in the form

$$|\alpha\rangle=D(\alpha)|0\rangle. \tag{2.6}$$

The operator $D(\alpha)$ may be shown to generate the displacement

$$D^{-1}(\alpha)aD(\alpha)=a+\alpha, \tag{2.7}$$

and to obey the multiplication rule

$$D(\alpha)D(\beta)=D(\alpha+\beta)e^{\frac{1}{2}(\beta^*\alpha-\alpha^*\beta)}. \tag{2.8}$$

It follows from this identity and Eq. (2.6) that the product of a displacement operator and a coherent state is a phase factor times a coherent state with a displaced complex amplitude,

$$D(\alpha)|\beta\rangle=|\alpha+\beta\rangle e^{\frac{1}{2}(\beta^*\alpha-\alpha^*\beta)}. \tag{2.9}$$

The displacement operators form a complete set in the sense that an arbitrary operator F has the representation[17]

$$F=\pi^{-1}\int d^2\eta\,\{\mathrm{tr}[FD(\eta)]\}D^{-1}(\eta), \tag{2.10}$$

the uniqueness of which follows from the identity

$$\pi^{-1}\,\mathrm{tr}[D(\eta)D^{-1}(\eta')]=\delta[\mathrm{Re}(\eta-\eta')]\delta[\mathrm{Im}(\eta-\eta')]\equiv\delta^{(2)}(\eta-\eta'). \tag{2.11}$$

The density operator for a single harmonic oscillator may thus be written in the form

$$\rho=\pi^{-1}\int d^2\eta\,\chi(\eta)D^{-1}(\eta), \tag{2.12}$$

where the characteristic function $\chi(\eta)$ is defined for arbitrary complex η as[18]

$$\chi(\eta)\equiv\mathrm{tr}[\rho D(\eta)]. \tag{2.13}$$

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[15] See, for example N. Bogolubov, J. Phys. U.S.S.R. **11**, 23 (1947); reprinted in *The Many-Body Problem*, edited by D. Pines, (W. A. Benjamin, Inc., New York, 1962), p. 292.

[16] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).

[17] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1953), p. 272–276.

[18] R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965), Lecture XIII.

In many cases of physical interest the density operator for harmonic-oscillator systems may be expressed as a statistical mixture of coherent states[6,7]

$$\rho=\int d^2\alpha\, P(\alpha)\,|\alpha\rangle\langle\alpha|\,, \tag{2.14}$$

which Glauber has called the P representation. The usefulness of the P representation derives from its role in evaluating normally ordered products. The expectation value of $a^{\dagger n}a^m$ in the P representation is given by the expression

$$\mathrm{tr}[\rho a^{\dagger n}a^m]=\int d^2\alpha\, \alpha^{*n}\alpha^m P(\alpha)\,, \tag{2.15}$$

the classical form of which permits one to investigate the coherence properties of electromagnetic fields in terms closely approximating those of classical theory.[6,18–21]

The P representation is far more general than its diagonal form would suggest. It may be shown that whenever the normally ordered characteristic function

$$\chi_N(\eta)\equiv\mathrm{tr}[\rho\,\exp(a^\dagger\eta)\,\exp(-\eta^* a)] \tag{2.16}$$

$$=e^{\frac{1}{2}|\eta|^2}\chi(\eta) \tag{2.17}$$

possesses a (two-dimensional) Fourier transform, then a P representation exists, and the weight function $P(\alpha)$ is given by the Fourier integral[22]

$$P(\alpha)=\pi^{-2}\int d^2\eta\, e^{\eta^*\alpha-\alpha^*\eta}\chi_N(\eta)\,. \tag{2.18}$$

For a wide class of density operators, this integral converges to a positive, square-integrable function $P(\alpha)$, and the representation of such quantum states by means of the expansion (2.14) is correspondingly straightforward. The function $P(\alpha)$ defined by Eq. (2.18) may, however, take on negative values for perfectly well-behaved quantum states, and may contain singularities at least as strong as those of a δ function and its derivatives.

A number of authors[23–25] have shown that if one allows the function $P(\alpha)$ to be even more singular than the tempered distributions, then in an abstract sense it may be said that a diagonal representation of the form (2.14) exists for all density operators. The mathematical operations one is allowed to perform on the weight function $P(\alpha)$ in such singular cases, however, are rather circumscribed,[26] and in many applications it is not possible to proceed without first ascertaining to which class of distribution $P(\alpha)$ belongs. In particular, in certain of the dynamical examples we shall discuss in this paper, the differential equation governing the time evolution of the P function for coupled oscillator systems leads to ambiguous results if it is construed as applying for times after the times at which singularities typically arise. Partly for this reason, and partly because the representation of density operators by highly singular weight functions is so very far from providing a means of understanding the classical aspects of quantum states, we shall say that a P representation exists only when the function $\chi_N(\eta)$ [and hence $P(\alpha)$] belongs to the class of tempered distributions.[27]

In more singular cases it is convenient to work with the Wigner function,[28] a species of quantum phase-space distribution which exists as a square-integrable, bounded function[10] for arbitrary density operators. The Wigner function may be defined for arbitrary complex argument α as the Fourier transform of the (ordinary) characteristic function[29,18]

$$W(\alpha)\equiv\pi^{-2}\int d^2\eta\, e^{\eta^*\alpha-\alpha^*\eta}\chi(\eta)\,. \tag{2.19}$$

Let us define the *symmetrized product* $\{a^{\dagger n}a^m\}_{\mathrm{sym}}$ as the sum of every possible ordering of n factors of $a^\dagger$ and m factors of a, divided by the total number $\binom{n+m}{m}$ of such orderings. Then it is easily shown[10] that the expectation value of such a product is

$$\mathrm{tr}[\rho\{a^{\dagger n}a^m\}_{\mathrm{sym}}]=\int d^2\alpha\, \alpha^{*n}\alpha^m W(\alpha)\,. \tag{2.20}$$

We may note that if a P representation exists, the Wigner function is given by the relation[18]

$$W(\alpha)=2\pi^{-1}\int d^2\alpha'\, e^{-2|\alpha-\alpha'|^2}P(\alpha')\,. \tag{2.21}$$

The functions $W(\alpha)$ and $P(\alpha)$, which Glauber has called quasiprobability functions, become equal to each other in the classical limit, and in that limit may be identified with the probability density for finding the oscillator with complex amplitude α.

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[23] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[24] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 734 (1965).
[25] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).
[26] K. Cahill, Phys. Rev. **138**, B1566 (1965).
[27] It should be borne in mind, however, that even the class of tempered distributions contains functions rather more singular than those which can be simply interpreted as analogs to the classical distribution function. In this connection, see R. Bonifacio, L. M. Narducci, and E. Montaldi, Phys. Rev. Letters **16**, 1125 (1966).
[28] E. Wigner, Phys. Rev. **40**, 749 (1932).
[29] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1948).

The identity (2.12) which expresses the density operator in terms of the characteristic function enables us to express it equally well in terms of the Wigner function. Let us define a Hermitian operator $T(\alpha)$ as the Fourier transform of the displacement operator $D(\eta)$:

$$T(\alpha)\equiv\pi^{-1}\int d^2\eta\; e^{\eta^*\alpha-\alpha^*\eta}D(\eta)\,. \tag{2.22}$$

Then it follows from Eqs. (2.19) and (2.13) that the Wigner function is given by

$$W(\alpha)=\pi^{-1}\,\mathrm{tr}[\rho T(\alpha)]\,, \tag{2.23}$$

and from Eq. (2.12) that the density operator has the representation[8–10]

$$\rho=\int d^2\alpha\; W(\alpha)T(\alpha)\,. \tag{2.24}$$

The latter relation is a consequence of the completeness of the set of operators $T(\alpha)$: It follows from Eq. (2.10) that any operator F may be written in the form

$$F=\pi^{-1}\int d^2\alpha\;\{\mathrm{tr}[FT(\alpha)]\}T(\alpha)\,. \tag{2.25}$$

The operator $T(\alpha)$ may be shown to obey the identities

$$\mathrm{tr}[\{a^{\dagger n}a^m\}_{\mathrm{sym}}T(\alpha)]=\alpha^{*n}\alpha^m \tag{2.26}$$

and

$$2\pi^{-1}\int d^2\alpha'\; e^{-2|\alpha-\alpha'|^2}T(\alpha')=|\alpha\rangle\langle\alpha|\,. \tag{2.27}$$

We may note that the operator $T(\alpha)$ is simply expressible in terms of a unitary reflection operator I, which may be defined by the relation

$$I|\alpha\rangle=|(-\alpha)\rangle\,, \tag{2.28}$$

or equivalently (to within a sign) by the relations

$$IaI=-a\,, \tag{2.29a}$$

$$I^2=1\,. \tag{2.29b}$$

It may be verified by direct evaluation of its coherent-state matrix elements that the operator $T(\alpha)$ is given in terms of the reflection operator I by the expression

$$T(\alpha)=2D(\alpha)ID^{-1}(\alpha)\,. \tag{2.30}$$

III. LINEAR TRANSFORMATIONS ON n-OSCILLATOR SYSTEMS

Let us consider a system consisting of n harmonic-oscillator modes; we denote by a_j and $a_j^\dagger$ the annihilation and creation operators, respectively, for the jth mode. These operators must satisfy the canonical commutation relations

$$[a_j,a_k^\dagger]=\delta_{jk}\,,\qquad [a_j,a_k]=0\,. \tag{3.1}$$

We wish to consider a linear inhomogeneous transformation on the annihilation and creation operators of the system. The most general such transformation may be written in the form

$$a_j'=\sum_{k=1}^{n}(u_{jk}a_k+v_{jk}a_k^\dagger)+\mathcal{A}_j'\,, \tag{3.2}$$

where u_{jk}, v_{jk}, and $\mathcal{A}_j'$ are c-numbers. We require that the transformation (3.2) be canonical; the operators a_j' and $a_j'^\dagger$ must therefore satisfy the commutation relations

$$[a_j',a_k'^\dagger]=\delta_{jk}\,,\qquad [a_j',a_k']=0\,. \tag{3.3}$$

Let us substitute Eq. (3.2) and its adjoint into these relations, and then make use of the commutation relations (3.1). If we consider the quantities u_{jk} and v_{jk} to be the matrix elements of the $n\times n$ matrices u and v, respectively, then the resulting equations take the form of the matrix conditions

$$uu^\dagger-vv^\dagger=1\,, \tag{3.4a}$$

$$u\tilde{v}-v\tilde{u}=0\,, \tag{3.4b}$$

where $\tilde{u}$ and $\tilde{v}$ are the transposes of the matrices u and v, and $u^\dagger$ and $v^\dagger$ are the corresponding Hermitian conjugates, i.e.,

$$\tilde{u}_{jk}\equiv u_{kj}\,,\quad \tilde{v}_{jk}\equiv v_{kj}\,,\quad u_{jk}^\dagger\equiv u_{kj}^*\,,\quad v_{jk}^\dagger\equiv v_{kj}^*\,.$$

It is convenient to think of the quantities a_j', a_j, and $\mathcal{A}_j'$ as the jth components of the column vectors a', a, and $\mathcal{A}'$, respectively. The transformation (3.2) may then be written in matrix notation in the form

$$a'=ua+va^\dagger+\mathcal{A}'\,. \tag{3.5a}$$

The Hermitian conjugate of this expression is

$$a'^\dagger=u^*a^\dagger+v^*a+\mathcal{A}'^*\,. \tag{3.5b}$$

These equations may be inverted without difficulty. We find that a and $a^\dagger$ are given in terms of a' and $a'^\dagger$ by the relations

$$a=u^\dagger a'-\tilde{v}a'^\dagger+\mathcal{A}\,, \tag{3.6a}$$

$$a^\dagger=\tilde{u}a'^\dagger-v^\dagger a'+\mathcal{A}^*\,, \tag{3.6b}$$

where the c-number quantity $\mathcal{A}$ is defined as

$$\mathcal{A}\equiv-(u^\dagger\mathcal{A}'-\tilde{v}\mathcal{A}'^*)\,. \tag{3.7}$$

The solutions (3.6) may be verified by substituting them directly into Eqs. (3.5) and making use of the matrix conditions (3.4). They have the same linear inhomogeneous form as the original transformation (3.5), but with the substitutions

$$u \to u^\dagger, \tag{3.8a}$$

$$v \to -\tilde{v}, \tag{3.8b}$$

$$\mathfrak{A}' \to \mathfrak{A}. \tag{3.9}$$

It follows that if we reverse the steps which led to Eqs. (3.4), i.e., if we begin with the commutation relations (3.3) for a' and $a'^\dagger$, and then use Eqs. (3.6) to evaluate the commutators which appear in Eqs. (3.1), we must find the relations

$$u^\dagger u - \tilde{v}v^* = 1, \tag{3.10a}$$

$$u^\dagger v - \tilde{v}u^* = 0, \tag{3.10b}$$

which are related to the conditions (3.4) through the substitutions (3.8). The conditions (3.10) on the matrices u and v follow from the canonical nature of the transformation we are considering, and hence from the matrix conditions (3.4).

It is worth noting that, according to Eq. (3.4a), the matrix $uu^\dagger - 1 = vv^\dagger$ is positive definite, and hence the (real) eigenvalues of the Hermitian matrix $uu^\dagger$ must be at least as great as unity. It follows that u is nonsingular. The same is not true of v, however. Indeed, a case of some interest is specified by the condition $v=0$, which corresponds to the transformation

$$a' = ua + \mathfrak{A}'. \tag{3.11}$$

The matrix conditions (3.4) then reduce to the single relation

$$uu^\dagger = 1, \tag{3.12}$$

which is the statement that the matrix u is unitary.

It is instructive to consider a system of n classical harmonic oscillators described by the c-number complex amplitudes α_j, and to subject these amplitudes to a transformation formally analogous to the operator transformation (3.5). The transformed amplitudes α_j' may then be expressed in terms of the original amplitudes in matrix notation as

$$\alpha' = u\alpha + v\alpha^* + \mathfrak{A}', \tag{3.13}$$

and α may be expressed in terms of α', α'^*, and the vector $\mathfrak{A}$ defined by Eq. (3.7) in the form

$$\alpha = u^\dagger\alpha' - \tilde{v}\alpha'^* + \mathfrak{A}, \tag{3.14}$$

which is the c-number analog of Eq. (3.6a).

It is not difficult to show that the element of phase-space volume

$$d^{2n}\alpha \equiv \prod_{j=1}^{n} d^2\alpha_j \equiv \prod_{j=1}^{n} d(\mathrm{Re}\alpha_j)d(\mathrm{Im}\alpha_j) \tag{3.15}$$

is preserved[30] under the canonical transformation (3.13), i.e., we have

$$d^{2n}\alpha' = d^{2n}\alpha. \tag{3.16}$$

The quantity

$$|\alpha|^2 \equiv \sum_{j=1}^{n} |\alpha_j|^2, \tag{3.17}$$

on the other hand, is not in general preserved, even under the homogeneous part of the transformation (3.13). The difference $|\alpha'|^2 - |\alpha|^2$ is easily evaluated with the aid of the identity (3.10a). If we denote by $\tilde{\alpha}$ and $\tilde{\alpha}^*$ the row vectors with jth components α_j and α_j^*, respectively, then we may express the result (for the homogeneous case) in the form

$$|u\alpha + v\alpha^*|^2 - |\alpha|^2 = 2\tilde{\alpha}^*\tilde{v}v^*\alpha + \tilde{\alpha}^*u^\dagger v\alpha^* + \tilde{\alpha}v^\dagger u\alpha, \tag{3.18}$$

which vanishes identically only if $v \equiv 0$.

IV. TRANSFORMATIONS ON STATE VECTORS

The canonical transformation of Eq. (3.5a) may be generated by a unitary operator U via the relation

$$U^{-1}aU = a' \tag{4.1}$$

$$= ua + va^\dagger + \mathfrak{A}'. \tag{4.2}$$

The operator U is determined, apart from a phase factor, by Eq. (4.2); its unitary character is guaranteed by the matrix conditions (3.4). The inverse relation to Eq. (4.2), which follows directly from Eq. (3.6a), is

$$UaU^{-1} = u^\dagger a - \tilde{v}a^\dagger + \mathfrak{A}. \tag{4.3}$$

We define a transformation on the state vectors of the system by the relation

$$|\psi'\rangle = U|\psi\rangle. \tag{4.4}$$

The mean value of any operator function $F(a,a^\dagger)$ in the state $|\psi'\rangle$ is then

$$\langle\psi'|F(a,a^\dagger)|\psi'\rangle = \langle\psi|F(a',a'^\dagger)|\psi\rangle. \tag{4.5}$$

We shall gain some insight into the transformation properties of the state vectors by first considering the effect of the transformation on the displacement operator appropriate to an n-mode system. This operator is defined as a function of n complex arguments α_j by the expression

$$D(\{\alpha_j\}) = \prod_{j=1}^{n} \exp(a_j^\dagger\alpha_j - \alpha_j^* a_j). \tag{4.6}$$

In matrix notation we may write

$$D(\alpha) = \exp(\tilde{a}^\dagger\alpha - \tilde{\alpha}^* a). \tag{4.7}$$

Here we are thinking of $\tilde{a}^\dagger$ as a row vector with jth component $a_j^\dagger$. We may note that the row vectors $\tilde{a}'$ and

[30] See, for example, H. C. Corben and P. Stehle, *Classical Mechanics* (John Wiley & Sons, Inc., New York, 1950), 2nd ed., Chap. 13.

$\tilde{a}'^\dagger$ are given, according to Eqs. (3.5), by the relations[31]

$$\tilde{a}'=\tilde{a}\tilde{u}+\tilde{a}^\dagger\tilde{v}+\mathcal{a}'^T, \tag{4.8a}$$

$$\tilde{a}'^\dagger=\tilde{a}^\dagger u^\dagger+\tilde{a}v^\dagger+\mathcal{a}'^{*T}. \tag{4.8b}$$

Let us now define the transformed displacement operator $D'(\alpha')$, for arbitrary α', by the relation

$$D'(\alpha')\equiv U^{-1}D(\alpha')U \tag{4.9}$$

$$=\exp(\tilde{a}'^\dagger\alpha'-\tilde{\alpha}'^*a'). \tag{4.10}$$

If we substitute Eqs. (4.8b) and (3.5a) into Eq. (4.10), we find the identity

$$\begin{aligned}D'(\alpha')&=\exp\{[\tilde{a}^\dagger(u^\dagger\alpha'-\tilde{v}\alpha'^*)+\mathcal{a}'^{*T}\alpha']-\text{H.c.}\}\\&=D(u^\dagger\alpha'-\tilde{v}\alpha'^*)\exp(\mathcal{a}'^{*T}\alpha'-\tilde{\alpha}'^*\mathcal{a}').\end{aligned} \tag{4.11}$$

The transformed displacement operator $D'(\alpha')$ is thus equal to a phase factor times the original displacement operator D, evaluated at the argument

$$\alpha=u^\dagger\alpha'-\tilde{v}\alpha'^*, \tag{4.12}$$

which is thus defined in terms of α' by the homogeneous part of the (inverse) transformation (3.14). The quantity α' may therefore be expressed in terms of α, as in Eq. (3.13), by means of the relation

$$\alpha'=u\alpha+v\alpha^*. \tag{4.13}$$

If we express the identity (4.11) in terms of α rather than α', we find, for arbitrary α,

$$D(\alpha)=D'(u\alpha+v\alpha^*)\exp(\mathcal{a}^{*T}\alpha-\tilde{\alpha}^*\mathcal{a}), \tag{4.14}$$

where $\mathcal{a}$ is defined by Eq. (3.7). We may note that Eq. (4.14) may be deduced more directly from Eq. (4.11) simply by interchanging D and D' in the latter equation, and making the substitutions (3.8) and (3.9) appropriate to the inverse transformation (3.6).

The phase factors which appear in Eqs. (4.11) and (4.14) are a simple consequence of the relationship between the homogeneous and inhomogeneous parts of the transformation (4.2). Let us suppose that the unitary operator U_{hom} generates the transformation

$$U_{\text{hom}}{}^{-1}aU_{\text{hom}}=ua+va^\dagger. \tag{4.15}$$

Then it follows from the relation (2.7) that the transformation (4.2) is generated by the unitary operator

$$U=D(\mathcal{a}')U_{\text{hom}}e^{i\varphi} \tag{4.16}$$

$$=U_{\text{hom}}D(-\mathcal{a})e^{i\varphi}, \tag{4.17}$$

where φ is an arbitrary real c-number. Either of these relations may be used, along with the identity (2.8), to deduce Eq. (4.11) from its form for $\mathcal{a}'=0$.

The identity (4.14) relating the displacement operators D and D' permits us to prove a useful theorem about the behavior of the state vectors of the system under the transformation (4.4). It follows from Eq. (4.14) and the definition (4.9) of D' that we may write, for an arbitrary state vector $|\psi\rangle$,

$$\begin{aligned}U[D(\alpha)|\psi\rangle]=D(u\alpha+v\alpha^*)[U|\psi\rangle]\\\times\exp(\mathcal{a}^{*T}\alpha-\tilde{\alpha}^*\mathcal{a}),\end{aligned} \tag{4.18}$$

so that the result of applying U to a displaced form of $|\psi\rangle$ is a displaced form of $U|\psi\rangle$.

Let us define the state $|\alpha\rangle$ as the state of the system in which each of the oscillators is in a coherent state, with complex eigenvalue α_j for the jth oscillator. The state $|\alpha\rangle$ may then be expressed in terms of the ground state $|0\rangle$ by means of the relation

$$|\alpha\rangle=D(\alpha)|0\rangle, \tag{4.19}$$

where $D(\alpha)$ is defined by Eq. (4.7). Arbitrary state vectors and operators may be expressed in terms of the states $|\alpha\rangle$ by means of the identity

$$\pi^{-n}\int d^{2n}\alpha\ |\alpha\rangle\langle\alpha|=1, \tag{4.20}$$

which is the generalization of the completeness relation (2.3) appropriate to an n-mode system. It is clear that if we know how U operates on the coherent states, then we can find out how it operates on an arbitrary state. The operator U may be expressed in terms of the coherent-state vectors in the form

$$U=\pi^{-2n}\int d^{2n}\alpha d^{2n}\beta\ |\beta\rangle\langle\beta|U|\alpha\rangle\langle\alpha|, \tag{4.21}$$

and is therefore uniquely determined by its coherent-state matrix elements $\langle\beta|U|\alpha\rangle$.

To obtain some insight into the transformation properties of the coherent states, let us begin by evaluating the mean value of a in the state $U|\alpha\rangle$. We have, according to Eq. (4.2),

$$\begin{aligned}\bar{\alpha}&\equiv\langle\alpha|U^{-1}aU|\alpha\rangle\\&=u\alpha+v\alpha^*+\mathcal{a}'.\end{aligned} \tag{4.22}$$

The sum of the variances $\langle(a_j^\dagger-\bar{\alpha}_j^*)(a_j-\bar{\alpha}_j)\rangle$ in the state $U|\alpha\rangle$ may be evaluated straightforwardly with the aid of Eq. (4.2) and the commutation relations for a and $a^\dagger$. We find[31]

$$\begin{aligned}|\Delta\alpha|^2&\equiv\langle\alpha|U^{-1}(\tilde{a}^\dagger-\tilde{\bar{\alpha}}^{*T})(a-\bar{\alpha})U|\alpha\rangle\\&=\mathrm{tr}(v^\dagger v),\end{aligned} \tag{4.23}$$

which is greater than zero unless the matrix v is identically zero. It follows that if v does not vanish identically, the state $U|\alpha\rangle$ is not coherent.

If v is identically equal to zero, on the other hand, then Eq. (4.23) implies that the state $U|\alpha\rangle$ is an eigenstate of a with eigenvalue $\bar{\alpha}=u\alpha+\mathcal{a}'$, so that we must have

$$U|\alpha\rangle=|u\alpha+\mathcal{a}'\rangle e^{i\varphi(\alpha,\alpha^*)}, \tag{4.24}$$

[31] For typographical reasons, we denote the transposes of $\mathcal{a}$, $\mathcal{K}$, and $\tilde{a}$ by $\mathcal{a}^T$, $\mathcal{K}^T$, and $\tilde{a}^T$.

where $\varphi(\alpha,\alpha^*)$ is a real c-number function of α and α^*. A coherent state thus remains coherent under the transformations we are considering if $v\equiv 0$.

The function $\varphi(\alpha,\alpha^*)$ in Eq. (4.24) may be determined to within an additive constant by the identity (4.18). Let us write $\varphi(0,0)=\varphi_0$, so that

$$U|0\rangle=|\mathfrak{A}'\rangle e^{i\varphi_0}. \tag{4.25}$$

Let us next evaluate Eq. (4.18) for the case $|\psi\rangle=|0\rangle$. Since we are assuming $v=0$, it follows from Eqs. (3.12) and (3.7) that $u^\dagger=u^{-1}$ and $\mathfrak{A}=-u^{-1}\mathfrak{A}'$. If we make use of these identities and the relations (4.19) and (4.25) in Eq. (4.18), we find

$$\begin{aligned}U|\alpha\rangle&=D(u\alpha)|\mathfrak{A}'\rangle\exp(\tilde{\alpha}^*u^{-1}\mathfrak{A}'-\mathfrak{A}'^{*T}u\alpha+i\varphi_0)\\&=|u\alpha+\mathfrak{A}'\rangle\exp[\tfrac{1}{2}(\tilde{\alpha}^*u^{-1}\mathfrak{A}'-\mathfrak{A}'^{*T}u\alpha)+i\varphi_0],\end{aligned}\tag{4.26}$$

where the n-mode generalization of the identity (2.9) was used to reach the latter expression. By comparing Eqs. (4.24) and (4.26), we see that the function $\varphi(\alpha,\alpha^*)$ is given by

$$\varphi(\alpha,\alpha^*)=\tfrac{1}{2}i(\mathfrak{A}'^{*T}u\alpha-\tilde{\alpha}^*u^{-1}\mathfrak{A}')+\varphi_0. \tag{4.27}$$

The quantity φ_0 remains undetermined, since the transformations we are considering determine U only to within a phase factor.

It is worth noting that Eq. (4.27) is a simple consequence of unitarity. According to Eq. (4.24) we must have, for arbitrary α and β,

$$\begin{aligned}\langle\beta|\alpha\rangle=&\langle u\beta+\mathfrak{A}'|u\alpha+\mathfrak{A}'\rangle\\&\times\exp\{i[\varphi(\alpha,\alpha^*)-\varphi(\beta,\beta^*)]\}.\end{aligned}\tag{4.28}$$

The form of $\varphi(\alpha,\alpha^*)$ given by Eq. (4.27) is easily deduced from this relation and Eq. (2.4).

Let us now return to the general transformation of Eq. (4.2), for which the matrix v does not vanish identically. It is not difficult in this case to solve for the function $\langle\beta|U|\alpha\rangle$ directly. To this end we first introduce the states[32,6]

$$\|\alpha\rangle\equiv\exp(\tilde{a}^\dagger\alpha)|0\rangle \tag{4.29}$$

$$=e^{\frac{1}{2}|\alpha|^2}|\alpha\rangle, \tag{4.30}$$

which have the convenient property

$$a^\dagger\|\alpha\rangle=\frac{\partial}{\partial\alpha}\|\alpha\rangle. \tag{4.31}$$

We next define the function

$$\mathfrak{U}(\beta^*,\alpha)\equiv\langle\beta\|U\|\alpha\rangle \tag{4.32}$$

$$=e^{\frac{1}{2}|\beta|^2+\frac{1}{2}|\alpha|^2}\langle\beta|U|\alpha\rangle, \tag{4.33}$$

which is an entire function of β^* and α. If we take the Hermitian conjugate of Eq. (4.2) and multiply the resulting equation on the left by $\langle\beta\|U$ and on the right by $\|\alpha\rangle$, then by making use of Eq. (4.31) we find that the function $\mathfrak{U}(\beta^*,\alpha)$ satisfies the partial differential equation

$$\left[\beta^*-u^*\frac{\partial}{\partial\alpha}-v^*\alpha-\mathfrak{A}'^*\right]\mathfrak{U}(\beta^*,\alpha)=0. \tag{4.34}$$

Similarly, by multiplying Eq. (4.3) on the left by $\langle\beta\|$ and on the right by $U\|\alpha\rangle$, we find

$$\left[\alpha-u^\dagger\frac{\partial}{\partial\beta^*}+\tilde{v}\beta^*-\mathfrak{A}\right]\mathfrak{U}(\beta^*,\alpha)=0. \tag{4.35}$$

Equations (4.34) and (4.35) are sufficient to determine the function $\mathfrak{U}(\beta^*,\alpha)$ to within a complex factor which is independent of β^* and α. Let us write this factor as $Ne^{i\theta}$, where N and θ are real. If we make use of Eqs. (3.4b) and (3.10b) to deduce the identities

$$u^{-1}v=\tilde{v}\tilde{u}^{-1}$$

and

$$vu^{*-1}=u^{\dagger-1}\tilde{v},$$

which are the statements that the matrices $u^{-1}v$ and vu^{*-1} are both symmetric, then by direct differentiation we may verify that the solution to Eqs. (4.34) and (4.35) is

$$\begin{aligned}\mathfrak{U}(\beta^*,\alpha)=N\exp[&\tilde{\beta}^*u^{\dagger-1}\alpha+\tfrac{1}{2}\tilde{\beta}^*vu^{*-1}\beta^*-\tfrac{1}{2}\tilde{\alpha}u^{*-1}v^*\alpha\\&-\tilde{\beta}^*u^{\dagger-1}\mathfrak{A}-\tilde{\alpha}u^{*-1}\mathfrak{A}'^*+i\theta].\end{aligned}\tag{4.36}$$

The (constant) phase θ which appears in this expression is undetermined. The constant N, on the other hand, is determined by the normalization condition

$$\begin{aligned}\pi^{-n}\int d^{2n}\beta\,&|\langle\beta|U|\alpha\rangle|^2\\&=\pi^{-n}\int d^{2n}\beta\,e^{-|\beta|^2-|\alpha|^2}|\mathfrak{U}(\beta^*,\alpha)|^2=1.\end{aligned}\tag{4.37}$$

V. TRANSFORMATIONS ON DENSITY OPERATORS

Let us now suppose that the state of the system of n oscillators is mixed, i.e., that it is specified by a density operator rather than by a state vector. The density operator for an n-mode system may be described by means of characteristic functions and quasiprobability functions, just as in the case of a single-mode system. We shall devote the present section to establishing the behavior of these functions under the transformation of Eq. (4.2).

Let us define the ordinary characteristic function, for n complex arguments η_j, by the n-mode generalization of Eq. (2.13),

$$\chi(\eta)\equiv\mathrm{tr}[\rho D(\eta)], \tag{5.1}$$

where D is defined by Eq. (4.7). The Wigner function,

[32] V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); Proc. Natl. Acad. Sci. U. S. 48, 199 (1962).

according to Eq. (2.19), is defined by the relation

$$W(\alpha)\equiv\pi^{-2n}\int d^{2n}\eta\,\exp(\tilde{\eta}^*\alpha-\tilde{\alpha}^*\eta)\chi(\eta)\,,\qquad(5.2)$$

which has the form of a $2n$-dimensional Fourier transform. If we now define the Hermitian operator $T(\alpha)$, as as in Eq. (2.22), as the Fourier transform of $D(\eta)$,

$$T(\alpha)=\pi^{-n}\int d^{2n}\eta\,\exp(\tilde{\eta}^*\alpha-\tilde{\alpha}^*\eta)D(\eta)\,,\qquad(5.3)$$

then the Wigner function associated with the density operator ρ is given by

$$W(\alpha)=\pi^{-n}\,\mathrm{tr}[\rho T(\alpha)].\qquad(5.4)$$

The density operator then has the representations

$$\rho=\pi^{-n}\int d^{2n}\eta\,\chi(\eta)D^{-1}(\eta)\qquad(5.5)$$

$$=\int d^{2n}\alpha\,W(\alpha)T(\alpha)\,,\qquad(5.6)$$

which are the n-mode generalizations of Eqs. (2.12) and (2.24), respectively.

Let us define a transformation on the density operator by the expression

$$\rho'=U\rho U^{-1}\,,\qquad(5.7)$$

so that the relation

$$\mathrm{tr}[\rho' F(a,a^\dagger)]=\mathrm{tr}[\rho F(a',a'^\dagger)]\qquad(5.8)$$

is satisfied for any function F of the annihilation and creation operators. We define the functions χ' and W' as the characteristic function and the Wigner function, respectively, corresponding to the transformed density operator ρ'. The functions $\chi'(\eta')$ and $W'(\alpha')$ are then defined for arbitrary η' and α' as

$$\chi'(\eta')\equiv\mathrm{tr}[\rho' D(\eta')]\qquad(5.9)$$

and

$$W'(\alpha')\equiv\pi^{-2n}\int d^{2n}\eta'\,\exp(\tilde{\eta}'^*\alpha'-\tilde{\alpha}'^*\eta')\chi'(\eta')\qquad(5.10)$$

$$=\pi^{-n}\,\mathrm{tr}[\rho' T(\alpha')].\qquad(5.11)$$

Let us substitute Eq. (5.7) for ρ' into Eq. (5.9). If we then make use of the cyclical symmetry of the traces of products, we find the relation

$$\chi'(\eta')=\mathrm{tr}[\rho D'(\eta')]\,,\qquad(5.12)$$

which expresses χ' in terms of the original density operator ρ and the transformed displacement operator D' defined by Eq. (4.9). It follows from Eqs. (5.1) and (5.12) that the functional relationship (4.11) between D' and D must also exist between χ' and χ. We have therefore

$$\chi'(\eta')=\chi(u^\dagger\eta'-\tilde{v}\eta'^*)\,\exp(\mathfrak{a}'^{*T}\eta'-\tilde{\eta}'^*\mathfrak{a}')\,.\qquad(5.13)$$

The transformation rule for the Wigner function is most easily found by making use of its expression in terms of the Hermitian operator $T(\alpha)$. The function W', according to Eqs. (5.11) and (5.7), may be expressed in terms of the original density operator ρ as

$$W'(\alpha')=\pi^{-n}\,\mathrm{tr}[\rho T'(\alpha')]\,,\qquad(5.14)$$

where the operator $T'(\alpha')$ is defined for arbitrary α' by the similarity transformation

$$T'(\alpha')\equiv U^{-1}T(\alpha')U\,.\qquad(5.15)$$

To obtain a functional identity relating T' to T, let us first note that the operator $T(\alpha')$ may be expressed, as in Eq. (2.30), in the form

$$T(\alpha')=2^nD(\alpha')ID^{-1}(\alpha')\,,\qquad(5.16)$$

where the reflection operator I is defined by the relations

$$Ia_jI=-a_j\qquad(5.17)$$

for $j=1,\cdots,n$, and

$$I^2=1\,.\qquad(5.18)$$

By substituting Eq. (5.16) for $T(\alpha')$ into Eq. (5.15) and making use of the definition (4.9) of D' and the relation (4.11), we find

$$\begin{aligned}T'(\alpha')&=2^nD'(\alpha')I'D'^{-1}(\alpha')\\&=2^nD(u^\dagger\alpha'-\tilde{v}\alpha'^*)I'D^{-1}(u^\dagger\alpha'-\tilde{v}\alpha'^*)\,,\end{aligned}\qquad(5.19)$$

where the transformed reflection operator I' is defined by the equation

$$I'\equiv U^{-1}IU\,,\qquad(5.20)$$

and is therefore determined by the relations

$$I'a_j'I'=-a_j'\qquad(5.21)$$

$$I'^2=1\,.\qquad(5.22)$$

It is not difficult to show with the aid of Eq. (3.5a) for a' and the relation (2.7) that the operator which satisfies Eqs. (5.21) and (5.22) is

$$I'=D(\mathfrak{a})ID^{-1}(\mathfrak{a})\,,\qquad(5.23)$$

where $\mathfrak{a}$ is defined by Eq. (3.7).

Let us substitute Eq. (5.23) for I' into Eq. (5.19). If we then make use of the identity (2.8) and the expression (5.16) for T, we find that T' may be expressed in terms of T in the form

$$T'(\alpha')=T(u^\dagger\alpha'-\tilde{v}\alpha'^*+\mathfrak{a})\,.\qquad(5.24)$$

It is clear from this equation and Eqs. (5.4) and (5.14) that W' may be expressed in terms of W by means of the similar relation

$$W'(\alpha')=W(u^\dagger\alpha'-\tilde{v}\alpha'^*+\mathfrak{a})\,.\qquad(5.25)$$

The transformed Wigner function W', for arbitrary argument α', is thus equal to the original Wigner function W evaluated at an argument related to α' through the (inverse) transformation (3.14) on the complex amplitudes of a classical system. It follows then from Eq. (3.13) that we may also write for arbitrary α,

$$W'(u\alpha+v\alpha^*+\mathcal{A}')=W(\alpha). \tag{5.26}$$

Thus the effect of the transformation on the Wigner function may be expressed by transforming its arguments by the c-number analog of the operator transformation (3.5). We may note that the Wigner function, under the linear transformations of variables we are considering, behaves exactly like the probability density $f(\alpha)$ for finding a system of n classical oscillators with complex amplitudes α_j. The classical function f, under a change of variables $\alpha\rightarrow\alpha'$, must be replaced by the function f' defined by the relation $f'(\alpha')=f(\alpha)$. It should be emphasized, however, that the quantum-mechanical relation $W'(\alpha')=W(\alpha)$ is valid only for linear (inhomogeneous) transformations.

Let us now consider the P representation. We shall say that the density operator ρ has a P representation if it can be expressed in the form

$$\rho=\int d^{2n}\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|. \tag{5.27}$$

If a P representation for ρ exists, then the normally ordered characteristic function, which is defined for the n-mode case as

$$\chi_N(\eta)\equiv\operatorname{tr}[\rho\exp(\tilde{a}^\dagger+\eta)\exp(-\tilde{\eta}^*a)] \tag{5.28}$$

$$=e^{\frac{1}{2}|\eta|^2}\chi(\eta), \tag{5.29}$$

has the Fourier transform

$$P(\alpha)=\pi^{-2n}\int d^{2n}\eta\,\exp(\tilde{\eta}^*\alpha-\tilde{\alpha}^*\eta)\chi_N(\eta), \tag{5.30}$$

which is the n-mode generalization of Eq. (2.18).

We shall say that ρ' has a P representation if it can be written in terms of some weight function P' in the form

$$\rho'=\int d^{2n}\alpha'\, P'(\alpha')|\alpha'\rangle\langle\alpha'|, \tag{5.31}$$

where we have used the symbol α' for the variables of integration to facilitate later work. If a P representation exists for ρ', then we must have

$$P'(\alpha')=\pi^{-2n}\int d^{2n}\eta'\,\exp(\tilde{\eta}'^*\alpha'-\tilde{\alpha}'^*\eta')\chi_N'(\eta'), \tag{5.32}$$

where χ_N' is the normally ordered characteristic function associated with ρ', and may therefore be expressed in terms of χ' in the form

$$\chi_N'(\eta')=e^{\frac{1}{2}|\eta'|^2}\chi'(\eta'). \tag{5.33}$$

The relationship between χ_N' and χ_N follows directly from the identity (5.13) relating χ' to χ, and from Eqs. (5.29) and (5.33), which relate χ_N to χ and χ_N' to χ', respectively. By combining Eqs. (5.33), (5.13), and (5.29) we obtain

$$\chi_N'(\eta')=\exp[\mathcal{A}'^{*T}\eta'-\tilde{\eta}'^*\mathcal{A}' +\tfrac{1}{2}(|\eta'|^2-|u^\dagger\eta'-\tilde{v}\eta'^*|^2)]\cdot\chi_N(u^\dagger\eta'-\tilde{v}\eta'^*). \tag{5.34}$$

Let us now substitute this expression for $\chi_N'(\eta')$ into the integral in Eq. (5.32), and then change the variables of integration from η' to

$$\eta=u^\dagger\eta'-\tilde{v}\eta'^*. \tag{5.35}$$

Then η' is given in terms of the new variables η by the relation

$$\eta'=u\eta+v\eta^*, \tag{5.36}$$

and we have, as in Eq. (3.16),

$$d^{2n}\eta'=d^{2n}\eta. \tag{5.37}$$

By substituting Eq. (5.34) into Eq. (5.32) and carrying out the indicated change of variables, we find

$$P'(\alpha')=\mathcal{P}(u^\dagger\alpha'-\tilde{v}\alpha'^*+\mathcal{A}), \tag{5.38}$$

where the function $\mathcal{P}(\alpha)$ is defined as

$$\mathcal{P}(\alpha)\equiv\pi^{-2n}\int d^{2n}\eta\,\exp(\eta^*\alpha-\tilde{\alpha}^*\eta) \times\{\exp[\tfrac{1}{2}|u\eta+v\eta^*|^2-\tfrac{1}{2}|\eta|^2]\chi_N(\eta)\}. \tag{5.39}$$

Equations (5.38) and (5.39) express the weight function P' for the P representation of the transformed density operator ρ' in terms of the normally ordered characteristic function χ_N associated with the original density operator ρ. The existence of a P representation for ρ' depends upon the convergence of the integral in Eq. (5.39), which defines the function $\mathcal{P}(\alpha)$. That integral is just the $2n$-dimensional Fourier transform of the product of $\chi_N(\eta)$ and the exponential of a quadratic form in η and η^*, which according to Eq. (3.18) is given by

$$\tfrac{1}{2}(|u\eta+v\eta^*|^2-|\eta|^2) =\tilde{\eta}^*\tilde{v}v^*\eta+\tfrac{1}{2}\tilde{\eta}^*u^\dagger v\eta^*+\tfrac{1}{2}\tilde{\eta}v^\dagger u\eta. \tag{5.40}$$

Let us now suppose that the density operator ρ has a P representation, so that the function $\chi_N(\eta)$ has a well-defined Fourier transform. If the expression evaluated in Eq. (5.40) were a negative semidefinite quadratic form in η and η^*, then the integral in Eq. (5.39) would necessarily converge. The existence of a P representation for ρ' would then follow from the existence of a P representation for ρ. It is not difficult to show, however, that the expression evaluated in Eq. (5.40) can not in general be negative semidefinite. Indeed, if we express the right-hand side of Eq. (5.40) as a quadratic form in the real and imaginary parts of η, then we can show from the canonical nature of the transformations we are considering that the $2n\times 2n$ real symmetric matrix which defines this quadratic form must have positive eigenvalues if it has negative ones. The existence of a P representation for ρ therefore does not guarantee the existence of a P representation for ρ', and indeed there are many density operators which do not retain their P representation under the transformation of Eq. (4.2).

The unique exception to this rule is specified by the condition $v=0$, for which the quadratic form evaluated in Eq. (5.40) vanishes identically. In that case the function $\mathcal{P}(\alpha)$ defined by Eq. (5.39) is equal to $P(\alpha)$, the weight function for the P representation of ρ given by Eq. (5.30). It follows then from Eqs. (5.38) and (3.12) that P' and P are related by the expression

$$P'(\alpha')=P(u^{-1}\alpha'+\mathfrak{A}). \tag{5.41}$$

For the case $v=0$, then, the weight function for the P representation has the same classical-type transformation property as the Wigner function. The identity (5.41) may easily be derived from the results of the preceding section, in which it was shown that a coherent state remains coherent under the transformation specified by $v=0$. If we substitute Eq. (5.27) for ρ into Eq. (5.7) and make use of Eq. (4.24), we find

$$\rho'=\int d^{2n}\alpha\, P(\alpha)\,|u\alpha+\mathfrak{A}'\rangle\langle u\alpha+\mathfrak{A}'|\,, \tag{5.42}$$

and if we then change the variables of integration from α to $\alpha'=u\alpha+\mathfrak{A}'$, we find

$$\rho'=\int d^{2n}\alpha'\, P(u^{-1}\alpha'+\mathfrak{A})\,|\alpha'\rangle\langle\alpha'|\,, \tag{5.43}$$

so that ρ' has a P representation, with the weight function $P'(\alpha')$ given by Eq. (5.41). The transformation rule (5.41) for the P function is thus a simple reflection of the fact that a coherent state is transformed into a coherent state, if $v=0$. The transformation rule (5.25) for the Wigner function, on the other hand, is valid even when $v\neq 0$, and thus even for transformations under which a coherent state does not retain its coherent character.

VI. DYNAMICALLY COUPLED OSCILLATOR SYSTEMS

A particularly important application of the results of the preceding sections is to dynamical problems for which the Heisenberg operators $a(t)$ and $a^\dagger(t)$ are related to their initial values a and $a^\dagger$ by linear inhomogeneous expressions. Let us assume that the Hamiltonian for the system of n oscillators is given by a quadratic form in $a^\dagger(t)$ and $a(t)$,

$$H(t)=\hbar[\tilde{a}^\dagger(t)\omega(t)a(t)+\tfrac{1}{2}\tilde{a}^\dagger(t)\sigma(t)a^\dagger(t) +\tfrac{1}{2}\tilde{a}(t)\sigma^*(t)a(t)+\tilde{a}^\dagger(t)\mathcal{K}(t)+\tilde{a}(t)\mathcal{K}^*(t)], \tag{6.1}$$

where the $n\times n$ matrices $\omega(t)$ and $\sigma(t)$ and the n-component vector $\mathcal{K}(t)$ are c-number functions of time. The hermiticity of $H(t)$ leads to the requirements

$$\omega(t)=\omega^\dagger(t) \tag{6.2a}$$

and

$$\sigma(t)=\tilde{\sigma}(t). \tag{6.2b}$$

These are the only conditions we need impose on the Hamiltonian (6.1). No assumption need be made about the way the functions $\omega(t)$, $\sigma(t)$, and $\mathcal{K}(t)$ depend on time, and the Hamiltonian may be allowed to take on negative values as well as positive ones.

The term involving $\omega(t)$ in Eq. (6.1) contains the free Hamiltonian for a system of uncoupled oscillators, but may also contain part of the coupling, e.g., in systems such as the frequency converter.[2] The terms involving $\sigma(t)$ are characteristic of parametric amplification processes; they will be discussed in some detail in the next section. The linear terms involving $\mathcal{K}(t)$ in Eq. (6.1) represent externally applied forces which drive the oscillators; if the oscillators are modes of oscillation of the electromagnetic field, terms of this kind arise from the presence of classical currents.

The Heisenberg equation of motion for $a(t)$ which follows from the Hamiltonian (6.1)is

$$i\frac{d}{dt}a(t)=\frac{1}{\hbar}[a(t),H(t)] =\omega(t)a(t)+\sigma(t)a^\dagger(t)+\mathcal{K}(t). \tag{6.3}$$

The solution to this equation may be expressed in terms of the initial operator

$$a(0)\equiv a \tag{6.4}$$

in the form

$$a(t)=u(t)a+v(t)a^\dagger+\mathfrak{A}'(t)\,, \tag{6.5}$$

where the matrices $u(t)$ and $v(t)$ are defined as the solutions to the differential equations

$$i\frac{d}{dt}u(t)=\omega(t)u(t)+\sigma(t)v^*(t)\,, \tag{6.6}$$

$$i\frac{d}{dt}v(t)=\omega(t)v(t)+\sigma(t)u^*(t)\,, \tag{6.7}$$

corresponding to the initial conditions

$$u(0)=1\,, \tag{6.8}$$

$$v(0)=0\,, \tag{6.9}$$

and the vector $\mathfrak{A}'(t)$ is defined as the solution to the differential equation

$$i\frac{d}{dt}\mathfrak{A}'(t)=\omega(t)\mathfrak{A}'(t)+\sigma(t)\mathfrak{A}'^*(t)+\mathcal{K}(t) \tag{6.10}$$

corresponding to the initial condition

$$\mathfrak{A}'(0)=0. \tag{6.11}$$

The canonical transformation (6.5) relating the Heisenberg operator $a(t)$ to the Schrödinger operators a and $a^\dagger$ has the same form as the linear inhomogeneous transformation of Eq. (3.5a): If we let the quantities u, v, and $\mathfrak{A}'$ which appear in Eq. (3.5a) be the functions $u(t)$, $v(t)$, and $\mathfrak{A}'(t)$ defined by Eqs. (6.6)–(6.11) for some time t, then the operator a' is the Heisenberg operator $a(t)$. The unitary operator $U(t)$ which generates the transformation (6.5) by means of the relation

$$U^{-1}(t)aU(t)=a(t) \tag{6.12a}$$

$$=u(t)a+v(t)a^\dagger+\mathfrak{A}'(t) \tag{6.12b}$$

is then the time translation operator which connects the Heisenberg and Schrödinger pictures of the motion. It may be defined by the equations

$$i\hbar\frac{d}{dt}U(t)=H_S(t)U(t) \tag{6.13}$$

and

$$U(0)=1\,, \tag{6.14}$$

where the Schrödinger Hamiltonian $H_S(t)$ is related to the Heisenberg Hamiltonian $H(t)$ by means of the identity

$$H_S(t)=U(t)H(t)U^{-1}(t)\,, \tag{6.15}$$

and is therefore given by the expression

$$H_S(t)=\hbar[\tilde{a}^\dagger\omega(t)a+\tfrac{1}{2}\tilde{a}^\dagger\sigma(t)a^\dagger+\tfrac{1}{2}\tilde{a}\sigma^*(t)a +\tilde{a}^\dagger\mathcal{K}(t)+\tilde{a}\mathcal{K}^*(t)]. \tag{6.16}$$

The work of Secs. IV and V may now be used to find the time evolution of the state of the system in the Schrödinger picture. If we let $|\psi\rangle$ represent the fixed Heisenberg state vector for the system, then the transformed state vector $|\psi'\rangle$ defined by Eq. (4.4) is

$$|\psi(t)\rangle=U(t)|\psi\rangle\,, \tag{6.17}$$

which is the Schrödinger state vector at time t. Similarly, if we let ρ represent the fixed Heisenberg density operator, then the operator ρ' defined by Eq. (5.7) is

$$\rho(t)=U(t)\rho U^{-1}(t)\,, \tag{6.18}$$

which is the Schrödinger density operator at time t. The solution to Schrödinger's equation corresponding to the Hamiltonian (6.16) may therefore be expressed in terms of the functions $u(t)$, $v(t)$, and $\mathfrak{A}'(t)$ defined by Eqs. (6.6)–(6.11). We may note that these functions also define the solutions to the equations of motion for a system of n classical oscillators with complex amplitudes α_j, governed by the Hamiltonian

$$H_{\text{cl.}}(t)=\hbar[\tilde{\alpha}^*\omega(t)\alpha+\tfrac{1}{2}\tilde{\alpha}^*\sigma(t)\alpha^*+\tfrac{1}{2}\tilde{\alpha}\sigma^*(t)\alpha +\tilde{\alpha}^*\mathcal{K}(t)+\tilde{\alpha}\mathcal{K}^*(t)], \tag{6.19}$$

which is the classical analog of the Hamiltonian (6.1). The equations of motion for the classical amplitudes $\alpha_j(t)$ which follow from the Hamiltonian (6.19) may be obtained by making the substitutions $a(t)\to\alpha(t)$, $a^\dagger(t)\to\alpha^*(t)$ in Eq. (6.3); the amplitudes at time t corresponding to the initial amplitudes α_0 are therefore given by the relation

$$\alpha_c(\alpha_0,\alpha_0^*,t)=u(t)\alpha_0+v(t)\alpha_0^*+\mathfrak{A}'(t)\,. \tag{6.20}$$

Before proceeding further, it is convenient to investigate the effect of the driving terms in the Hamiltonian (6.1). Let us define the operator $U_{\text{hom}}(t)$ as the unitary time translation operator in the absence of driving terms, i.e., by the equations

$$i\frac{d}{dt}U_{\text{hom}}(t) =[\tilde{a}^\dagger\omega(t)a+\tfrac{1}{2}\tilde{a}^\dagger\sigma(t)a^\dagger+\tfrac{1}{2}\tilde{a}\sigma^*(t)a]U_{\text{hom}}(t) \tag{6.21}$$

and

$$U_{\text{hom}}(0)=1\,, \tag{6.22}$$

so that $U_{\text{hom}}(t)$ generates the homogeneous transformation

$$U_{\text{hom}}^{-1}(t)aU_{\text{hom}}(t)=u(t)a+v(t)a^\dagger\,. \tag{6.23}$$

The generator of the inhomogeneous transformation (6.12b) may then be expressed, as in Eq. (4.16), in the form

$$U(t)=D(\mathfrak{A}'(t))U_{\text{hom}}(t)e^{i\varphi(t)}\,, \tag{6.24}$$

where $\varphi(t)$ is a real function of time. The function $\varphi(t)$ may be evaluated by substituting Eq. (6.24) for $U(t)$ into Eq. (6.13), and making use of Eq. (6.21) and the identity

$$\frac{d}{dt}D(\mathfrak{A}'(t))=\left\{[\tilde{a}^\dagger-\tfrac{1}{2}\mathfrak{A}'^{*T}(t)]\frac{d}{dt}\mathfrak{A}'(t)-\text{H.C.}\right\} \times D(\mathfrak{A}'(t))\,. \tag{6.25}$$

If we then make use of Eq. (6.10) and the property of the displacement operators expressed in Eq. (2.7), we find that Eq. (6.13) reduces to the relation[31]

$$\frac{d}{dt}\varphi(t)=-\tfrac{1}{2}[\mathfrak{A}'^{*T}(t)\mathcal{K}(t)+\mathcal{K}^{*T}(t)\mathfrak{A}'(t)]. \tag{6.26}$$

The function $\varphi(t)$ is then given by

$$\varphi(t)=-\frac{1}{2}\int_0^t dt'\,[\mathfrak{A}'^{*T}(t')\mathcal{K}(t')+\mathcal{K}^{*T}(t')\mathfrak{A}'(t')]\,, \tag{6.27}$$

since $\varphi(0)=0$.

As a simple illustration of the way the results of the preceding sections may be used to solve dynamical problems, let us consider the case $\sigma(t)\equiv0$, so that the Hamiltonian (6.1) takes the form

$$H(t)=\hbar[\tilde{a}^\dagger(t)\omega(t)a(t)+\tilde{a}^\dagger(t)\mathcal{K}(t)+\tilde{a}(t)\mathcal{K}^*(t)]\,, \tag{6.28}$$

and the solution to the Heisenberg equations of motion is

$$a(t)=u(t)a+\mathfrak{A}'(t)\,, \tag{6.29}$$

where $u(t)$ and $\mathcal{A}'(t)$ are defined by the differential equations

$$i\frac{d}{dt}u(t)=\omega(t)u(t)\,, \tag{6.30}$$

$$i\frac{d}{dt}\mathcal{A}'(t)=\omega(t)\mathcal{A}'(t)+\mathcal{K}(t)\,, \tag{6.31}$$

and the initial conditions (6.8) and (6.11). The solution to the corresponding classical equations of motion is then

$$\alpha_c(\alpha_0,t)=u(t)\alpha_0+\mathcal{A}'(t)\,. \tag{6.32}$$

Let us now suppose that the initial state of the system is the pure coherent state $|\alpha_0\rangle$. Since $v(t)=0$, we may use the result found in Eq. (4.24) to evaluate the Schrödinger state vector at time t. If we make use of Eq. (6.32), we find

$$U(t)|\alpha_0\rangle=|\alpha_c(\alpha_0,t)\rangle e^{i\varphi(\alpha_0,\alpha_0^*,t)}\,, \tag{6.33}$$

where the real function $\varphi(\alpha_0,\alpha_0^*,t)$ is determined by Eq. (4.27) to have the form

$$\varphi(\alpha_0,\alpha_0^*,t) =\tfrac{1}{2}i[\mathcal{A}'^{*T}(t)u(t)\alpha_0-\tilde{\alpha}_0^*u^{-1}(t)\mathcal{A}'(t)]+\varphi_0(t)\,. \tag{6.34}$$

Equation (6.33) states that if the Hamiltonian for a system of n oscillators has the form given by Eq. (6.28), then an initially coherent state vector retains its coherent character at all times, and its complex eigenvalues obey the equations of motion for the analogous classical system.[13,14,9]

The function $\varphi_0(t)$ in Eq. (6.34) is easily evaluated with the aid of Eqs. (6.24) and (6.27). Let us first note that if $\sigma(t)$ and $\mathcal{K}(t)$ both vanish identically, the Schrödinger Hamiltonian is just $\tilde{a}^\dagger\omega(t)a$, so that the vacuum state remains invariant, and we have

$$U_{\text{hom}}(t)|0\rangle=|0\rangle. \tag{6.35}$$

Let us use this result in Eq. (6.24) to evaluate the state which evolves from the vacuum when the driving terms are present, i.e., when the Hamiltonian is given by Eq. (6.28). We have then

$$U(t)|0\rangle=|\mathcal{A}'(t)\rangle e^{i\varphi(t)}. \tag{6.36}$$

If we compare this equation to the form Eq. (6.33) takes for $\alpha_0=0$, we see that $\varphi_0(t)=\varphi(t)$, and the function $\varphi(\alpha_0,\alpha_0^*,t)$ may therefore be evaluated by replacing the function $\varphi_0(t)$ in Eq. (6.34) by the expression for $\varphi(t)$ given by Eq. (6.27). It is interesting to observe that the result may be expressed, with the aid of the relations (6.30) and (6.31), in the form

$$\varphi(\alpha_0,\alpha_0^*,t)=-\frac{1}{2}\int_0^t dt' \times[\tilde{\alpha}_c^*(\alpha_0,t')\mathcal{K}(t')+\mathcal{K}^{*T}(t')\alpha_c(\alpha_0,t')]\,, \tag{6.37}$$

which is the integral of $(-1/2\hbar)$ times the linear part of the classical analog of the Hamiltonian (6.28), evaluated along a classical trajectory beginning at α_0 and ending at $\alpha_c(\alpha_0,t)$.

Let us now return to the general case, $\sigma(t)\neq 0$. The matrix $v(t)$ is then not identically zero, and it follows from the work of Sec. IV that a Schrödinger state vector which is initially coherent does not retain its coherent character at later times. We have seen in Sec. V that even in this more general case, the transformation properties of the Wigner function are particularly simple. Let us define the function $W(\alpha,t)$ as the Wigner function corresponding to the time-dependent Schrödinger density operator $\rho(t)$ given by Eq. (6.18). If we similarly define the function $\chi(\eta,t)$ as the ordinary characteristic function corresponding to $\rho(t)$, i.e., by the relation

$$\chi(\eta,t)\equiv\text{tr}[\rho(t)D(\eta)]\,, \tag{6.38}$$

then $W(\alpha,t)$ is the Fourier transform of $\chi(\eta,t)$,

$$W(\alpha,t)\equiv\pi^{-2n}\int d^{2n}\eta\,\exp(\tilde{\eta}^*\alpha-\tilde{\alpha}^*\eta)\chi(\eta,t)\,. \tag{6.39}$$

It is clear that the functions $\chi(\eta,t)$ and $W(\alpha,t)$, respectively, are to be identified with the functions $\chi'(\eta)$ and $W'(\alpha)$ of Sec. V, when the quantities u, v, and $\mathcal{A}'$ which define the operator transformation (4.2) are taken to be equal to the functions $u(t)$, $v(t)$, and $\mathcal{A}'(t)$ defined for some particular time by Eqs. (6.6)–(6.11). It follows then from Eq. (5.26) that the time-dependent Wigner function obeys the functional identity

$$W(\alpha_c(\alpha_0,\alpha_0^*,t),t)=W(\alpha_0,0)\,, \tag{6.40}$$

where the function $\alpha_c(\alpha_0,\alpha_0^*,t)$ is defined by Eq. (6.20) as the (n-component) complex amplitude at time t, corresponding to the initial value α_0, for a classical system governed by the Hamiltonian (6.19). Equation (6.40) therefore states that the Wigner function for a system of quantum-mechanical oscillators governed by a Hamiltonian at most quadratic in the creation and annihilation operators is constant along the classical trajectories for the corresponding classical system.[33,9] This is a property which the Wigner function shares with the probability density $f(\alpha,t)$ for finding a system of classical oscillators with the complex amplitudes α_j at time t. The classical function $f(\alpha,t)$, however, satisfies the relation $f(\alpha_c(t),t)=f(\alpha_c(0),0)$ even when the equations of motion for the classical amplitudes $\alpha_c(t)$ are not linear.

We may note that the function $W(\alpha,t)$ may be expressed for arbitrary α, as in Eq. (5.25), in the form

$$W(\alpha,t)=W(\alpha_{0c}(\alpha,\alpha^*,t),0)\,, \tag{6.41}$$

[33] This result has also been obtained by W. H. Wells [Ann. Phys. (N. Y.) 12, 1 (1961)] and by R. P. Feynman and F. L. Vernon, Jr. [*ibid.* 24, 118 (1963)].

where the function $\alpha_{0c}(\alpha,\alpha^*,t)$ is defined in terms of the function

$$\mathcal{A}(t)\equiv-[u^\dagger(t)\mathcal{A}'(t)-\tilde{v}(t)\mathcal{A}'^*(t)]$$

by the relation

$$\alpha_{0c}(\alpha,\alpha^*,t)\equiv u^\dagger(t)\alpha-\tilde{v}(t)\alpha^*+\mathcal{A}(t)\,, \tag{6.42}$$

and is therefore the initial (n-component) complex amplitude for a classical system governed by the Hamiltonian (6.19), corresponding to the amplitude α at time t. We may verify by direct differentiation of either Eq. (6.40) or Eq. (6.41) that $W(\alpha,t)$ satisfies Liouville's equation

$$\frac{d}{dt}W(\alpha,t)=0\,, \tag{6.43}$$

where the total time derivative d/dt is defined as

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\left\{i[\tilde{\alpha}^*\omega(t)+\tilde{\alpha}\sigma^*(t)+\mathcal{K}^{*T}(t)]\frac{\partial}{\partial\alpha^*}+\text{c.c.}\right\}. \tag{6.44}$$

Let us now consider the time evolution of the P function. Let us assume that the normally ordered characteristic function

$$\chi_N(\eta,t)=\text{tr}[\rho(t)\exp(\tilde{a}^\dagger\eta)\exp(-\tilde{\eta}^*a)]\,, \tag{6.45}$$

corresponding to the Schrödinger density operator $\rho(t)$, has the Fourier transform

$$P(\alpha,t)=\pi^{-2n}\int d^{2n}\eta\,\exp(\tilde{\eta}^*\alpha-\tilde{\alpha}^*\eta)\chi_N(\eta,t) \tag{6.46}$$

during some interval of time including the initial time $t=0$. Then $\rho(t)$ has the P representation

$$\rho(t)=\int d^{2n}\alpha\,P(\alpha,t)|\alpha\rangle\langle\alpha|\,. \tag{6.47}$$

The function $P(\alpha,t)$ may be expressed in terms of quantities defined in terms of the initial state of the system, with the aid of Eqs. (5.38) and (5.39). Let us define the function $\mathcal{P}(\alpha_0,t)$, for n complex arguments α_{0j}, as the Fourier integral

$$\mathcal{P}(\alpha_0,t)\equiv\pi^{-2n}\int d^{2n}\eta_0\,\exp(\tilde{\eta}_0^*\alpha_0-\tilde{\alpha}_0^*\eta_0)\times\exp[\tfrac{1}{2}|u(t)\eta_0+v(t)\eta_0^*|^2-\tfrac{1}{2}|\eta_0|^2]\chi_N(\eta_0,0)\,. \tag{6.48}$$

Then $P(\alpha,t)$ is given, as in Eq. (5.38), by the expression

$$P(\alpha,t)=\mathcal{P}(\alpha_{0c}(\alpha,\alpha^*,t),t)\,, \tag{6.49}$$

where $\alpha_{0c}(\alpha,\alpha^*,t)$ is defined by Eq. (6.42). The time dependence of the function $\mathcal{P}(\alpha_0,t)$ given by Eq. (6.48) implies that $P(\alpha,t)$ has an explicit time dependence, in addition to the implicit time dependence involved in the definition of $\alpha_{0c}(\alpha,\alpha^*,t)$. Unlike the Wigner function, the function $P(\alpha,t)$ is then not constant along classical trajectories.

To obtain a differential equation governing the time evolution of $P(\alpha,t)$, let us first note the identity

$$\frac{d}{dt}\alpha_{0c}(\alpha,\alpha^*,t)=0\,, \tag{6.50}$$

which follows directly from the definitions (6.44) and (6.42) of d/dt and $\alpha_{0c}(\alpha,\alpha^*,t)$, respectively. The total time derivative of $P(\alpha,t)$ is then, according to Eq. (6.49),

$$\frac{d}{dt}P(\alpha,t)=\frac{\partial}{\partial t}\mathcal{P}(\alpha_0,t)\,|_{\alpha_0=\alpha_{0c}(\alpha,\alpha^*,t)}\,. \tag{6.51}$$

Let us introduce the function

$$\alpha_{c,h}(\eta_0,\eta_0^*,t)\equiv u(t)\eta_0+v(t)\eta_0^*\,, \tag{6.52}$$

which is thus defined as the homogeneous part of the classical solution (6.20), corresponding to the initial value η_0. The quantity $|\alpha_{c,h}(\eta_0,\eta_0^*,t)|^2$, which is the classical analog of the total number of quanta in the system, may be shown with the aid of Eqs. (6.6) and (6.7) to have the time derivative

$$\frac{\partial}{\partial t}|\alpha_{c,h}(\eta_0,\eta_0^*,t)|^2=i\tilde{\alpha}_{c,h}(\eta_0,\eta_0^*,t)\sigma^*(t)\alpha_{c,h}(\eta_0,\eta_0^*,t)+\text{c.c.}\,, \tag{6.53}$$

which vanishes only if $\sigma(t)=0$. We note the relations

$$\frac{\partial}{\partial\alpha}[\tilde{\eta}_0^*\alpha_{0c}(\alpha,\alpha^*,t)-\text{c.c.}]=\alpha_{c,h}^*(\eta_0,\eta_0^*,t)\,, \tag{6.54a}$$

$$\frac{\partial}{\partial\alpha^*}[\tilde{\eta}_0^*\alpha_{0c}(\alpha,\alpha^*,t)-\text{c.c.}]=-\alpha_{c,h}(\eta_0,\eta_0^*,t)\,. \tag{6.54b}$$

If we substitute Eq. (6.48) for $\mathcal{P}(\alpha_0,t)$ into Eq. (6.51) and make use of the definition (6.52) and the identities (6.53) and (6.54), we find that $P(\alpha,t)$ obeys the partial differential equation

$$\left\{\frac{d}{dt}+\tfrac{1}{2}i\left[\frac{\tilde{\partial}}{\partial\alpha}\sigma(t)\frac{\partial}{\partial\alpha}-\frac{\tilde{\partial}}{\partial\alpha^*}\sigma^*(t)\frac{\partial}{\partial\alpha^*}\right]\right\}P(\alpha,t)=0\,, \tag{6.55}$$

in which d/dt is defined by Eq. (6.44). The weight function $P(\alpha,t)$ for the P representation of $\rho(t)$ then does not satisfy Liouville's equation. The terms involving $\sigma(t)$ in Eq. (6.55), however, vanish in the classical limit, since in that limit the function $P(\alpha,t)$ varies by very small amounts over unit distances in the α_j planes. We may note that if $\sigma(t)=0$, $P(\alpha,t)$ satisfies Liouville's equation exactly. This result also follows from Eq. (6.33), which states that a coherent state remains coherent when the Hamiltonian takes the form given by Eq. (6.28).

VII. PARAMETRIC COUPLINGS

In nearly all cases of physical interest, the dominant term in the Hamiltonian for coupled oscillator systems is the free Hamiltonian

$$H_0(t)=\hbar\sum_{j=1}^{n}\omega_j a_j^\dagger(t)a_j(t)\,, \tag{7.1}$$

where ω_j is the natural frequency of oscillation of the jth oscillator. The remaining terms in the Hamiltonian (6.1) are typically small compared to $H_0(t)$, and give rise to appreciable changes in the oscillator amplitudes only after many periods of oscillation.

A situation in which the effect of the coupling between the modes is particularly pronounced is the one in which the average value of the interaction part of the Hamiltonian is nearly constant during a small number of oscillations. The interaction Hamiltonian in this case has an explicit time dependence which cancels the time dependence due to the free Hamiltonian H_0. The matrix $\sigma_{jk}(t)$, for example, is given in terms of its initial value $\sigma_{jk}(0)\equiv\sigma_{jk}$ by the relation

$$\sigma_{jk}(t)=\sigma_{jk}e^{-i(\omega_j+\omega_k)t}\,, \tag{7.2}$$

and the terms involving $\sigma_{jk}(t)$ in the Hamiltonian (6.1) take the form

$$\sigma_{jk}a_j^\dagger(t)a_k^\dagger(t)e^{-i(\omega_j+\omega_k)t}+\text{H.c.} \tag{7.3}$$

Couplings of this kind characterize a number of physical processes, including the coherent Raman and Brillouin effects, and the frequency splitting of light beams in media with nonlinear dielectric susceptibilities.[34] Louisell, Yariv, and Siegman[2] have proposed a theoretical model of the parametric amplifier consisting of two electromagnetic cavity modes coupled by an interaction of the form (7.3). The coupling between the modes of frequency ω_j and ω_k is produced by an intense pump field oscillating at the frequency $\omega=\omega_j+\omega_k$ within a cavity filled with a nonlinear dielectric substance. References 12 are devoted to an analysis of this model within the context of the coherent states and the representation of quantum states by means of quasiprobability functions. In this section we shall generalize the treatment of Ref. 12 to include the case in which more than two modes are coupled by terms of the form (7.3); our analysis will therefore be adequate to describe the case in which some of the modes are degenerate, or in which the pump field has more than one Fourier component of oscillation. The results obtained provide an instructive illustration of the analysis of the preceding section.

[34] See, for example, N. Bloembergen, *Nonlinear Optics* (W. A. Benjamin, Inc., New York, 1965), Chap. IV.

Let us assume, then, that the system of n oscillators is governed by the Hamiltonian

$$H(t)=\hbar\{\tilde{a}^\dagger(t)\omega a(t)+[\tfrac{1}{2}\tilde{a}^\dagger(t)e^{-i\omega t}\sigma e^{-i\omega t}a^\dagger(t)+\text{H.c.}]\}\,, \tag{7.4}$$

where the matrix elements of the time-independent diagonal matrix ω are

$$\omega_{jk}=\delta_{jk}\omega_j\,. \tag{7.5}$$

It will somewhat simplify our calculations if we assume that σ is real,

$$\sigma=\sigma^*. \tag{7.6}$$

The time dependence of the various quantities which describe the state of the system takes an especially simple form in the interaction picture. The interaction-picture unitary time translation operator is defined as

$$U'(t)\equiv e^{iH_0(0)t/\hbar}U(t)\,, \tag{7.7}$$

where $U(t)$ is defined by Eqs. (6.13) and (6.14). Let us define $H_{1S}(t)$ as the interaction part of the Hamiltonian (7.4), evaluated in the Schrödinger picture, i.e., by the expression

$$H_{1S}(t)=\tfrac{1}{2}\hbar\tilde{a}^\dagger e^{-i\omega t}\sigma e^{-i\omega t}a^\dagger+\text{H.c.} \tag{7.8}$$

Then it follows from Eqs. (6.13) and (6.14) that $U'(t)$ satisfies the equations

$$i\hbar\frac{d}{dt}U'(t)=H_{1I}(t)U'(t)\,, \tag{7.9}$$

$$U'(0)=1\,, \tag{7.10}$$

where $H_{1I}(t)$, the interaction Hamiltonian in the interaction picture, is defined as

$$H_{1I}(t)=e^{iH_0(0)t/\hbar}H_{1S}(t)e^{-iH_0(0)t/\hbar}. \tag{7.11}$$

The special form we have assumed for the explicit time dependence of the coupling implies that $H_{1I}(t)$ is independent of time. If we substitute Eq. (7.8) for $H_{1S}(t)$ into Eq. (7.11) and then make use of the identity

$$e^{iH_0(0)t/\hbar}ae^{-iH_0(0)t/\hbar}=e^{-i\omega t}a\,, \tag{7.12}$$

we find

$$H_{1I}(t)=H_{1I}(0)=\tfrac{1}{2}\hbar(\tilde{a}^\dagger\sigma a^\dagger+\tilde{a}\sigma a)\,. \tag{7.13}$$

The solution to Eqs. (7.9) and (7.10) is therefore

$$U'(t)=\exp[-\tfrac{1}{2}i(\tilde{a}^\dagger\sigma a^\dagger+\tilde{a}\sigma a)t]\,. \tag{7.14}$$

Let us define the interaction-picture density operator $\rho'(t)$ in terms of the Heisenberg density operator ρ by the equation

$$\rho'(t)\equiv U'(t)\rho U'^{-1}(t)\,. \tag{7.15}$$

Then $\rho'(t)$ may be expressed in terms of the Schrödinger density operator $\rho(t)$ as

$$\rho'(t)=e^{iH_0(0)t/\hbar}\rho(t)e^{-iH_0(0)t/\hbar}. \tag{7.16}$$

It follows from this equation and the identity (7.12) that the mean value of any operator function $F(a,a^\dagger)$ at time t is

$$\mathrm{tr}[\rho(t)F(a,a^\dagger)]=\mathrm{tr}[\rho'(t)F(e^{-i\omega t}a,e^{i\omega t}a^\dagger)], \quad (7.17)$$

so that the variables appropriate to the interaction picture are the uncoupled operators $e^{-i\omega t}a$.

It is useful to consider the behavior of the operators a_j under the similarity transformation generated by $U'(t)$. Let us define the operator $a'(t)$ in terms of the Schrödinger operator a by the relation

$$a'(t)\equiv U'^{-1}(t)aU'(t). \quad (7.18)$$

It follows from this definition and from Eqs. (7.7), (7.12), and (6.12a) that $a'(t)$ may be expressed in terms of the Heisenberg operator $a(t)$ as

$$a'(t)=e^{i\omega t}a(t). \quad (7.19)$$

It is important to distinguish between the operator $a'(t)$ and the interaction-picture operator $e^{-i\omega t}a$. The time dependence of $a'(t)$ is governed completely by the interaction part of the Hamiltonian, and $a'(t)$ therefore reduces to the Schrödinger operator a when the coupling between the modes vanishes. It follows from Eqs. (7.18) and (7.14) that $a'(t)$ satisfies the differential equation

$$i\frac{d}{dt}a'(t)=\sigma a'^\dagger(t). \quad (7.20)$$

The solution to this equation is, by virtue of Eq. (7.6),

$$a'(t)=u'(t)a+v'(t)a^\dagger, \quad (7.21)$$

where

$$u'(t)=\cosh\sigma t, \quad (7.22)$$

$$v'(t)=-i\sinh\sigma t. \quad (7.23)$$

The Heisenberg operator $a(t)$ is then given, according to Eq. (7.19), by

$$a(t)=e^{-i\omega t}[(\cosh\sigma t)a-i(\sinh\sigma t)a^\dagger]. \quad (7.24)$$

Let us now suppose that the interaction-picture density operator $\rho'(t)$ has a P representation, i.e., that it can be written in terms of some weight function $P'(\alpha',t)$ in the form

$$\rho'(t)=\int d^{2n}\alpha'\, P'(\alpha',t)|\alpha'\rangle\langle\alpha'|. \quad (7.25)$$

It then follows from Eq. (7.16) and the identity

$$\exp[-iH_0(0)t/\hbar]|\alpha'\rangle=|e^{-i\omega t}\alpha'\rangle \quad (7.26)$$

that the Schrödinger density operator $\rho(t)$ has a P representation as given by Eq. (6.47), in which the weight function $P(\alpha,t)$ satisfies the relation

$$P'(\alpha',t)=P(e^{-i\omega t}\alpha',t). \quad (7.27)$$

Thus the operator $\rho'(t)$ has a P representation if and only if $\rho(t)$ does, and the relationship between the weight functions $P'(\alpha',t)$ and $P(\alpha,t)$ may be expressed by subjecting the arguments of P to the complex rotations associated with uncoupled oscillators.

The value of $P'(\alpha',t)$ at some time t may be found either by making use of Eq. (7.27) in Eqs. (6.48) and (6.49), or alternatively by noting that the operator $U'(t)$ which defines $\rho'(t)$ in terms of ρ generates the transformation

$$U'^{-1}(t)aU'(t)=u'(t)a+v'(t)a^\dagger, \quad (7.28)$$

and then making use of Eqs. (5.38) and (5.39). It follows then from the solutions (7.22) and (7.23) for $u'(t)$ and $v'(t)$ that $P'(\alpha',t)$ is given by the relation

$$P'(\alpha',t)=\mathcal{P}([(\cosh\sigma t)\alpha'+i(\sinh\sigma t)\alpha'^*],t), \quad (7.29)$$

where the function $\mathcal{P}(\alpha_0,t)$ is defined as the Fourier integral

$$\mathcal{P}(\alpha_0,t)=\pi^{-2n}\int d^{2n}\eta_0 \exp(\tilde{\eta}_0^*\alpha_0-\tilde{\alpha}_0^*\eta_0)$$
$$\times\exp[\tfrac{1}{2}|(\cosh\sigma t)\eta_0-i(\sinh\sigma t)\eta_0^*|^2-\tfrac{1}{2}|\eta_0|^2]$$
$$\cdot\chi_N(\eta_0,0). \quad (7.30)$$

The function $P'(\alpha',t)$ satisfies the partial differential equation

$$\left\{\left(\frac{d}{dt}\right)'+\left[\tfrac{1}{2}i\frac{\tilde{\partial}}{\partial\alpha'}\sigma\frac{\partial}{\partial\alpha'}-\tfrac{1}{2}i\frac{\tilde{\partial}}{\partial\alpha'^*}\sigma\frac{\partial}{\partial\alpha'^*}\right]\right\}P'(\alpha',t)=0, \quad (7.31)$$

where the differential operator $(d/dt)'$ is defined as

$$\left(\frac{d}{dt}\right)'\equiv\frac{\partial}{\partial t}+i\tilde{\alpha}'\sigma\frac{\partial}{\partial\alpha'^*}-i\tilde{\alpha}'^*\sigma\frac{\partial}{\partial\alpha'}. \quad (7.32)$$

The time dependence of $P'(\alpha',t)$ takes a somewhat simpler form when P' is expressed in terms of variables which obey decoupled equations of motion. Since the matrix σ is real and symmetric, it can be diagonalized by means of a real orthogonal transformation. We may therefore write

$$\sigma=\tau^{-1}\kappa\tau, \quad (7.33)$$

where κ is a real diagonal matrix with eigenvalues κ_j,

$$\kappa_{jk}=\kappa_{jk}^*=\delta_{jk}\kappa_j, \quad (7.34)$$

and τ is a real orthogonal matrix,

$$\tau=\tau^*=\tilde{\tau}^{-1}. \quad (7.35)$$

Let us define the operator $b(t)$ by the expression

$$b(t)\equiv e^{-i\pi/4}\tau a'(t). \quad (7.36)$$

The operators $b_j(t)$ and $b_j^\dagger(t)$ then satisfy the canonical commutation relations

$$[b_j(t),b_k^\dagger(t)]=\delta_{jk}, \quad (7.37)$$
$$[b_j(t),b_k(t)]=0.$$

It follows from the definition (7.36) of $b(t)$ and Eqs. (7.21)–(7.23) and (7.33) that the operators $b_j(t)$ are given in terms of their initial values by the decoupled relations

$$b_j(t)=(\cosh\kappa_j t)b_j(0)-(\sinh\kappa_j t)b_j^\dagger(0). \qquad (7.38)$$

We have introduced the phase factor $e^{-i\pi/4}$ into the definition (7.36) of $b(t)$ so that the Hermitian operators

$$\hat{q}_j(t)\equiv 2^{-1/2}[b_j^\dagger(t)+b_j(t)], \qquad (7.39a)$$

$$\hat{p}_j(t)\equiv i2^{-1/2}[b_j^\dagger(t)-b_j(t)], \qquad (7.39b)$$

have the simple exponential time dependence

$$\hat{q}_j(t)=e^{-\kappa_j t}\hat{q}_j(0), \qquad (7.40a)$$

$$\hat{p}_j(t)=e^{\kappa_j t}\hat{p}_j(0). \qquad (7.40b)$$

Let us now define real variables q and p in terms of the arguments α' of $P(\alpha',t)$ by the c-number analogs of Eqs. (7.36) and (7.39), i.e., by the relations

$$q=2^{-1/2}\tau[e^{i\pi/4}\alpha'^*+e^{-i\pi/4}\alpha'], \qquad (7.41a)$$

$$p=i2^{-1/2}\tau[e^{i\pi/4}\alpha'^*-e^{-i\pi/4}\alpha']. \qquad (7.41b)$$

If we similarly define real variables x_0 and y_0 in terms of the arguments η_0 of χ_N,

$$x_0=2^{-1/2}\tau[e^{i\pi/4}\eta_0^*+e^{-i\pi/4}\eta_0], \qquad (7.42a)$$

$$y_0=i2^{-1/2}\tau[e^{i\pi/4}\eta_0^*-e^{-i\pi/4}\eta_0], \qquad (7.42b)$$

then we find from Eqs. (7.29) and (7.30) that $P'(\alpha',t)$ may be expressed in terms of the real variables q and p in the form

$$P'(q,p,t)=\mathcal{P}(e^{\kappa t}q,e^{-\kappa t}p,t), \qquad (7.43)$$

where the function $\mathcal{P}(q_0,p_0,t)$ is defined for arbitrary real arguments q_{0j} and p_{0j} as

$$\mathcal{P}(q_0,p_0,t)=(2\pi^2)^{-n}\int\prod_{j=1}^{n}[dx_{0j}dy_{0j}\,e^{i(x_{0j}p_{0j}-y_{0j}q_{0j})}]$$
$$\times\exp\{\sum_{j=1}^{n}[\tfrac{1}{4}(e^{-2\kappa_j t}-1)x_{0j}^2+\tfrac{1}{4}(e^{2\kappa_j t}-1)y_{0j}^2]\}$$
$$\cdot\chi_N(x_0,y_0,0). \qquad (7.44)$$

It is not difficult to see that this integral will not in general converge for all times t. The coefficient of x_{0j}^2 in the exponential function becomes infinite as $t\to-\infty$ if $\kappa_j>0$, or as $t\to\infty$ if $\kappa_j<0$. Similarly, the coefficient of y_{0j}^2 becomes infinite as $t\to\infty$ if $\kappa_j>0$, or as $t\to-\infty$ if $\kappa_j<0$. It is therefore clear that unless $\chi_N(\eta_0,0)$ approaches zero as $|\eta_0|\to\infty$ more rapidly than the exponential of any (negative-definite) quadratic form in η_0 and η_0^*, a P representation can exist during a finite time interval at most.

This is also clear from the form the differential equation (7.31) takes when it is expressed in terms of the real variables q and p:

$$\left\{\frac{\partial}{\partial t}+\sum_{j=1}^{n}\kappa_j\left[\left(p_j\frac{\partial}{\partial p_j}-q_j\frac{\partial}{\partial q_j}\right)-\frac{1}{2}\left(\frac{\partial^2}{\partial p_j^2}-\frac{\partial^2}{\partial q_j^2}\right)\right]\right\}$$
$$\times P'(q,p,t)=0. \qquad (7.45)$$

This equation has solutions of the form

$$P'(q,p,t)=\prod_j f_j(q_j,t)g_j(p_j,t),$$

where $f_j(q_j,t)$ and $g_j(p_j,t)$ obey the differential equations

$$\left[\frac{\partial}{\partial t}-\kappa_j\left(q_j\frac{\partial}{\partial q_j}-\frac{1}{2}\frac{\partial^2}{\partial q_j^2}\right)\right]f_j(q_j,t)=0, \qquad (7.46a)$$

$$\left[\frac{\partial}{\partial t}+\kappa_j\left(p_j\frac{\partial}{\partial p_j}-\frac{1}{2}\frac{\partial^2}{\partial p_j^2}\right)\right]g_j(p_j,t)=0, \qquad (7.46b)$$

each of which resembles a Fokker-Planck equation in one variable. It is not difficult to show that for reasonably well-behaved initial functions $f_j(q_j,0)$ and $g_j(p_j,0)$, these equations lead as $t\to\infty$ to highly singular functions $f_j(q_j,t)$ if $\kappa_j>0$, and to highly singular functions $g_j(p_j,t)$ if $\kappa_j<0$.

As an example of some interest, let us consider the case in which the initial state of the system is the product of chaotic mixtures for each mode. To simplify calculations, let us assume that the mean quantum numbers for all of the modes in this initial state are equal. The initial density operator then has a P representation,[6] and the weight function is

$$P(\alpha_0,0)=(\pi\langle m\rangle)^{-n}\exp[-|\alpha_0|^2/\langle m\rangle], \qquad (7.47)$$

where $\langle m\rangle$ is the initial mean quantum number for each mode. The normally ordered characteristic function at $t=0$ is then

$$\chi_N(\eta_0,0)=e^{-\langle m\rangle|\eta_0|^2}$$
$$=\exp[-\tfrac{1}{2}\langle m\rangle\sum_{j=1}^{n}(x_{0j}^2+y_{0j}^2)]. \qquad (7.48)$$

If we substitute this expression into Eq. (7.44) and perform the indicated integrations, we find, by virtue of Eq. (7.43),

$$P'(q,p,t)=\prod_{j=1}^{n}\{[\pi^2\lambda_j(t)\lambda_j(-t)]^{-1/2}$$
$$\times\exp[-\tfrac{1}{2}q_j^2/\lambda_j(-t)-\tfrac{1}{2}p_j^2/\lambda_j(t)]\}, \qquad (7.49)$$

where

$$\lambda_j(t)\equiv\tfrac{1}{2}[(1+2\langle m\rangle)e^{2\kappa_j t}-1]. \qquad (7.50)$$

The Fourier integral which leads to Eq. (7.49) converges only when the conditions

$$\lambda_j(t)\geq 0, \qquad (7.51a)$$

$$\lambda_j(-t)\geq 0, \qquad (7.51b)$$

are satisfied for $j=1, \cdots, n$, and thus only within the time interval

$$-t_1 \leq t \leq t_1, \tag{7.52}$$

where the time $t_1 \geq 0$ is defined by the equation

$$\exp(2|\kappa_j|_{\max} t_1) = 1 + 2\langle m\rangle. \tag{7.53}$$

For $|t| > t_1$, the integral which results from the substitution of Eq. (7.48) into Eq. (7.44) is strongly divergent, and no P representation exists in the sense we have defined.

It is interesting to observe that the Wigner function remains well behaved at all times. When the Wigner function $W'(\alpha',t)$ corresponding to the interaction-picture density operator $\rho'(t)$ is expressed in terms of the real variables q and p defined by Eqs. (7.41), it may be shown to obey the functional identity

$$W'(q,p,t) = W'(e^{\kappa t}q, e^{-\kappa t}p, 0), \tag{7.54}$$

which is easily deduced from Eq. (6.41) and the expressions (7.40) for $\hat{q}(t)$ and $\hat{p}(t)$. For the initial state (7.47), the Wigner function is given by

$$W'(q,p,t) = \prod_{j=1}^{n} \{[\pi^2\langle\hat{q}_j^2(t)\rangle\langle\hat{p}_j^2(t)\rangle]^{-1/2} \times \exp[-\tfrac{1}{2}q_j^2/\langle\hat{q}_j^2(t)\rangle - \tfrac{1}{2}p_j^2/\langle\hat{p}_j^2(t)\rangle]\}, \tag{7.55}$$

in which $\langle\hat{q}_j^2(t)\rangle$ and $\langle\hat{p}_j^2(t)\rangle$ are the second moments of the operators $\hat{q}_j(t)$ and $\hat{p}_j(t)$, and are given by the relations

$$\langle\hat{q}_j^2(t)\rangle = \lambda_j(-t) + \tfrac{1}{2}, \tag{7.56a}$$

$$\langle\hat{p}_j^2(t)\rangle = \lambda_j(t) + \tfrac{1}{2}. \tag{7.56b}$$

It may be noted that for an arbitrary initial density operator, the variances

$$\Delta q_j^2(t) \equiv \mathrm{tr}\{\rho[\hat{q}_j(t) - \langle\hat{q}_j(t)\rangle]^2\}, \tag{7.57a}$$

$$\Delta p_j^2(t) \equiv \mathrm{tr}\{\rho[\hat{p}_j(t) - \langle\hat{p}_j(t)\rangle]^2\}, \tag{7.57b}$$

may be expressed in terms of the Wigner function by means of the relations

$$\Delta q_j^2(t) = 2^{-n}\int d^nq\,d^np\,[q_j - \langle\hat{q}_j(t)\rangle]^2 W'(q,p,t), \tag{7.58a}$$

$$\Delta p_j^2(t) = 2^{-n}\int d^nq\,d^np\,[p_j - \langle\hat{p}_j(t)\rangle]^2 W'(q,p,t), \tag{7.58b}$$

which follow simply from the formula (2.20) for symmetrized products. The relation (2.15) for normally ordered products implies that the corresponding expressions in the P representation are

$$\Delta q_j^2(t) = 2^{-n}\int d^nq\,d^np \times [q_j - \langle\hat{q}_j(t)\rangle]^2 P'(q,p,t) + \tfrac{1}{2}, \tag{7.59a}$$

$$\Delta p_j^2(t) = 2^{-n}\int d^nq\,d^np \times [p_j - \langle\hat{p}_j(t)\rangle]^2 P'(q,p,t) + \tfrac{1}{2}. \tag{7.59b}$$

It is clear from these relations that a positive P representation can exist only when the conditions

$$\Delta q_j^2(t) \geq \tfrac{1}{2}, \tag{7.60a}$$

$$\Delta p_j^2(t) \geq \tfrac{1}{2}, \tag{7.60b}$$

are satisfied for $j=1, \cdots, n$. That these conditions can all be satisfied during a finite time interval at most is implied by the relations

$$\Delta q_j^2(t) = e^{-2\kappa_j t}\Delta q_j^2(0), \tag{7.61a}$$

$$\Delta p_j^2(t) = e^{2\kappa_j t}\Delta p_j^2(0), \tag{7.61b}$$

which follow from Eqs. (7.40). If a P representation were to exist outside the time interval defined by Eqs. (7.60), the weight function would have to have negative variances. In the example we have discussed, the P representation ceases to exist, in the sense we have defined, at the instant when one of the inequalities (7.60) fails to be satisfied.

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