

where we now use the propagator \mathcal{G}_\pm in place of G in front of the integral so as to obtain correct external-line wave functions for the S matrix.

The only vertices which get inserted by the factor in square brackets in (20.36b) are the bare vertices S_n and $V_{(\alpha i)\beta}$. Therefore $\langle 0, \infty | \phi^i | 0, -\infty \rangle$ may be expressed in the compact form (12.20), but with G replaced by \mathcal{G}_\pm , provided the symbol $\delta/\delta\varphi$ is no longer taken literally but is understood to yield $GS_\beta G$ when acting on G and S_{n+1} when acting on S_n , to have no effect on $V_{(\alpha i)\beta}$, and to insert (in all possible ways) into any fictitious quantum loop merely one more vertex $V_{(\alpha i)\beta}$ having the same orientation as all the other vertices already in the loop.⁵⁸ With this understanding it is easy to see that (20.35) then yields also Eqs. (12.21), (12.22), and (12.23), with the modification $G \rightarrow \mathcal{G}_\pm$ applied to all external lines. Chronological product amplitudes defined in this way may be used directly in (13.6) to calculate the S matrix.

The consistency of these simple rules with previously obtained results is readily checked. For example, if non-causal chains are reinserted into Figs. 2(b) and 3(b) the resulting primary diagrams for the lowest-order radiative corrections to the one- and two-quantum amplitudes are precisely those obtained by the present prescription. We note in particular the sufficiency of the vertices S_n and $V_{(\alpha i)\beta}$ and the uniform orientation of the latter around any fictitious quantum loop.

⁵⁸ It will be noted that the operators $\delta/\delta\varphi^i$, when redefined in this way, are still commutative.

Quantum Theory of Gravity. III. Applications of the Covariant Theory*

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The basic momentum-space propagators and vertices (including those for the fictitious quanta) are given for both the Yang-Mills and gravitational fields. These propagators are used to obtain the cross sections for gravitational scattering of two scalar particles, scattering of gravitons by scalar particles, graviton-graviton scattering, two-graviton annihilation of scalar-particle pairs, and graviton bremsstrahlung. Special features of these cross sections are noted. Problems arising in renormalization theory and the role of the Planck length are discussed. The gravitational Ward identity is derived, and the structure of the radiatively corrected 1-graviton vertex for a scalar particle is displayed. The Ward identity is only one of an infinity of identities relating the many-graviton vertex functions of the theory. The need for such identities may be eliminated in principle by computing radiative corrections directly in coordinate space, using the theory of manifestly covariant Green's functions. As an example of such a calculation, the contribution of conformal metric fluctuations to the vacuum-to-vacuum amplitude is summed to all orders. The physical significance of the renormalization terms is discussed. Finally, Weinberg's treatment of the infrared problem is examined. It is not difficult to show that the fictitious quanta contribute negligibly to infrared amplitudes, and hence that Weinberg's use of the DeDonder gauge is justified. His proof that the infrared problem in gravodynamics can be handled just as in electrodynamics is thereby made rigorous.

1. INTRODUCTION

IN the first two papers of this series¹ two distinct mathematical approaches to the quantum theory of gravity were developed, one based on the so-called

canonical or Hamiltonian theory and the other on the manifestly covariant theory of propagators and diagrams. So far no rigorous mathematical link between the two has been established. In part this is due to the kinds of questions each asks. The canonical theory leads almost unavoidably to speculations about the meaning of "amplitudes for different 3-geometries" or "the wave function of the universe." The covariant theory, on the other hand, concerns itself with "micro-processes" such as scattering, vacuum polarization, etc. Some of the questions raised by the canonical theory were explored in I. In this third and final paper of the series we examine some of the consequences of the covariant theory.

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¹ B. S. DeWitt, Phys. Rev. **160**, 1113 (1967); preceding paper, *ibid.* **162**, 1195 (1967). These papers will be referred to as I and II, respectively. The notation of the present paper is the same as that of II, which should be consulted for the definition of unfamiliar symbols, e.g., S_n for the n -pronged bare vertex and $V_{(\alpha i)\beta}$ for the asymmetric vertex coupling real and fictitious quanta.

Armed with the formalism constructed in II one can in principle carry out the calculation of any "micro-process" to any order of perturbation theory in a manner which is completely invariant and unambiguous except for the arbitrary high-energy cutoff which must be introduced to render divergent integrals finite. A few of these calculations have actually been performed, and the only thing which prevents more of them from being done is the extreme tediousness of the algebra involved and the lack of any experimental motivation for them. It is a pity that Nature displays such indifference to so intriguing and beautiful a subject, for the calculations themselves are of considerable intrinsic interest. The present paper contains several examples. They are by no means exhaustive but have been selected as useful landmarks in a still largely unexplored territory. Not all of these were originally carried out by the author, but it is hoped that their unified presentation here will make their results more accessible than hitherto.

Section 2 begins with the rules of calculation in momentum space. The basic structural elements of the theory, namely the propagators for real and fictitious quanta, the vertices S_3 , S_4 , $V_{(\alpha i)\beta}$, and the coupling with matter fields, are given for both the Yang-Mills and gravitational fields. The standard Feynman rules are summarized. The results of a few lowest-order scattering calculations based on these rules are given and discussed in Sec. 3. Included are the cross section for gravitational scattering of two scalar particles, the cross section for scattering of gravitons by scalar particles, the corresponding annihilation cross section, and the graviton-graviton cross section. Section 4 is devoted to the problem of gravitational bremsstrahlung. The role of the energy quadrupole moment tensor and the absence of the forward peak at high energies, characteristic of photon bremsstrahlung, are noted.

Section 5 discusses some of the problems which arise in renormalization theory. Although the Yang-Mills theory looks as if it may be renormalizable (provided its infrared difficulties can be disposed of), quantum gravodynamics is definitely not renormalizable in the usual sense. Tentative proposals for dealing with this situation are briefly described, as is also the evidence that gravity contains its own cutoff—at the Planck length. Illustration of the actual details of the renormalization program, by explicit calculation of a radiative correction, is postponed to Sec. 7.

The gravitational Ward identity and its implications for gravitational form factors are derived in Sec. 6. The general structure of the radiatively corrected 1-graviton vertex is displayed in the case of a scalar particle. It is emphasized that the gravitational Ward identity is only one of an infinity of identities relating the many-graviton vertex functions of the theory. The need for Ward identities can be eliminated by computing radiative corrections directly in coordinate space

rather than in momentum space. An example of such a calculation is given in Sec. 7, where the contribution of conformal metric fluctuations to the vacuum-to-vacuum amplitude is summed to all orders. The calculation, which is manifestly covariant throughout, makes use of an integral representation for the amplitude. A resumé is given of that part of the mathematical theory of covariant Green's functions which is needed.

Section 8 concludes the paper with a review of Weinberg's treatment of the infrared problem (see Ref. 37). If Yang-Mills quanta are assumed to be massless then, since they can act as their own sources, they give rise to the special infrared divergences which plague massless electrodynamics. Weinberg showed that gravity miraculously escapes these difficulties; its infrared divergences can be handled by the standard methods familiar in ordinary quantum electrodynamics. His proofs, however, were incomplete, since he did not have available a fully elaborated quantum theory. In particular he used the DeDonder gauge without taking into account the fictitious quanta. It is not difficult to show that the fictitious quanta contribute negligibly in the infrared limit. Weinberg's results are therefore rigorous.

2. RULES OF CALCULATION IN MOMENTUM SPACE

We begin with the vertex functions for the Yang-Mills field interacting with itself. We have seen in II that when the standard field variables are used only S_3 and S_4 are nonvanishing for this case. In momentum space these become (apart from a δ function expressing conservation of momentum)

$$\frac{\delta^3 S}{\delta A^\alpha_\mu \delta A^{\beta'}_{\nu'} \delta A^{\gamma''}_{\sigma''}} \rightarrow i c_{\alpha\beta\gamma} [(p'^\mu - p''^\mu) \eta^{\nu\sigma} + (p''^\nu - p^\nu) \eta^{\sigma\mu} + (p^\sigma - p'^\sigma) \eta^{\mu\nu}], \quad (2.1)$$

$$\frac{\delta^4 S}{\delta A^\alpha_\mu \delta A^{\beta'}_{\nu'} \delta A^{\gamma''}_{\sigma''} \delta A^{\delta'''}_{\tau'''}} \rightarrow -c_{\alpha\beta\epsilon} c^\epsilon_{\gamma\delta} (\eta^{\mu\sigma} \eta^{\nu\tau} - \eta^{\mu\tau} \eta^{\nu\sigma}) - c_{\alpha\gamma\epsilon} c^\epsilon_{\beta\delta} (\eta^{\mu\nu} \eta^{\sigma\tau} - \eta^{\mu\tau} \eta^{\sigma\nu}) - c_{\alpha\delta\epsilon} c^\epsilon_{\gamma\beta} (\eta^{\mu\sigma} \eta^{\tau\nu} - \eta^{\mu\nu} \eta^{\tau\sigma}). \quad (2.2)$$

The correspondence of momenta with indices is $p_{\alpha\mu}$, $p'^{\beta\nu'}$, $p''^{\gamma''\sigma''}$, $p'''^{\delta'''\tau''''}$. All momenta are incoming (to the vertex), and momentum conservation implies $p + p' + p'' = 0$ for S_3 and $p + p' + p'' + p''' = 0$ for S_4 . Indices on the structure constants are raised and lowered by means of the Cartan metric $\gamma_{\alpha\beta}$. When all indices are in the lower position the structure constants are completely antisymmetric.

In addition to the above vertices, the fictitious vertex $V_{(\alpha i)\beta}$ is needed for the calculation of radiative

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma',\sigma'')\beta'} \rightarrow -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma} + p''^{\sigma}). \quad (2.3)$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \rightarrow \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \quad (2.4)$$

$$\hat{G} \rightarrow \gamma^{\alpha\beta}/p^2, \quad (2.5)$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational

field are much more complicated. In this case we shall employ the momentum-index combinations $p_{\mu\nu}$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''l'''k'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms² and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

$$\begin{array}{l} \delta^3 S \\ \hline \delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''} \\ \text{Sym}[-\frac{1}{4}P_3(p \cdot p' \eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}) - \frac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}) + \frac{1}{4}P_3(p \cdot p' \eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}) + \frac{1}{2}P_6(p \cdot p' \eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}) + P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho}) \\ - \frac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda}) + \frac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}) + \frac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}) + P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho}) + P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}) \\ - P_3(p \cdot p' \eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})], \quad (2.6) \end{array}$$

$$\begin{array}{l} \delta^4 S \\ \hline \delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{l''k''} \\ \text{Sym}[-\frac{1}{8}P_6(p \cdot p' \eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) - \frac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) - \frac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) + \frac{1}{8}P_6(p \cdot p' \eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) \\ + \frac{1}{4}P_6(p \cdot p' \eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\kappa\epsilon}) + \frac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) + \frac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) - \frac{1}{4}P_6(p \cdot p' \eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\kappa\epsilon}) \\ + \frac{1}{4}P_{24}(p \cdot p' \eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\kappa\epsilon}) + \frac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\kappa\epsilon}) + \frac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\kappa\epsilon}) + \frac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\kappa\epsilon}) \\ - \frac{1}{2}P_{12}(p \cdot p' \eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\kappa\epsilon}) - \frac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\nu\rho}\eta^{\lambda\nu}\eta^{\kappa\epsilon}) + \frac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\kappa\epsilon}) - \frac{1}{2}P_{24}(p \cdot p' \eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\kappa}\eta^{\epsilon\sigma}) \\ - P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\kappa}\eta^{\epsilon\mu}) - P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\kappa}\eta^{\epsilon\sigma}\eta^{\tau\mu}) - P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\kappa}\eta^{\epsilon\mu}\eta^{\nu\lambda}) - P_{12}(p^{\rho}p'^{\kappa}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\epsilon}) \\ + P_6(p \cdot p' \eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\kappa}\eta^{\epsilon\mu}) - P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\kappa}\eta^{\epsilon\lambda}) - \frac{1}{2}P_{12}(p \cdot p' \eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\kappa}\eta^{\tau\epsilon}) - P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\kappa}\eta^{\nu\epsilon}) \\ - P_6(p^{\rho}p'^{\kappa}\eta^{\lambda\epsilon}\eta^{\mu\sigma}\eta^{\nu\tau}) - P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\kappa}\eta^{\epsilon\lambda}) - P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\kappa}\eta^{\nu\epsilon}) + 2P_6(p \cdot p' \eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\kappa}\eta^{\epsilon\mu})]. \quad (2.7) \end{array}$$

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once S_2^0 is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits

him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$\begin{array}{l} V_{(\mu \sigma''\tau'')\nu'} \rightarrow \\ \frac{1}{2}\text{Sym}[2p''_{\mu}p'^{\sigma}\delta_{\nu'}\tau - p''_{\mu}p'^{\nu}\eta^{\sigma\tau} \\ + (p_{\nu}p'^{\sigma} - p'_{\nu}p^{\sigma})\delta_{\mu'}\tau + p \cdot p'\delta_{\mu'}\delta_{\nu'}\tau], \quad (2.8) \end{array}$$

where the momentum-index combinations are p_{μ} , p'_{ν} , $p''_{\sigma''}$, τ'' , and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \rightarrow (\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \quad (2.9)$$

$$\hat{G} \rightarrow \eta^{\mu\nu}/p^2. \quad (2.10)$$

² The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

If one wishes to calculate processes involving the interaction of the Yang-Mills and/or gravitational field with matter, additional vertices describing this interaction must be included. As prototypes of such vertices, we shall display those which arise from interactions with scalar (or pseudoscalar) particles. The latter particles contribute to the total action functional an expression of the form

$$S_\varphi = -\frac{1}{2} \int g^{1/2} (\bar{\varphi}_{;\mu} \varphi^{;\mu} + m^2 \bar{\varphi} \varphi) dx, \quad (2.11)$$

where the covariant derivative is defined in Table I of II and where

$$\bar{\varphi} \equiv \varphi \tilde{\gamma}, \quad \gamma G_\alpha \gamma^{-1} = -G_\alpha \tilde{\gamma}, \quad (2.12)$$

γ being the matrix which connects the two forms of a self-contragredient representation (of the Yang-Mills Lie group) generated by the matrices G_α and $-G_\alpha \tilde{\gamma}$, respectively. We find

$$\frac{\delta^3 S_\varphi}{\delta \bar{\varphi}' \delta \varphi \delta A^{\alpha'}{}_{\mu'}} \rightarrow -i(p'^\mu - p^\mu) G_\alpha, \quad (2.13)$$

$$\frac{\delta^4 S_\varphi}{\delta \bar{\varphi}' \delta \varphi \delta A^{\alpha'}{}_{\mu'} \delta A^{\beta''}{}_{\nu''}} \rightarrow (G_\alpha G_\beta + G_\beta G_\alpha) \eta^{\mu\nu}, \quad (2.14)$$

$$\frac{\delta^3 S_\varphi}{\delta \bar{\varphi}' \delta \varphi \delta \varphi_{\mu''}{}_{\nu''}} \rightarrow -\frac{1}{2} [\eta^{\mu\nu} (m^2 - p \cdot p') + p^\mu p'^\nu + p^\nu p'^\mu], \quad (2.15)$$

$$\begin{aligned} & \frac{\delta^4 S_\varphi}{\delta \bar{\varphi}' \delta \varphi \delta \varphi_{\mu''}{}_{\nu''} \delta \varphi_{\sigma''}{}_{\tau''}} \rightarrow \\ & \frac{1}{4} [(\eta^{\mu\sigma} \eta^{\nu\tau} + \eta^{\mu\tau} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\sigma\tau}) (m^2 - p \cdot p') \\ & + (p^\mu p'^\sigma + p^\sigma p'^\mu) \eta^{\nu\tau} + (p^\nu p'^\tau + p^\tau p'^\nu) \eta^{\mu\sigma} \\ & + (p^\mu p'^\tau + p^\tau p'^\mu) \eta^{\nu\sigma} + (p^\nu p'^\sigma + p^\sigma p'^\nu) \eta^{\mu\tau} \\ & - (p^\mu p'^\nu + p^\nu p'^\mu) \eta^{\sigma\tau} - (p^\sigma p'^\tau + p^\tau p'^\sigma) \eta^{\mu\nu}]. \end{aligned} \quad (2.16)$$

The corresponding vertices which describe the interaction of the gravitational and/or Yang-Mills fields with particles having spin are obtained by straightforward computation from the pertinent action functional. The latter is obtained in each case via the "principle of minimal coupling" (which, in the case of gravity, is nothing but the "strong equivalence principle") from the corresponding action functional in the absence of gravitational and Yang-Mills fields, by replacing ordinary derivatives by covariant derivatives, the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$, and the volume element dx by $g^{1/2} dx$. We do not give here the results of such calculations for particles with spin but merely point out (what is more useful for the reader) that the three-pronged vertices, when sandwiched between normalized wave functions, always reduce, in the limit of zero momentum transfer, with particle momenta on

the mass shell, to precisely the forms (2.13) and (2.15) regardless of the magnitude of the particle spin. This may be proved in each instance as a straightforward consequence of the gauge invariance of the theory and, when extended to the radiatively corrected vertices, constitutes a boundary condition on the Yang-Mills and gravitational form factors.³ [See also Sec. 6.]

It is to be emphasized that the inclusion of additional fields in no way affects the formal theoretical structure developed in II. The topology and invariance properties of diagrams remain completely unchanged. One simply permits the field indices i, j , etc., to extend over a greater range of values in order to accommodate the components of the new fields which have been added. The only differences are differences of detail such as, for example, the sign modifications due to statistics which appear when some of the added components are those of fermion fields, or changes in the structure of the invariance group which arise from having both the Yang-Mills and gravitational fields simultaneously present and interacting with each other.⁴

The rules for combining vertices and propagators into transition amplitudes are completely standard. With the notational conventions of the present paper they may be summarized as follows: (1) An expression such as (2.1), (2.2), (2.3), (2.6), etc., for each vertex; (2) an expression such as (2.4), (2.5), (2.9), (2.10), etc. for each propagator; (3) a factor $(-i)/(2\pi)^4$ for each independent closed loop; (4) an additional factor (-1) for each closed fermion or fictitious-quantum loop, or when necessary to assure antisymmetry of fermion amplitudes; (5) an over-all factor $i(2\pi)^4$ times a δ function assuring total energy-momentum conservation; (6) a wave function u^i_A (see Table II of II) or its complex conjugate evaluated at $x=0$ for each external line; (7) integration over all the independent momenta.

Gauge invariance may be invoked as a useful consistency check in all calculations. However, it must be applied to the entire amplitude for a given process and not merely to a single diagram. It is therefore algebraically more laborious than corresponding checks in electrodynamics. It is no longer possible to exploit charge conservation by following individual lines

³ These are analogs of the electromagnetic form factors. The gravitational form factors are also sometimes referred to as stress-energy, mass, or mechanical form factors.

⁴ These are, in fact, the most important differences. It is worth mentioning that when fermion fields are included it is usually convenient to replace the metric field $g_{\mu\nu}$ by a *vierbein* field. Otherwise the group transformation laws are no longer linear. [See B. S. DeWitt and C. M. DeWitt, Phys. Rev. **87**, 116 (1952).] We also mention that the combined *vierbein*-general-coordinate-transformation group has the structure of a semi-direct product based on the automorphisms of the *vierbein* group under general coordinate transformations. In the combined group only the *vierbein* group is an invariant subgroup. The coordinate transformation group is its factor group. Similar statements apply to the combined Yang-Mills-general coordinate-transformation group. The analysis of these cases is therefore correspondingly complicated.

through diagrams, for now the conserved quantity—Yang-Mills charge, energy-momentum—leaks all over every diagram. Moreover, when Yang-Mills quanta or gravitons interact with themselves, the closed loops form traffic jams of spurious charge which can be unsnarled only by calling the fictitious quanta to the rescue.

3. SCATTERING CROSS SECTIONS

We now display some of the lowest-order amplitudes and scattering cross sections which the covariant theory yields. One of the simplest is the amplitude for the scattering of two identical scalar particles by exchange of a single Yang-Mills quantum. This has the form

$$i(2\pi)^{-2}\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)j_\alpha(1)\cdot j^\alpha(2)/q^2 \\ +\text{exchange and virtual annihilation terms,} \quad (3.1)$$

where

$$q=\mathbf{p}_1'-\mathbf{p}_1=\mathbf{p}_2-\mathbf{p}_2', \quad (3.2)$$

$$j_{\alpha\mu}=\frac{1}{2}i(E'E)^{-1/2}\chi^\dagger\gamma G_\alpha\chi(\mathbf{p}'_\mu+\mathbf{p}_\mu), \quad (3.3)$$

the χ 's being the internal (Yang-Mills group) states of the particles and the remaining notation being conventional. The same form (3.1) also holds for particles with spin, but the expression for the current $j_{\alpha\mu}$ is then more complicated.

Since the initial and final momenta are on the mass shell we have the conservation laws

$$j_\alpha(1)\cdot q=j_\alpha(2)\cdot q=0, \quad (3.4)$$

which permit the scattering amplitude to be reexpressed in the form

$$\frac{j_{\alpha 0}(1)j^{\alpha 0}(2)}{q^2}+\frac{j_{\alpha 1}(1)j^{\alpha 1}(2)+j_{\alpha 2}(1)j^{\alpha 2}(2)}{q^2}, \\ +\text{exchange and virtual annihilation terms,} \quad (3.5)$$

where a factor $i(2\pi)^{-2}\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)$ has been removed, and the 3-axis has been chosen in the direction of the spatial part \mathbf{q} of the space-like 4-vector q . The first term of (3.5) represents the instantaneous "Coulomb" interaction of the particles; the second represents a "delayed" interaction propagated by transverse quanta, the factors $j_{\alpha 1}$ and $j_{\alpha 2}$ being separately coupled to the two states of linear polarization of these quanta.

The corresponding amplitude arising from exchange of a graviton is

$$i(4\pi)^{-2}\delta(\mathbf{p}_1'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2)T_{\mu\nu}(1) \\ \times(\eta^{\mu\sigma}\eta^{\nu\tau}+\eta^{\mu\tau}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\sigma\tau})T_{\sigma\tau}(2)/q^2 \\ +\text{exchange and virtual annihilation terms,} \quad (3.6)$$

where

$$T_{\mu\nu}=\frac{1}{2}(E'E)^{-1/2}[\mathbf{p}_\mu\mathbf{p}'_\nu+\mathbf{p}_\nu\mathbf{p}'_\mu-\eta_{\mu\nu}(\mathbf{p}\cdot\mathbf{p}'+m^2)]. \quad (3.7)$$

Again we have conservation laws

$$T_{\mu\nu}(1)q^\nu=T_{\mu\nu}(2)q^\nu=0, \quad (3.8)$$

which permit the amplitude to be recast in the form

$$\frac{1}{4}\{T_{00}(1)T_{00}(2)-4T_{01}(1)T_{01}(2)-4T_{02}(1)T_{02}(2) \\ -T_{03}(1)T_{03}(2)-T_{00}(1)[T_{11}(2)+T_{22}(2)] \\ -[T_{11}(1)+T_{22}(1)]T_{00}(2)\}/q^2+\frac{1}{4}\{[T_{11}(1)-T_{22}(1)] \\ \times[T_{11}(2)-T_{22}(2)]+4T_{12}(1)T_{12}(2)\}/q^2 \\ +\text{exchange and virtual annihilation terms.} \quad (3.9)$$

The first term yields an instantaneous "Newtonian" interaction, while the second gives rise to a "delayed" interaction propagated by transverse gravitons. In this case the factors which couple separately to the two states of linear polarization are $T_{11}-T_{22}$ and $2T_{12}$, respectively.

From (3.6) it is straightforward to compute the differential cross section for gravitational scattering of identical scalar particles in the center-of-mass frame. One finds⁵

$$\frac{d\sigma}{d\Omega}=\frac{G^2E^2}{16}\left[\frac{(1+3v^2)(1-v^2)+4v^2(1+v^2)\cos^2(\theta/2)}{v^2\sin^2(\theta/2)}\right. \\ \left.+\frac{(1+3v^2)(1-v^2)+4v^2(1+v^2)\sin^2(\theta/2)}{v^2\cos^2(\theta/2)}\right. \\ \left.+ (3-v^2)(1+v^2)+2v^4\sin^2\theta\right]^2, \quad (3.10)$$

where $v=|\mathbf{p}|/E$ and the gravitation constant has been reintroduced through the units convention $16\pi G=1$. The nonrelativistic and extreme relativistic limits of this cross section are, respectively,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{NR}}=\frac{G^2m^2}{16}\left[\frac{1}{v^2\sin^2\frac{1}{2}\theta}+\frac{1}{v^2\cos^2\frac{1}{2}\theta}+3\right]^2, \quad (3.11)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{ER}}=4G^2E^2[\cot^2\frac{1}{2}\theta+\tan^2\frac{1}{2}\theta+\frac{1}{2}+\frac{1}{4}\sin^2\theta]^2. \quad (3.12)$$

In a similar manner one may compute the cross section for scattering of gravitons by scalar particles. The relevant diagrams are shown in Fig. 1, the heavy lines denoting particles and the light lines gravitons. Diagrams (a) and (b) vanish in the rest frame of the target particle, and one finds for unpolarized gravitons⁵

$$\frac{d\sigma}{d\Omega}=\frac{G^2m^2}{[1+2\epsilon\sin^2\frac{1}{2}\theta]^2\sin^4\frac{1}{2}\theta} \\ \left[\cos^8\frac{1}{2}\theta+\sin^8\frac{1}{2}\theta\right. \\ \left.+2\epsilon(\cos^6\frac{1}{2}\theta+\sin^6\frac{1}{2}\theta)\sin^2\theta\right. \\ \left.+ \epsilon^2(\cos^4\frac{1}{2}\theta+\sin^4\frac{1}{2}\theta)\sin^4\theta\right], \quad (3.13)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{NR}}=\frac{G^2m^2}{\sin^4\frac{1}{2}\theta}(\cos^8\frac{1}{2}\theta+\sin^8\frac{1}{2}\theta), \quad (3.14)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{ER}}=4G^2m^2(\cos^4\frac{1}{2}\theta+\sin^4\frac{1}{2}\theta)\cot^4\frac{1}{2}\theta, \quad (3.15)$$

⁵ C. F. Cooke, Ph.D. thesis, University of North Carolina, 1964 (unpublished).

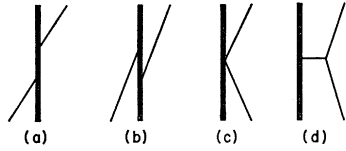


FIG. 1. Lowest-order diagrams for scattering of a graviton by a material particle. The heavy line denotes the particle and the light lines denote gravitons.

where ϵ is the energy of the graviton measured in units of m . It will be noted that these cross sections have no resemblance to those for Compton scattering, but on the contrary, continue to display the sharp forward "Rutherford peak" characteristic of long-range interactions. This feature is due to diagram (d) of Fig. 1 whose presence, as may be readily checked, is essential for the gauge invariance of the scattering amplitude. Owing to the equivalence principle gravitons, like photons, are deflected by a gravitational field (in particular by the long-range static field of any material particle), and the above cross sections are dominated by this effect.

By the well-known "substitution rule" the diagrams of Fig. 1 yield also the amplitude for annihilation of a pair of scalar particles into gravitons. We record here only the low- and high-energy limits of the total annihilation cross section in the center-of-mass frame:

$$\sigma_{\text{NR}} = 2\pi G^2 m^2 / v, \quad (3.16)$$

$$\sigma_{\text{ER}} = (38\pi/3)G^2 E^2. \quad (3.17)$$

The cross section for the inverse process, namely, the production of a scalar pair by colliding gravitons (again in the center-of-mass frame) is identical with (3.17) at high energies. Near threshold, on the other hand, it is given by

$$\sigma = 2\pi G^2 m^2 (\epsilon^2 - 1)^{1/2} / \epsilon, \quad \epsilon \geq 1. \quad (3.18)$$

The only elastic process which remains to be considered is the scattering of one graviton by another. This process has some unusual features. It turns out that the helicity of the colliding gravitons is *individually* conserved. That is, there is no spin flip, in spite of the presence of derivative coupling. If both gravitons are right (left) handed before collision then both are right (left) handed after the collision. If one is right handed and the other left handed then they maintain this relationship also, through the collision.

The helicity of extremely relativistic particles, and of massless quanta in particular, is notoriously rigid. In the classical theory, for example, the spin of such a particle suffers no precession under geodesic motion in an external gravitational field but remains always pointing parallel or antiparallel to the trajectory. However, no general principle has yet been discovered which implies that helicity conservation must hold to all orders of perturbation theory. It has so far been

demonstrated only in lowest order, by carrying out a brute-force computation of the relevant amplitudes.

The tediousness of the algebra involved in obtaining the graviton-graviton cross section may be inferred from the complexity of the vertex functions (2.6) and (2.7) which are involved in the diagrams which represent the amplitude (Fig. 1 with the heavy lines replaced by graviton lines). Fortunately, the presence of the polarization tensors in the external-line wave functions, and the momentum condition $p^2=0$ for free quanta, eliminate many of the terms from these expressions. Nevertheless, a large amount of cancellation between terms still has to be dug out of the algebra, and this, combined with the fact that the final results are ridiculously simple, leads one to believe that there must be an easier way. The cross sections which one finds are

$$\frac{d\sigma_{++}}{d\Omega} = \frac{d\sigma_{--}}{d\Omega} = 4G^2 E^2 \left[\frac{\cos^6 \frac{1}{2}\theta}{\sin^2 \frac{1}{2}\theta} + \frac{\sin^6 \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta} + 4 - \frac{1}{2} \sin^2 \theta \right]^2, \quad (3.19)$$

$$\frac{d\sigma_{+-}}{d\Omega} = 4G^2 E^2 \frac{\cos^{12} \frac{1}{2}\theta}{\sin^4 \frac{1}{2}\theta}, \quad (3.20)$$

showing again the forward Rutherford peaking.

We shall not record here the corresponding cross sections involving Yang-Mills quanta, since these depend, in their finer details, on which Lie group is chosen as generator of the Yang-Mills group and on which representations are chosen for the material particles. There is also a serious difficulty with the Yang-Mills field in regard to the infrared catastrophe, which will be discussed in Sec. 8. Since our primary interest in this article is the gravitational field, we refer the reader with a special interest in Yang-Mills cross sections to the dissertations of Remler⁶ and Dotson.⁷ It is, however, perhaps worth remarking that in the case of the scattering of one Yang-Mills quantum by another the phenomenon of helicity conservation is again found to hold, with or without the inclusion of graviton exchange forces in the total amplitude, and regardless of the choice of the Lie group. Moreover, an extension of the helicity conservation rule to processes involving real gravitons in interaction with Yang-Mills quanta apparently exists. Thus individual helicities remain unchanged when a graviton and a Yang-Mills quantum collide elastically. If the diagrams contributing to this process are turned on their sides so as to yield the amplitudes for annihilation of two Yang-Mills quanta into a pair of gravitons (or the reverse process) further selection rules emerge. One

⁶ E. A. Remler, Ph.D. thesis, University of North Carolina, 1964 (unpublished).

⁷ A. C. Dotson, Ph.D. thesis, University of North Carolina, 1964 (unpublished).

finds that it is impossible to produce two gravitons having opposite helicities by annihilation of Yang-Mills quanta, or conversely to produce a Yang-Mills pair having opposite helicities by the reverse process. The quanta in both the initial and final states must have identical helicities if the amplitude is to be nonvanishing. Helicity selection rules exist even for the process in which two Yang-Mills quanta coalesce to produce a single Yang-Mills quantum and a graviton. If both initial quanta have the same helicity the final quanta must have this helicity too; if the initial helicities are opposite the final helicities must be opposite. The same obviously holds for the reverse process.

4. GRAVITATIONAL BREMSSTRAHLUNG

Since the problem of gravitational radiation from accelerating masses has bedeviled classical relativists for years it is a pleasant surprise to discover that its treatment within the quantum framework is quite simple.⁸ Consider a scattering diagram in which one of the lines represents a scalar particle (real or virtual) of momentum p . Let the diagram be modified by the emission of a graviton of momentum q from this line. If the momenta of all lines subsequent to the inserted graviton vertex are held fixed while those prior to the vertex are adjusted in such a way as to conserve momentum and keep external lines on the mass shell, then the only additional effect of the graviton emission is to introduce into the corresponding amplitude, a factor

$$\frac{e_{\pm}^{\mu*} e_{\pm}^{\nu*}}{(2\pi)^{3/2} \sqrt{q^0}} \times \frac{p_{\mu}(p_{\nu} + q_{\nu}) + p_{\nu}(p_{\mu} + q_{\mu}) - \eta_{\mu\nu}[m^2 + p \cdot (p + q)]}{(p + q)^2 + m^2 - i0}, \quad (4.1)$$

which follows from Eq. (2.15) and Table II of II. Alternatively, if the momenta *prior* to the vertex are held fixed we get a factor which differs from (4.1) by the replacement $q \rightarrow -q$.

If the graviton is emitted from an external line these factors reduce to

$$\frac{e_{\pm}^{\mu*} e_{\pm}^{\nu*}}{(2\pi)^{3/2} \sqrt{q^0}} \frac{p_{\mu} p_{\nu} + \frac{1}{2} \eta (\dot{p}_{\mu} q_{\nu} + \dot{p}_{\nu} q_{\mu} - \eta_{\mu\nu} \dot{p} \cdot q)}{q^2 + 2\eta \dot{p} \cdot q - i0}, \quad (4.2)$$

where $\eta = +1$ or -1 according as the external line is outgoing or incoming, and \dot{p} is held fixed on the mass shell. In the long-wavelength limit $q \rightarrow 0$ (4.2) itself reduces to

$$\frac{\eta}{2(2\pi)^{3/2} \sqrt{q^0}} \frac{(e_{\pm}^* \cdot \dot{p})^2}{\dot{p} \cdot q - i\eta 0}, \quad (4.3)$$

⁸ The comparison is actually unfair. The questions which classical relativists ask are usually quite detailed—e.g., the precise damped motion of radiating sources, or the nonlinear properties of coherent waves of large amplitude—and are inevitably much more difficult.

and simply multiplies the original amplitude. This limiting form actually holds for *all* external lines, regardless of the spin character of their associated particles. It even holds when the external line is a graviton line, provided the emission vertex is inserted not merely into a single diagram but into the sum of all diagrams contributing to the original amplitude. This may be verified in a straightforward manner by plugging in the 3-pronged graviton vertex (2.6) and eliminating the terms involving q . Of the remaining terms only those survive which yield a net contribution of the form (4.3); the rest disappear in virtue of the gauge invariance of the total original amplitude.⁹

The multiplicative factor (4.3) exhibits the well-known infrared divergence and can be obtained from a purely classical model. We note that the infrared divergence shows up only when the emission takes place from lines on the mass shell; it does not occur when the emission is from internal lines of a scattering diagram. The external lines therefore dominate the soft graviton emission. This means that the precise details of the scattering process have little relevance in the limit $q \rightarrow 0$, and that the long-wavelength end of the emission spectrum is determined primarily by the asymptotic trajectories of the incoming and outgoing particles, just as in the case of photon bremsstrahlung. For wavelengths large compared to the space-time region in which the collision takes place (the size of this region is determined by the magnitudes of typical energies exchanged in the collision) the effective graviton source is a stress tensor of the form

$$T^{\mu\nu}(x) = \sum_n \eta_n m_n V_n^{\mu} V_n^{\nu} \int_0^{\eta_n \infty} \delta(x - V_n \tau) d\tau, \quad (4.4)$$

which idealizes the particles to classical points colliding at the coordinate origin. Here m_n and V_n are, respectively, the mass and 4-velocity of the n th particle, and the sign factor η_n tells whether the particle is incoming or outgoing. The summation is over all the external lines, and the velocities are subject to the energy-momentum conservation law:

$$\sum_n \eta_n m_n V_n = 0. \quad (4.5)$$

The classical emission spectrum is obtained by projecting (4.4) onto the graviton wave functions $u_{\mu\nu\pm}(x, \mathbf{q})$ (see Table II of II). The corresponding quantum amplitude is

$$\begin{aligned} & \frac{1}{2} i \int u_{\mu\nu\pm}^*(x, \mathbf{q}) T^{\mu\nu}(x) dx \\ &= i \sum_n \frac{\eta_n m_n (e_{\pm}^* \cdot V_n)^2}{2(2\pi)^{3/2} \sqrt{q^0}} \int dx \int_0^{\eta_n \infty} d\tau e^{-i\mathbf{q} \cdot x} \delta(x - V_n \tau) \\ &= \sum_n \frac{\eta_n m_n}{2(2\pi)^{3/2} \sqrt{q^0}} \frac{(e_{\pm}^* \cdot V_n)^2}{V_n \cdot q - i\eta_n 0}, \end{aligned} \quad (4.6)$$

⁹ The gauge invariance holds in every order of perturbation theory.

which, in view of the relation $p_n = m_n V_n$, is just (4.3) summed over all the external lines.

When the collision is nonrelativistic (4.6) reduces to

$$\frac{1}{2}(2\pi q^0)^{-3/2} \mathbf{e}_{\pm}^* \cdot \Delta \mathfrak{T} \cdot \mathbf{e}_{\pm}^*, \quad (4.7)$$

where the graviton gauge is chosen so that the components e_{\pm}^0 of the polarization vectors e_{\pm} vanish, and $\Delta \mathfrak{T}$ is the change in the spatial integral of the total 3-stress dyadic as a result of the collision:

$$\Delta \mathfrak{T} = \Delta \int \mathbf{T} d\mathbf{x} = \sum_n \eta_n \mathbf{p}_n \mathbf{v}_n, \quad (4.8)$$

$$\mathbf{T} = \sum_n \theta(\eta_n x^0) \mathbf{p}_n \mathbf{v}_n \delta(\mathbf{x} - \mathbf{v}_n x^0), \quad (4.9)$$

$$\mathbf{v}_n \equiv \mathbf{V}_n / V_n^0 = \mathbf{p}_n / E_n \approx \mathbf{p}_n / m_n. \quad (4.10)$$

Now it is well known¹⁰ that energy-momentum conservation permits the integral of the 3-stress dyadic to be reexpressed as one half the second time derivative of the second moment $\int \mathbf{x} \mathbf{x} T_{00} d\mathbf{x}$ of the energy density. Moreover, since $\mathbf{e}_{\pm}^* \cdot \mathbf{e}_{\pm}^* = 0$, the trace of $\Delta \mathfrak{T}$ may be removed from (4.7).¹¹ Therefore the emission amplitude may be written in the alternative form:

$$\frac{1}{4}(2\pi q^0)^{-3/2} \mathbf{e}_{\pm}^* \cdot \Delta(d^2 \mathbf{Q} / dt^2) \cdot \mathbf{e}_{\pm}^*, \quad (4.11)$$

where $\Delta(d^2 \mathbf{Q} / dt^2)$ denotes the change in the second time derivative of the energy quadrupole moment tensor

$$\mathbf{Q} = \int (\mathbf{x} \mathbf{x} - \frac{1}{3} \mathbf{1} x^2) T_{00} d\mathbf{x}, \quad (4.12)$$

showing that soft gravitons are emitted predominantly in the quadrupole mode.

It is of interest to examine the angular distribution of the emitted radiation. From (4.6) one sees that each external line makes a contribution to the emission amplitude, which has an angular distribution of the form

$$\frac{\sin^2 \theta}{1 - v \cos \theta} e^{2i\varphi}, \quad (4.13)$$

where θ is the angle between \mathbf{v} and \mathbf{q} , and φ is a helicity phase angle. In the case of photon bremsstrahlung the $\sin \theta$ appears linearly instead of quadratically in the numerator, with the consequence that for relativistic collisions ($v \approx 1$) the emission is concentrated sharply in the forward directions of all the particles (initial as well as final). This peaking may be attributed to the individual Lorentz-contracted Coulomb fields, which

¹⁰ See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), rev. 2nd Ed.

¹¹ In view of the nonrelativistic energy conservation law $\sum_n \eta_n (m_n + \frac{1}{2} \mathbf{p}_n \cdot \mathbf{v}_n) = 0$, this trace is just twice the rest mass lost in the collision and already vanishes for elastic collisions.

resemble bundles of plane waves having momenta confined to narrow cones. These bundles (particularly their outer regions) have difficulty readjusting to the altered particle trajectories arising from the collision and hence partly escape as radiation.

In the gravitational case the sharp forward emission is absent.¹² In fact for an extremely relativistic collision ($|\mathbf{p}_n| \approx E_n$) which is confined to a plane (e.g., 2-particle scattering) it is easy to verify that the total sum (4.6) yields an amplitude which vanishes for emission *in* the plane.¹³ This implies that, unlike photon emission, graviton emission is a cooperative phenomenon which cannot be traced to the individual particle fields. Indeed the *real* gravitational field of a particle, namely the Riemann tensor, falls off as the inverse cube rather than the inverse square of the distance, and hence its outer regions contribute negligibly to the emission. This has obvious implications for investigations of classical 2-body radiation as well as for attempts to introduce Weizsäcker-Williams approximation schemes into quantum calculations.

5. RENORMALIZATION AND THE PLANCK LENGTH

In lowest-order perturbation theory the formal rules of the manifestly covariant theory yield results which agree with the classical theory in the correspondence principle limit. In higher orders, divergences appear, just as they do for other field theories, and almost nothing is known about how to extract finite and physically meaningful radiative corrections from the results. In the case of quantum gravodynamics the severity of the divergences is such that the theory is not, by standard criteria, renormalizable. This is due to the quadratic momentum dependence of the vertices S_n ($n \geq 3$), which in turn may be traced to the dependence of the light cone on the background field, i.e., to the field dependence of the coefficients of the second time derivatives appearing in S_2 . Thus by counting momentum powers one finds for the superficial degree of divergence of any diagram

$$D = -2L_i + 2 \sum_n V_n + 4K, \quad (5.1)$$

where L_i denotes the number of internal lines, V_n the number of n -pronged vertices, and K the number of independent momentum integrations. Now it is not difficult to show that

$$K = L_i - \sum_n V_n + 1. \quad (5.2)$$

¹² This was first pointed out by R. P. Feynman in a mimeographed letter to V. F. Weisskopf dated January 4 to February 11, 1961 (unpublished).

¹³ Introducing unit vectors Ω and Ω_n in the directions of \mathbf{q} and \mathbf{p}_n , respectively, one may write the amplitude in this case in the form

$$\text{const} \times \sum_n \eta_n E_n \frac{(\Omega \times \Omega_n)^2}{1 - \Omega \cdot \Omega_n} = \text{const} \times \sum_n \eta_n E_n (1 + \Omega \cdot \Omega_n),$$

which vanishes by energy-momentum conservation.

Therefore

$$D=2(K+1), \quad K \geq 1, \quad (5.3)$$

which increases without limit as the order of the diagram increases.

In the case of the Yang-Mills field the situation is better. Here we have

$$D=-2L_i+V_3+4K, \quad (5.4)$$

which, in combination with the readily verified combinatorial law

$$L_e+2L_i=\sum_n nV_n=3V_3+4V_4, \quad (5.5)$$

yields

$$D=4-L_e, \quad (5.6)$$

where L_e is the number of external lines. In order to compensate the divergences one may introduce counter terms into the original Lagrangian in the conventional manner. The most divergent counter term is always the simplest. It is necessarily of the form $\text{const} \times F_{\alpha\mu\nu} F^{\alpha\mu\nu}$, provided the divergence has been handled in a manifestly group-invariant manner. But such a counter term can be detected by as many as four external lines. Hence D is never actually greater than zero. This is quite analogous to the situation which occurs with vacuum polarization diagrams in quantum electrodynamics: Although $L_e=2$, D is reduced from 2 to 0 by gauge invariance. One therefore expects that, with proper handling of the overlapping divergences, a careful analysis will show that none of the ultraviolet divergences of the Yang-Mills theory is worse than logarithmic, and that only a single counter term, of the above mentioned type, is needed for each group-invariant set of diagrams. If that is true it is then not difficult to show that renormalization merely rescales the structure constants $c^{\alpha}_{\beta\gamma}$. Mathematically, this is equivalent to a rescaling of coordinates in the group manifold; physically, it corresponds to a change in the strength of the coupling of the Yang-Mills field to itself and to other Yang-Mills-charge bearing fields.

It should be remarked that although the above results [Eqs. (5.3) and (5.6)] have been stated for the case in which each field interacts only with itself, they hold also when other fields bearing stress-energy or Yang-Mills charge are present, provided the spins of the added fields are not greater than $\frac{1}{2}$ and their other mutual interactions are renormalizable.¹⁴ Unfortunately, in the case of the Yang-Mills field the ultraviolet divergences are not the whole story; infrared difficulties of a special type also make their appearance. These will be discussed in Sec. 8.

In the case of gravity there are no infrared problems beyond those which can be handled by conventional methods. Equation (5.3), however, casts a rather dismal

light on the ultraviolet problem. Faced with its brutal consequences there are several paths one may try to follow to make life bearable. One of these is to soften the degree of divergence by abandoning S -matrix unitarity (i.e., positive definiteness of Hilbert space) through the introduction of field equations of the fourth differential order. This may be accomplished by adding terms of the form $\int g^{1/2(4)} R^2 dx$ and $\int g^{1/2} R_{\mu\nu} R^{\mu\nu} dx$ to the Einstein action, which changes Eqs. (5.1) and (5.3) to

$$D=-4L_i+4\sum_n V_n+4K=4 \quad (5.7)$$

for all diagrams of order greater than zero. Such a procedure in effect introduces a separate unit of mass (or length) into the theory, and if this mass is chosen sufficiently big, the S matrix will be *nearly* unitary, significant departures from unitarity occurring only under extreme conditions, when collision energies approach that of the unit of mass.

Nevertheless, it would be nice if any breakdown in conventional ideas which may be necessary were to emerge from the quantized Einstein theory in its unutilized form. There is already a unit of mass in the theory: the absolute unit $(\hbar c / 16\pi G)^{1/2} = 3.07 \times 10^{-6} g \sim 10^{18}$ BeV, and one is loath to introduce another. One might try to use the extra mass merely as a regulator, in a spirit similar to that of the ξ -limiting proposal of Lee and Yang¹⁵ for the charged vector boson. Equation (5.7) suggests that the regulated theory may be renormalizable, requiring only three counter terms, respectively, quartically, quadratically, and logarithmically divergent. If this is so one might attempt to let the regulator mass become infinite *after* the renormalization has been performed. However, there is no guarantee that the renormalized amplitudes will themselves remain finite in the limit, nor that unitarity, which is violated by the regulator, will then be restored. If unitarity stays violated one is not sure whether this represents a fundamental feature of the quantized Einstein theory or is merely a consequence of the regulator approach; one would be inclined to suspect the latter. Halpern¹⁶ has criticized unorthodox uses of regulators in handling nonrenormalizable field theories, and has shown that they often lead to illegal modifications of analytic properties.

If regulators are to be excluded then perturbation theory cannot be used except in a formal way. One must necessarily sum infinite classes of diagrams and hope that the increasingly strong divergence of the successive terms of the series, as expressed by Eq. (5.3), will lead to high-energy damping and a finite result for the total amplitude. The author has shown¹⁷ that this hope is actually fulfilled for at least one

¹⁴ T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); **128**, 899 (1962).

¹⁵ M. B. Halpern, Phys. Rev. **140**, B1570 (1965).

¹⁷ B. S. DeWitt, Phys. Rev. Letters **13**, 114 (1964).

¹⁴ In the case of gravity additional fields of spin 1 are allowed if they are massless.

simple class of diagrams, namely, those which represent two scalar particles exchanging gravitons in the ladder approximation. It turns out that the "leading terms" (i.e., the most divergent) of the Bethe-Salpeter amplitude can be summed exactly, and, owing to certain remarkable cancellations, the sum of the ladder-type contributions to the gravitational self-energy can be expanded in a power series in the bare mass, with no approximations whatever. The method can also be extended to the case of charged scalar particles, with one or more of the graviton ladder rungs replaced by photons, and a simple expression can be obtained for the lowest-order electromagnetic self-energy. The self-energies and renormalization constants found in this way are all *finite*.

The finiteness of these quantities may be traced to the behavior of the particle-particle scattering amplitude. In the limit of very high momentum transfer the singularity of the gravitational interaction kernel is displaced off the light cone in coordinate space and onto a hyperboloid lying at a distance $\lambda = (4\hbar G/\pi c^3)^{1/2} = 1.82 \times 10^{-33}$ cm in spacelike directions. This is roughly equivalent to endowing the scalar particles with the properties of hard spheres of diameter λ , and may be regarded as a manifestation of the smearing out of the light cone due to quantum fluctuations.

Similar results have been found for spin- $\frac{1}{2}$ electrons by Khriplovich,¹⁸ and there seems to be no reason why, with enough labor, they may not also be extended to particles of higher spin, including the graviton in interaction with itself. Thus gravity may indeed prove to be the universal regulator which renders all field theories finite.

It should be remarked that the self-energy functions which are obtained by summing ladder graphs appear to correspond to "good" spectral functions, which do a minimum of violence to unitarity. This suggests that no illegal analytic operations have inadvertently crept into the summation procedure. An improved calculational method, which insures analytic legality in general, has been developed by Halpern.¹⁶ He sums first the absorptive parts of any amplitude and then obtains the full amplitude by a dispersion integral. The technique is applicable to gravity theory as well as to other nonrenormalizable theories, and is amenable to N/D approximation schemes. It is probably the safest method currently available, but it is very complicated to apply.

Although the finite results which have been obtained thus far are very suggestive, one must remember that they derive from restricted classes of diagrams. They are therefore not γ -invariant but depend on the particular gauge chosen for the internal graviton lines. So far calculations have been restricted to those gauges which avoid "dangerous" singularities in the resulting

integral equations, or otherwise simplify the computational labor. It is clear that the results can give at best only a qualitative insight into the true analytic structure of the theory.

6. THE GRAVITATIONAL WARD IDENTITY

Although the computational difficulties involved in extracting physical information from quantum gravodynamics are formidable, the theory has a redeeming feature in its general covariance, which serves as a cross check on the consistency of various calculations and imposes constraints on the permissible forms of various amplitudes. One of these constraints has recently been discussed by Brout and Englert.¹⁹ These authors derive a generalized Ward identity relating the gravitational vertex function of a scalar particle to the self-energy function arising from all its interactions. Their derivation is easily generalized to the case of a particle of arbitrary spin.

Denote the field of the particle by φ^A . In addition to the functions R^i_α (or, in expanded notation, $R_{\mu\nu\rho}$) characterizing the coordinate transformation behavior of the gravitational field (see II) we now have corresponding functions R^A_α for φ^A . The explicit structure of these functions may be inferred from Table I of II:

$$R^A_{\mu'} = -\varphi^A_{,\mu}\delta(x,x') + G^{\nu A}_{\mu B}\varphi_{B,\nu}(x,x'). \quad (6.1)$$

We note that $R^A_{\mu'}$ vanishes in the limit $\varphi^A \rightarrow 0$, and that its functional derivative has the momentum-space form

$$R^A_{\mu',B''} \rightarrow -i\delta^A_B p''_\mu + iG^{\nu A}_{\mu B} p'_\nu, \quad (6.2)$$

in which the association of momenta with indices is $pA, p'_\mu, p''B''$ ($p+p'+p''=0$).

Let us denote the full (radiatively corrected) propagator for the particle by S^{AB} . It is the sum of the bare propagator G^{AB} and a function obtained by applying the operator $G^{AB}\delta/\delta\varphi^B$ twice to the vacuum-to-vacuum amplitude. Since the vacuum-to-vacuum amplitude is an invariant the propagator S^{AB} , like G^{AB} , transforms in the manner indicated by the position of its indices.²⁰ Its inverse must transform contragrediently:

$$S^{-1}_{AB,i}R^i_\alpha + S^{-1}_{AB,C}R^C_\alpha = -S^{-1}_{CB}R^B_{\alpha,A} - S^{-1}_{AC}R^C_{\alpha,B}. \quad (6.3)$$

Equation (6.3) is the gravitational Ward identity. To get it into more familiar form one must reexpress it in momentum space, with all the background fields set equal to zero. In this limit $S^{-1}_{AB,i}$ becomes the negative of the gravitational vertex function, which is conven-

¹⁹ R. Brout and F. Englert, Phys. Rev. **141**, 1231 (1966). See also K. Just and K. Rossberg, Nuovo Cimento **40**, 1077 (1965).

²⁰ This will be true even if φ^A possesses a gauge group of its own, provided the gauge conditions which determine G^{AB} are covariant. Note that the "background field" now includes φ^A in addition to the metric field.

¹⁸ I. B. Khriplovich, report, Siberian Section, Academy of Science, USSR, Novosibirsk, 1965 (unpublished).

tionally denoted by $\Gamma^{\mu\nu}$, the particle indices being suppressed and the index i being replaced by the more explicit $\mu\nu$. Making use of (6.2) and the momentum space form of R^i_α , which is given in Table II of II, one readily finds

$$2\Gamma_{\mu\nu}(\not{p}', \not{p})q_\nu = S^{-1}(\not{p}')\not{p}_\mu - S^{-1}(\not{p})\not{p}'_\mu - [S^{-1}(\not{p}')G_{\nu\mu} - G_{\nu\mu} \tilde{S}^{-1}(\not{p})]q_\nu, \quad (6.4)$$

where \not{p} and \not{p}' are, respectively, the incoming and outgoing particle momenta and $q = \not{p}' - \not{p}$ is the incoming graviton momentum. This, with the spin terms involving $G_{\nu\mu}$ omitted is the equation given by Brout and Englert. It holds, as a simple consequence of general covariance, no matter how many other fields are coupled to the field φ^A and involved in the structure of the vertex function.

Now introduce the vertex and wave-function renormalization constants Z_1 and Z_2 . They are defined by

$$u^\dagger(\not{p})\Gamma_{\mu\nu}(\not{p}, \not{p})u(\not{p}) = Z_1^{-1}u^\dagger(\not{p})\gamma_{\mu\nu}(\not{p}, \not{p})u(\not{p}), \quad (6.5)$$

$$\not{p}^2 = -m^2,$$

$$S^{-1}(\not{p}) = Z_2^{-1}[G^{-1}(\not{p}) + \bar{\Sigma}(\not{p})], \quad (6.6)$$

$$[\partial\bar{\Sigma}(\not{p})/\partial\not{p}^\mu]_{\not{p}^2 = -m^2} = 0,$$

where $\gamma_{\mu\nu}$ and G are the bare vertex and propagation functions, respectively, m is the particle rest mass, and $u(\not{p})$ is a particle wave function satisfying

$$S^{-1}(\not{p})u(\not{p}) = G^{-1}(\not{p})u(\not{p}) = 0 \quad (6.7)$$

on the mass shell. From (6.6) we may infer

$$[\partial S^{-1}(\not{p})/\partial\not{p}^\mu]_{\not{p}^2 = -m^2} = Z_2^{-1}[\partial G^{-1}(\not{p})/\partial\not{p}^\mu]_{\not{p}^2 = -m^2}. \quad (6.8)$$

On the other hand, (6.4) yields, in the limit $\not{p}' \rightarrow \not{p}$,

$$2\Gamma_{\mu\nu}(\not{p}, \not{p}) = \not{p}_\mu \partial S^{-1}(\not{p})/\partial\not{p}^\nu - \eta_{\mu\nu} S^{-1}(\not{p}) - S^{-1}(\not{p})G_{\nu\mu} - G_{\nu\mu} \tilde{S}^{-1}(\not{p}), \quad (6.9)$$

whence, in virtue of (6.7),

$$2u^\dagger(\not{p})\Gamma_{\mu\nu}(\not{p}, \not{p})u(\not{p}) = \not{p}_\mu u^\dagger(\not{p})[\partial S^{-1}(\not{p})/\partial\not{p}^\nu]u(\not{p}), \quad (6.10)$$

$$\not{p}^2 = -m^2.$$

Now, since (6.4) is a consequence simply of general covariance, it holds also if $\Gamma_{\mu\nu}$ and S^{-1} are replaced by $\gamma_{\mu\nu}$ and G^{-1} , respectively. Therefore we have

$$2u^\dagger(\not{p})\gamma_{\mu\nu}(\not{p}, \not{p})u(\not{p}) = \not{p}_\mu u^\dagger(\not{p})[\partial G^{-1}(\not{p})/\partial\not{p}^\nu]u(\not{p}), \quad (6.11)$$

$$\not{p}^2 = -m^2.$$

From (6.5), (6.8), and (6.11) it follows that

$$Z_1 = Z_2. \quad (6.12)$$

When both vertex and wave-function radiative corrections are taken into account the two renormalizations cancel, and there remains only the graviton renormalization Z_3 arising from vacuum polarization,²¹ which has the effect of modifying the gravitation constant.

²¹ The polarization of the vacuum by a gravitational field is of the quadrupole type. Examples of renormalization terms to which it leads are given in Sec. 7.

The cancellation of *divergences* which is implied by (6.12) applies only to the leading term of the vertex function, in the limit $\not{p}' \rightarrow \not{p}$, and only on the mass shell. In order that no divergences occur in the remaining terms, or off the mass shell, the interactions which the field φ^A experiences with other fields must be of the renormalizable type (or else summable to finite values). The example of the scalar particle provides an adequate illustration of the conditions which must be satisfied. In this case we have²²

$$G^{-1}(\not{p}) = \not{p}^2 + m^2, \quad (6.13)$$

$$\gamma_{\mu\nu}(\not{p}', \not{p}) = \frac{1}{2}[\not{p}_\mu \not{p}'_\nu + \not{p}'_\nu \not{p}_\mu - \eta_{\mu\nu}(\not{p} \cdot \not{p}' + m^2)] \quad (6.14a)$$

$$= \gamma^0_{\mu\nu}(\not{p}', \not{p}) - \frac{1}{2}\eta_{\mu\nu}\delta m^2, \quad (6.14b)$$

where the index 0 refers to the bare mass, and we may write

$$S^{-1}(\not{p}) = \not{p}^2 + m_0^2 + \Sigma(\not{p}^2), \quad (6.15)$$

$$\delta m^2 = m^2 - m_0^2 = \Sigma(-m^2),$$

$$\Gamma_{\mu\nu}(\not{p}', \not{p}) = \gamma^0_{\mu\nu}(\not{p}', \not{p}) + \Lambda_{\mu\nu}(\not{p}', \not{p}). \quad (6.16)$$

The functions Σ and Λ are related by the Ward identity as follows:

$$2\Lambda_{\mu\nu}(\not{p}', \not{p})q^\nu = \Sigma(\not{p}'^2)\not{p}_\mu - \Sigma(\not{p}^2)\not{p}'_\mu. \quad (6.17)$$

It is not hard to show that the general solution of (6.17) is

$$\Lambda_{\mu\nu}(\not{p}', \not{p}) = \frac{1}{2} \frac{\Sigma(\not{p}'^2) - \Sigma(\not{p}^2)}{\not{p}'^2 - \not{p}^2} (\not{p}_\mu \not{p}'_\nu + \not{p}'_\nu \not{p}_\mu + \frac{1}{2}q_\mu q_\nu) - \frac{1}{4}[\Sigma(\not{p}'^2) + \Sigma(\not{p}^2)]\eta_{\mu\nu} + F(\not{p}'^2, \not{p}^2, q^2)(q^2\eta_{\mu\nu} - q_\mu q_\nu), \quad (6.18)$$

where F is an arbitrary function. Therefore the graviton vertex of a scalar particle is characterized on the mass shell, by a single function of q^2 . This is the gravitational form factor.

Now introduce the renormalized self-energy function $\bar{\Sigma}$, defined by

$$\Sigma(\not{p}^2) = \delta m^2 + (Z_2^{-1} - 1)(\not{p}^2 + m^2) + Z_2^{-1}\bar{\Sigma}(\not{p}^2), \quad (6.19)$$

$$\bar{\Sigma}(-m^2) = 0, \quad [d\bar{\Sigma}(\not{p}^2)/d\not{p}^2]_{\not{p}^2 = -m^2} = 0.$$

In terms of this function Eq. (6.18) takes the form

$$\Lambda_{\mu\nu}(\not{p}', \not{p}) = (Z_2^{-1} - 1)\gamma_{\mu\nu}(\not{p}', \not{p}) - \frac{1}{2}\delta m^2\eta_{\mu\nu} + \frac{1}{2}Z_2^{-1} \frac{\bar{\Sigma}(\not{p}'^2) - \bar{\Sigma}(\not{p}^2)}{\not{p}'^2 - \not{p}^2} (\not{p}_\mu \not{p}'_\nu + \not{p}'_\nu \not{p}_\mu + \frac{1}{2}q_\mu q_\nu) - \frac{1}{4}Z_2^{-1}[\bar{\Sigma}(\not{p}'^2) + \bar{\Sigma}(\not{p}^2)]\eta_{\mu\nu} + [F(\not{p}'^2, \not{p}^2, q^2) - \frac{1}{4}(Z_2^{-1} - 1)] \times (q^2\eta_{\mu\nu} - q_\mu q_\nu), \quad (6.20)$$

²² Equation (6.14a) is obtained from (2.15) by making the replacement $\not{p}' \rightarrow -\not{p}'$, since \not{p}' is here an outgoing and not an incoming momentum.

which suggests that we also introduce a renormalized form factor \bar{F} , defined by

$$F(p'^2, p^2, q^2) = \frac{1}{4}(Z_2^{-1} - 1) + Z_2^{-1} \bar{F}(p'^2, p^2, q^2). \quad (6.21)$$

Combining (6.14b), (6.16), and (6.20) we then get

$$\bar{\Gamma}_{\mu\nu}(p', p) \equiv Z_2 \Gamma_{\mu\nu}(p', p) \quad (6.22a)$$

$$\begin{aligned} &= \gamma_{\mu\nu}(p', p) + \frac{1}{2} \frac{\bar{\Sigma}(p'^2) - \bar{\Sigma}(p^2)}{p'^2 - p^2} (p_\mu p'_\nu + p_\nu p'_\mu + \frac{1}{2} q_\mu q_\nu) \\ &\quad - \frac{1}{4} [\bar{\Sigma}(p'^2) + \bar{\Sigma}(p^2)] \eta_{\mu\nu} + \bar{F}(p'^2, p^2, q^2) \\ &\quad \times (q^2 \eta_{\mu\nu} - q_\mu q_\nu), \quad (6.22b) \end{aligned}$$

which reduces, on the mass shell, to

$$\bar{\Gamma}_{\mu\nu}(p', p) = \gamma_{\mu\nu}(p', p) + \bar{F}(-m^2, -m^2, q^2) (q^2 \eta_{\mu\nu} - q_\mu q_\nu), \quad (6.23)$$

$$p'^2 = p^2 = -m^2.$$

The Z_2 factor in (6.22a) takes into account the wavefunction renormalization arising from self-energy insertions in the external lines.

If the scalar particle is coupled to other fields through nonrenormalizable interactions then the functions $\bar{\Sigma}$ and \bar{F} will diverge in perturbation theory. In particular, they will diverge if virtual gravitons are permitted to contribute to the vertex function. Thus unless an arbitrary cutoff is used, or someone discovers a way to sum gravitational interactions to all orders, the gravitational field must be allowed to act only through the external graviton line. Although the identity (6.12) continues to hold formally in the nonrenormalizable case, it is then of no utility. Because of the divergence which remains in \bar{F} , Eq. (6.23) will yield an infinite cross section for the scattering of the particle in an external gravitational field.

In the renormalizable case $\bar{\Sigma}$ and \bar{F} are finite, and expression (6.23) has a well-defined limit as $q \rightarrow 0$, namely,

$$\bar{\Gamma}_{\mu\nu}(p, p) = p_\mu p_\nu, \quad p^2 = -m^2. \quad (6.24)$$

More generally, with particles of arbitrary spin one finds

$$u^\dagger(p) \bar{\Gamma}_{\mu\nu}(p, p) u(p) = (2\pi)^{-3} p_\mu p_\nu / 2E, \quad p^2 = -m^2, \quad (6.25)$$

when the wave functions $u(p)$ are chosen to correspond to δ -function normalization with respect to 3-momentum. As Brout and Englert point out,¹⁹ the universality of (6.25) implies that the equivalence principle relating gravitational and inertial mass holds in the quantum theory as well as the classical theory. In particular the motion of a nonrelativistic particle in a slowly varying gravitational field is independent of its mass.

If a high-energy cutoff is permitted then the Ward identity may be applied to gravity itself, i.e., to the three-graviton vertex. In this case the wave function renormalization constants Z_2 and Z_3 coincide, and Eq. (6.12) tells us that $Z_1 = Z_2 = Z_3$. This leaves only the

magnitude of the gravitation constant G , in terms of arbitrarily chosen (e.g., international mks) mass standards, to be determined by experiment.²³

It is clear that the gravitational Ward identity is only one of an infinity of identities, derivable from Eq. (17.31) of II, which relate vertex functions involving n gravitons to those involving $n-1$ gravitons. Such identities become superfluous if calculations are performed in coordinate space rather than in momentum space, for then the general covariance of the theory can be kept constantly manifest. That such calculations are actually feasible will be demonstrated in the next section.

7. RENORMALIZATION IN COORDINATE SPACE. CONFORMAL VACUUM FLUCTUATIONS

The chief tool for studying quantum gravodynamics directly in space-time is the theory of Green's functions in hyperbolic Riemannian manifolds developed by Hadamard.²⁴ The basic structural element of this theory is the *geodesic interval*, denoted by σ ,²⁵ which is defined as one half the square of the distance along the geodesic between any two space-time points x and x' . The geodesic interval is a symmetric function of x and x' which transforms as a *biscalar*, i.e., as a scalar separately at x and x' . It satisfies the differential equation²⁶

$$\sigma = \frac{1}{2} \sigma_{;\mu} \sigma^{;\mu} = \frac{1}{2} \sigma_{;\mu'} \sigma^{;\mu'}, \quad (7.1)$$

and the boundary condition

$$\lim_{x' \rightarrow x} \sigma_{;\mu\nu} = - \lim_{x' \rightarrow x} \sigma_{;\mu'\nu'} = g_{\mu\nu}. \quad (7.2)$$

In a general Riemannian manifold σ is not single-valued, except when x and x' are sufficiently close to one another.²⁷ The geodesics emanating from a given point will often, beyond a certain distance, begin to cross over one another. The locus of points at which the onset of overlap occurs forms an envelope of the

²³ The necessity of measuring G disappears if absolute units are adopted, with $\hbar = c = 16\pi G = 1$. However, the masses of the elementary particles must then be measured in absolute units, which is operationally the same thing as measuring both G and the masses in mks units.

²⁴ J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, Connecticut, 1923).

²⁵ B. S. DeWitt and R. W. Brehme, *Ann. Phys. (N. Y.)* **9**, 220 (1960). See also J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960). Synge calls this function the *world function* and denotes it by the symbol Ω .

²⁶ The semicolons denote covariant differentiation. For a scalar this is the same as ordinary differentiation. $\sigma_{;\mu}$ is a vector of length equal to the distance along the geodesic between x and x' , tangent to geodesic at x , and oriented in the direction $x' \rightarrow x$. $\sigma_{;\mu'}$ is a vector of equal length, tangent to the geodesic at x' , and oriented in the opposite direction.

²⁷ In some manifolds (e.g., some compact manifolds) every pair of points may be linked by more than one geodesic. It is always possible, however, to define a single-valued function σ in the neighborhood of x by starting at x and following each geodesic emanating from x until it hits a caustic.

family of geodesics, known as a *caustic surface*. The equation for the caustic surface relative to a given point is $D^{-1}=0$, where

$$D \equiv -\det(-\sigma_{;\mu\nu}). \quad (7.3)$$

D is a *bidensity* of unit weight at both x and x' , which satisfies the boundary condition

$$\lim_{x' \rightarrow x} D = g. \quad (7.4)$$

It is convenient to replace D by the biscalar

$$\Delta \equiv g^{-1/2} D g'^{-1/2}, \quad \lim_{x' \rightarrow x} \Delta = 1, \quad (7.5)$$

whose values at given points are independent of the choice of coordinate system. By covariantly differentiating Eq. (7.1), one can derive the differential equation²⁵

$$\Delta^{-1}(\Delta\sigma_{;\mu})_{;\mu} = 4 \quad \text{or} \quad \sigma_{;\mu} = 4 - \sigma_{;\mu}(\ln\Delta)_{;\mu}, \quad (7.6)$$

which shows that Δ increases or decreases along each geodesic from x' according as the rate of divergence of the neighboring geodesics from x' , which is measured by $\sigma_{;\mu}$, is less than or greater than 4, the rate in flat space-time. If the divergence rate becomes negatively infinite a caustic surface develops and Δ blows up.

We shall illustrate the use of σ and Δ in the theory of Green's functions by considering the Feynman propagator of the simplest of all fields: the massless scalar field. The defining equation is²⁸

$$g^{1/2} G_{;\mu}{}^{\mu}(x, x') = -\delta(x, x'), \quad (7.7)$$

together with appropriate boundary conditions. The introduction of boundary conditions is most easily accomplished in the abstract formalism which replaces (7.7) by

$$FG = -1, \quad (7.8)$$

where

$$F = -\not{p}_{\mu} g^{1/2} g^{\mu\nu} \not{p}_{\nu}, \quad (7.9)$$

$$G(x, x') = \langle x | G | x' \rangle, \quad (7.10)$$

the $|x'\rangle$ being eigenvectors of a commuting set of Hermitian operators x^{μ} in a fictitious Hilbert space,²⁹ and the \not{p} 's being Hermitian "momenta" "canonically conjugate" to the x 's. The formal solution of (7.8) which incorporates the Feynman boundary conditions is

$$g^{1/4} G g^{1/4} = \frac{-1}{g^{-1/4} F g^{-1/4} + i0} = i \int_0^{\infty} \exp(ig^{-1/4} F g^{-1/4} s) ds, \quad (7.11)$$

the factors $g^{\pm 1/4}$ being inserted to insure the covariance of operator functions.³⁰

Taking matrix elements of (7.11) one obtains

$$g^{1/4} G(x, x') g^{1/4} = i \int_0^{\infty} \langle x, s | x', 0 \rangle ds, \quad (7.12)$$

where

$$\langle x, s | x', 0 \rangle \equiv \langle x | \exp(ig^{-1/4} F g^{-1/4} s) | x' \rangle, \quad (7.13)$$

satisfying

$$-i \frac{\partial}{\partial s} \langle x, s | x', 0 \rangle = \langle x, s | x', 0 \rangle_{;\mu}{}^{\mu} \quad (7.14)$$

and the boundary condition

$$\langle x, 0 | x', 0 \rangle = \langle x | x' \rangle = \delta(x, x'). \quad (7.15)$$

The "Schrödinger equation" (7.14) is solved by the ansatz

$$\langle x, s | x', 0 \rangle = -i(4\pi)^{-2} D^{1/2} s^{-2} e^{(i/2s)\sigma} \sum_{n=0}^{\infty} a_n (is)^n, \quad (7.16)$$

$$a_0 = 1, \quad (7.17)$$

which is suggested by its known solution in flat space-time.²⁹

Inserting (7.16) into (7.14) and making use of (7.1) and (7.6), one finds

$$\sigma_{;\mu} a_{n;\mu} + n a_n = \Delta^{-1/2} (\Delta^{1/2} a_{n-1})_{;\mu}{}^{\mu}, \quad n = 1, 2, 3 \dots \quad (7.18)$$

These recursion relations may be solved by successive quadratures along each geodesic emanating from x' . Hadamard²⁴ and Riesz³¹ have shown that the solutions as well as the series (7.16) converge up to the first caustic. Formally we may write

$$a_n = \prod_{m=1}^n [(i\sigma_{;\mu} \not{p}_{\mu} + m)^{-1} \Delta^{-1/2} g^{-1/4} F g^{-1/4} \Delta^{1/2}] \cdot 1, \quad (7.19)$$

where $i\not{p}_{\mu}$ and $g^{-1/4} F g^{-1/4}$ are now to be understood as the gradient and Laplace-Beltrami operators, respectively. Setting

$$(i\sigma_{;\mu} \not{p}_{\mu} + m)^{-1} = \int_{-\infty}^0 [\exp(i\sigma_{;\mu} \not{p}_{\mu} + m) t'_m] dt'_m, \quad (7.20)$$

and making the variable transformation

$$\begin{aligned} t'_1 &= t_1, & t_1 &= t'_1, \\ t'_2 &= t_2 - t_1, & t_2 &= t'_1 + t'_2, \\ &\dots & & \dots \\ t'_n &= t_n - t_{n-1}, & t_n &= t'_1 + t'_2 + \dots + t'_n, \end{aligned} \quad (7.21)$$

³⁰ For example

$$\langle x | (g^{1/4} G g^{1/4})^2 | x' \rangle = g^{1/4} g'^{1/4} \int G(x, x'') g'^{1/2} G(x'', x') dx''.$$

³¹ M. Riesz, Acta Math. 81 (1949).

²⁸ In Eq. (7.7) $G(x, x')$ is to be understood as a biscalar and the δ function $\delta(x, x')$ as a bidensity of unit weight at x and zero weight at x' .

²⁹ Cf., J. Schwinger, Phys. Rev. 82, 664 (1951).

we may recast (7.19) into the form

$$a_n = \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \mathfrak{D}(t_1) \cdots \mathfrak{D}(t_n) \cdot 1, \quad (7.22)$$

where

$$\mathfrak{D}(t) \equiv \exp[(i\sigma,{}^\mu p_\mu + 1)t] \Delta^{-1/2} g^{-1/4} F g^{-1/4} \Delta^{1/2} \times \exp(-i\sigma,{}^\mu p_\mu t). \quad (7.23)$$

Substitution into (7.16) then yields

$$\langle x, s | x', 0 \rangle = -i(4\pi)^{-2} D^{1/2} s^{-2} T[\exp i(\mathfrak{R}s + \sigma/2s)] \cdot 1, \quad (7.24)$$

where

$$\mathfrak{R} \equiv \int_{-\infty}^0 \mathfrak{D}(t) dt, \quad (7.25)$$

$$G(x, x') = \frac{i\Delta^{1/2}}{8\pi^2} T \left\{ \frac{1}{\sigma + i0} - \mathfrak{R} [2\gamma - \ln 2 + \ln(-\mathfrak{R} - i0) + \ln(\sigma + i0)] \left[\frac{1}{2} - \frac{2\mathfrak{R}\sigma}{2^2 \cdot 4} + \frac{(2\mathfrak{R}\sigma)^2}{2^2 \cdot 4^2 \cdot 6} - \cdots \right] - 2\mathfrak{R} \left[\frac{1}{4} - \frac{2\mathfrak{R}\sigma}{2^2 \cdot 4} \left(1 + \frac{1}{4}\right) + \frac{(2\mathfrak{R}\sigma)^2}{2^2 \cdot 4^2 \cdot 6} \left(1 + \frac{1}{2} + \frac{1}{6}\right) - \cdots \right] \right\} \cdot 1, \quad (7.27)$$

$$\gamma = 0.5772 \cdots,$$

where the instructions “ $\sigma + i0$ ” and “ $-\mathfrak{R} - i0$ ” indicate what is evident from (7.24), namely, that $G(x, x')$ is the boundary value of a function of σ and \mathfrak{R} which is analytic in the upper-half σ plane and the upper-half \mathfrak{R} plane. The singularity structure in σ reflects the usual behavior of the Feynman propagator on the light cone ($\sigma = 0$). The remaining singularity structure symbolized by the logarithm of $-\mathfrak{R} - i0$, on the other hand, is far from simple owing to the presence of the chronological ordering operation.

In the perturbative approach to quantum gravidynamics we must deal not with the scalar propagator (7.27) but with the vector and tensor propagators $\hat{G}^{\alpha\beta}$ and G^{ij} . However, the latter have structures closely similar to (7.27); the only difference is that the operators $\mathfrak{D}(t)$ out of which \mathfrak{R} is built are slightly more complicated, and the “1” standing on the right of Eqs. (7.19), (7.22), (7.24), (7.26), and (7.27) is replaced by the *geodesic parallel displacement function*.²⁵ Therefore we can gain a qualitative understanding of the renormalization program in coordinate space already by studying the scalar propagator. Moreover, there is an interesting nonperturbative treatment of the vacuum-to-vacuum amplitude in which the scalar propagator itself directly enters:

Consider the Feynman functional integral, Eq. (20.33) of II, which may be rewritten in the form $\exp i\bar{w}[\varphi]$

$$= Z \int \exp i[S[\varphi + \phi] - S[\varphi] - S_{,i}[\varphi]\phi^i] d\phi, \quad (7.29)$$

where S is the Einstein action and $\varphi_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ (see

and where the symbol T indicates that the operators $\mathfrak{D}(t)$ appearing in the exponential, via (7.25), are to be “chronologically ordered” with respect to the parameters t .

Now the chronological ordering operation commutes with differentiation or integration with respect to the parameter s . Hence Eq. (7.24) may be inserted directly into (7.12). The result is a formal generalization of a well-known expression

$$G(x, x') = -\frac{\Delta^{1/2}}{8\pi} T \left[\frac{H_1^{(2)}((2\mathfrak{R}\sigma)^{1/2})}{(2\mathfrak{R}\sigma)^{1/2}} \right] \cdot 1, \quad (7.26)$$

$H_1^{(2)}$ being the Hankel function of the second kind of order 1. This formula has the series expansion

$$\left\{ \frac{1}{\sigma + i0} - \mathfrak{R} [2\gamma - \ln 2 + \ln(-\mathfrak{R} - i0) + \ln(\sigma + i0)] \left[\frac{1}{2} - \frac{2\mathfrak{R}\sigma}{2^2 \cdot 4} + \frac{(2\mathfrak{R}\sigma)^2}{2^2 \cdot 4^2 \cdot 6} - \cdots \right] - 2\mathfrak{R} \left[\frac{1}{4} - \frac{2\mathfrak{R}\sigma}{2^2 \cdot 4} \left(1 + \frac{1}{4}\right) + \frac{(2\mathfrak{R}\sigma)^2}{2^2 \cdot 4^2 \cdot 6} \left(1 + \frac{1}{2} + \frac{1}{6}\right) - \cdots \right] \right\} \cdot 1, \quad (7.28)$$

Table I of II). Because of the coordinate invariance of the theory the functional integration is redundant and ambiguous, and since no one has yet discovered an analytically accessible nonredundant subspace for the integration, we are forced to accept Eq. (20.12) of II as the effective definition of the integral. However, there is an *incomplete* nonredundant subspace which is easily accessible, namely, the subspace of all conformally equivalent geometries. One may simply set

$$\phi_{\mu\nu} = \chi g_{\mu\nu}, \quad \varphi_{\mu\nu} + \phi_{\mu\nu} = \bar{g}_{\mu\nu} - \eta_{\mu\nu}, \quad \bar{g}_{\mu\nu} = (1 + \chi)g_{\mu\nu}, \quad (7.30)$$

and integrate over χ , to obtain the partial contribution to $\langle 0, \infty | 0, -\infty \rangle$ arising from conformal fluctuations in the vacuum geometry. The special interest of this integration is that it can be performed exactly, giving the conformal contribution to all orders of perturbation theory. The only “fly in the ointment” is that this is the *one* contribution for which high-energy damping cannot be expected to produce a finite cutoff. There is no smearing out of the light cone, because conformal metric fluctuations leave the light cone invariant.

It is easy to show that

$$\bar{g}^{1/2} {}^{(4)}\bar{R} = (1 + \chi)g^{1/2} {}^{(4)}R - 3g^{1/2}(1 + \chi)^{-1}\chi_{;\mu}\chi^{;\mu} - 3g^{1/2}\chi_{;\mu}{}^{;\mu}, \quad (7.31)$$

and hence

$$\begin{aligned} S[\varphi + \phi] - S[\varphi] - S_{,i}[\varphi]\phi^i &= \int [\bar{g}^{1/2} {}^{(4)}\bar{R} - g^{1/2} {}^{(4)}R + g^{1/2}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} {}^{(4)}R)\phi_{\mu\nu}] dx \\ &= -3 \int g^{1/2}(1 + \chi)^{-1}\chi_{;\mu}\chi^{;\mu} dx. \end{aligned} \quad (7.32)$$

The following change of variables then suggests itself:

$$\chi = \phi + \frac{1}{4}\phi^2, \quad 1 + \chi = (1 + \frac{1}{2}\phi)^2. \quad (7.33)$$

This change not only simplifies expression (7.32) but at the same time guarantees the integrity of the signature of space-time. We may allow ϕ to range from $-\infty$ to ∞ without danger of encountering unphysical geometries and at the trivial cost of counting each distinct geometry twice at every point instead of only once. Thus we write

$$\begin{aligned} \exp(i\bar{w}_{\text{conformal}}) \\ = Z \int \exp\left(-3i \int g^{1/2} \phi_{;\mu} \phi^{;\mu} dx\right) d\phi, \end{aligned} \quad (7.34)$$

from which we immediately obtain

$$\begin{aligned} \bar{W}_{\text{conformal}} &\equiv \bar{w}_{\text{conformal}}[\varphi] - \bar{w}_{\text{conformal}}[0] \\ &= -\frac{1}{2}i \ln \frac{\det(g^{1/4} G g^{1/4})}{\det G_0}. \end{aligned} \quad (7.35)$$

The formal determinant may be evaluated by a variational technique with the aid of Eq. (7.11). Under a change in the background field (7.35) suffers the variation

$$\begin{aligned} \delta \bar{W}_{\text{conformal}} &= -\frac{1}{2}i \operatorname{tr} [g^{1/4} G g^{1/4} \delta(g^{-1/4} F g^{-1/4})] \\ &= \frac{1}{2} \operatorname{tr} \int_0^\infty \exp(i g^{-1/4} F g^{-1/4} s) \delta(g^{-1/4} F g^{-1/4}) ds \\ &= -\frac{1}{2}i \delta \operatorname{tr} \int_0^\infty s^{-1} \exp(i g^{-1/4} F g^{-1/4} s) ds, \end{aligned} \quad (7.36)$$

which may be immediately integrated to yield

$$\bar{W}_{\text{conformal}} = -\frac{1}{2}i \operatorname{tr} \int_0^\infty s^{-1} \exp(i g^{-1/4} F g^{-1/4} s) ds + \text{constant}. \quad (7.37)$$

The trace symbol here means "integrate the diagonal matrix element over space-time." Hence, making use of (7.12), (7.13), (7.16) and (7.27), we find

$$\bar{W}_{\text{conformal}} = \int \mathcal{L}_{\text{conformal}} dx + \text{constant}, \quad (7.38)$$

where

$$\begin{aligned} \mathcal{L}_{\text{conformal}} \\ = -\frac{1}{2}i \int_0^\infty s^{-1} \langle x, s | x, 0 \rangle ds = g^{1/2} [i \partial G(x, x') / \partial \sigma]_{x'=x} \\ = \frac{g^{1/2}}{8\pi^2} \left[T \left\{ \frac{1}{(\sigma + i0)^2} + \frac{1}{2(\sigma + i0)} \mathfrak{R} \right. \right. \\ \left. \left. + [2\gamma - \ln 2 - \frac{5}{16} + \ln(-\mathfrak{R} - i0) \right. \right. \\ \left. \left. + \ln(\sigma + i0)] \mathfrak{R}^2 \right\} \cdot 1 \right]_{x'=x}. \end{aligned} \quad (7.39)$$

Several comments are now in order. First we remark that although the final result is divergent, *the degree of divergence is bounded*. The singularity at $x' = x$ is therefore not an essential one as one might have expected on the basis of Eq. (5.3). As a matter of fact (7.39) is identical in structure with the contributions which the propagators G^{ij} and $\hat{G}^{\alpha\beta}$ of the full theory make in lowest perturbation order (i.e., the single closed loops of $\bar{W}_{(1)}$).³²

It may be conjectured that inclusion of the non-conformal vacuum fluctuations will eliminate the divergences altogether, and that a rough approximation to the exact vacuum-to-vacuum amplitude can be obtained simply by making the replacement $\sigma(x, x) \rightarrow \frac{1}{2}\Lambda^{-2}$ in (7.39), where Λ is a high-energy cutoff of the order of unity in absolute units. The "i0" attached to each σ in (7.39) reflects the presence of unremoved noncausal chains. In passing from \bar{W} to W these imaginary infinitesimals should be discarded. We obtain, therefore, the estimate

$$W = \int \mathcal{L}' dx + \text{constant}, \quad (7.40)$$

$$\begin{aligned} \mathcal{L}' \sim \frac{g^{1/2}}{8\pi^2} \left\{ T \left[4\Lambda^4 + \Lambda^2 \mathfrak{R} + \left(\ln \left(\frac{-\mathfrak{R} - i0}{\Lambda^2} \right) \right. \right. \right. \\ \left. \left. \left. - 0.545 \dots \right) \mathfrak{R}^2 \right] \cdot 1 \right\}_{x'=x}. \end{aligned} \quad (7.41)$$

Still cruder estimates of W can be obtained by finding approximations to the complicated quantity \mathfrak{R} . By repeated covariant differentiation of Eqs. (7.1) and (7.6), and use of the commutation laws for the indices thereby induced, one can show that

$$\begin{aligned} \lim_{x' \rightarrow x} \Delta^{-1/2} g^{-1/4} F g^{-1/4} \Delta^{1/2} \cdot 1 \\ = \lim_{x' \rightarrow x} \Delta^{-1/2} \Delta^{-1/2} \cdot \mu^\mu = \frac{1}{6} {}^{(4)}R. \end{aligned} \quad (7.42)$$

This quantity raised to the n th power can be extracted from expression (7.19) or (7.22) for a_n . Moreover, it is clear that the operator $\mathfrak{Q}(t)$ has the dimensions of the curvature scalar and in the limit $x' \rightarrow x$, is a kind of nonlocal, or mean curvature averaged over a certain neighborhood of x . If we represent the purely nonlocal part schematically by $\Delta \mathfrak{R}$, we may write

$$\mathfrak{Q}(t) \underset{x' \rightarrow x}{\sim} e^t \left(\frac{1}{6} {}^{(4)}R + \Delta \mathfrak{R} \right), \quad (7.43)$$

$$\mathfrak{R} \underset{x' \rightarrow x}{\sim} \frac{1}{6} {}^{(4)}R + \Delta \mathfrak{R}. \quad (7.44)$$

The crudest approximation to \mathcal{L}' is then obtained

³² The manifestly covariant occurrence of three distinct types of divergences: quartic, quadratic, and logarithmic, already in lowest order, implies that the conjecture of Brout and Englert (Ref. 19) that quantum gravodynamics is conventionally renormalizable is unfounded.

simply by omitting $\Delta\mathfrak{R}$ altogether:

$$\mathcal{L}' \sim \frac{\Lambda^4}{2\pi^2} g^{1/2} + \frac{\Lambda^2}{48\pi^2} g^{1/2} {}^{(4)}R + \frac{1}{288\pi^2} g^{1/2} {}^{(4)}R^2 \\ \times \left[\ln \left(\frac{-{}^{(4)}R - i0}{6\Lambda^2} \right) - 0.545 \dots \right]. \quad (7.45)$$

Expression (7.45) is prototypical of the contributions which *all* fields make to the geometrical part of the vacuum-to-vacuum amplitude. (The only deviations from it occur with massive fields, for which $\frac{1}{6} {}^{(4)}R$ gets replaced by $-m^2 + \frac{1}{6} {}^{(4)}R$, and with fermion fields, for which the sign of each term is reversed.) These contributions originate in the vacuum polarization which the background geometry induces, and give rise to nonobservable renormalizations as well as physically real radiative corrections.

The first term in (7.45) is a "cosmological" term representing the zero-point vacuum energy which every field, including the gravitational field itself, possesses. It is eliminated by redefining the zero point.

The second term in (7.45) renormalizes the gravitational interaction strength. The relation between the renormalized and "bare" gravitation constants (G and G_0 , respectively) is

$$G = ZG_0, \quad (7.46)$$

$$Z \sim (1 + \Lambda^2/48\pi^2)^{-1}. \quad (7.47)$$

In the theory of the pure gravitational field Z is the only renormalization constant which occurs (provided, of course, the exact theory is really finite.) Because of the manifest covariance of (7.45) it is clear that the same renormalization applies to all vertex functions no matter how many graviton prongs they possess. No Ward identity is needed.

The third term in (7.45) is the only one having observable physical consequences.³³ In the classical limit of long wavelengths and large coherent amplitudes it may be regarded as a correction to the Einstein Lagrangian. Hill³⁴ has applied such a correction term to the problem of gravitational collapse of the Friedmann universe, with encouraging results. He finds that if the sign of the coefficient in front is negative, as would be the case for the contribution from a fermion field, this term succeeds in turning the collapse cycle around before infinite curvature is reached.³⁵ It may be objected that in applying the correction to the Friedmann model one violates the boundary conditions of asymptotic flatness which were assumed to get it in the first place. However, vacuum polarization is

³³ The nonlocal part of the Λ^2 term, which has been omitted from (7.45), also has observable consequences.

³⁴ T. W. Hill, Ph.D. thesis, University of North Carolina, 1965 (unpublished).

³⁵ Not, however, until a density of the order of unity in absolute units is reached. At this density all the matter in the visible universe has been compressed to a region the size of a nucleon.

basically a local phenomenon, and global conditions should have little relevance here.

8. THE INFRARED PROBLEM

The most important contributors to the gravitational polarization of the vacuum, and to the modifications in Einstein's equations which this polarization produces, are the massless fields, including gravity itself. These are also the fields which most readily yield real quanta. The effect of real quantum production on the vacuum-to-vacuum amplitude is taken into account by the " $-i0$ " attached to $-\mathfrak{R}$ in the logarithm of (7.41). Owing to the complexity of \mathfrak{R} , however, the branch-point behavior of the logarithm is very involved, and it is not easy to investigate directly in coordinate space whether or not serious infrared difficulties lie hidden in this expression.

In a closed finite world such difficulties cannot arise since there is a natural low-energy cutoff; troubles occur only in infinite worlds. Let us for simplicity confine our attention to flat backgrounds.³⁶ It is then appropriate to revert to momentum space to study the problem. The analysis for this case is straightforward and has been carried out by Weinberg;³⁷ we shall summarize his results.

The amplitude for a single soft graviton to be produced in a given process has already been derived [Eq. (4.6)]. The corresponding amplitude for the emission of N soft gravitons in all possible ways from a given diagram is just the product of N single-graviton amplitudes. The form of these amplitudes is such that an infrared divergence arises in the computation of the rate at which any physical process takes place when arbitrary numbers of soft gravitons having total energy less than E are simultaneously emitted. This divergence disappears if the contributions from virtual soft gravitons are also included. Weinberg shows that the correct total rate is given by an expression of the form

$$\Gamma(E) = \Gamma_0(E/\Lambda)^B b(B), \quad (8.1)$$

$$b(B) \equiv - \int_{-\infty}^{\infty} \frac{\sin \sigma}{\sigma} \exp \left(B \int_0^1 \frac{e^{i\omega\sigma} - 1}{\omega} d\omega \right) d\sigma \quad (8.2)$$

$$= 1 - \frac{1}{12} \pi^2 B^2 + \dots, \quad (8.3)$$

where Γ_0 is the rate without graviton emission and Λ is a parameter marking the dividing line between "soft" and "hard" virtual gravitons. If Λ is chosen to be of the order of the typical energies involved in the physical process, Eq. (8.1) gives a fair estimate of the rigorous value which would be obtained for $\Gamma(E)$ if the contributions from ultraviolet virtual gravitons were also included and appropriate renormalizations performed. The soft gravitons make appreciable contributions only if attached to the external lines of the Γ_0

³⁶ Conclusions reached for this case are presumably valid also for infinite worlds having other background geometries.

³⁷ S. Weinberg, Phys. Rev. **140**, B516 (1965).

diagrams, and hence the only radiative corrections which should be included in Γ_0 are those which involve internal lines and vertices. The quantity B is given by

$$B = \frac{G}{2\pi} \sum_{m,n} \frac{1+v_{nm}^2}{(1-v_{nm}^2)^{1/2}} \frac{\eta_n \eta_m m_n m_m}{v_{nm}} \ln \frac{1+v_{nm}}{1-v_{nm}}, \quad (8.4)$$

$$v_{nm} \equiv [1 - (m_n m_m / \mathbf{p}_n \cdot \mathbf{p}_m)^2]^{1/2}; \quad (8.5)$$

it depends only on the parameters of the external lines.

These results are completely standard. Except for the detailed form of the quantity B they are identical with the corresponding results in quantum electrodynamics. The question which now must be asked is: What happens when emission takes place from particles which are themselves massless? In quantum electrodynamics such emission is known to give rise to a new and more serious kind of infrared divergence which cannot be removed in any simple or completely natural way. This circumstance has been invoked as the "reason" why massless charged particles do not occur in Nature. In the case of the Yang-Mills and gravitational fields the difficulty presents itself in a peculiarly acute form since these fields are themselves both massless and "charged." Moreover, although there is no experimental evidence for the existence of the Yang-Mills field, gravity is an established fact, as is also its interaction with photons.

Since the Yang-Mills field is a vector field its divergence difficulties are similar to those of massless electrodynamics and hence are difficult if not impossible to remove. Because of the noncommutativity of the emission vertices for Yang-Mills quanta it is not possible to sum the effect of arbitrary numbers of real and virtual quanta into a closed expression like (8.1). Moreover, the situation is further complicated by the fact that there is a non-negligible amplitude for the soft quanta themselves to emit soft quanta. However, there is no evidence whatever that the situation would improve if one could find a way to take all these extra complications rigorously into account.

In the case of the gravitational field, on the other hand, the difficulties miraculously disappear. This happy state of affairs is a consequence of the detailed structure of expression (8.4), which in turn derives from the special form of the graviton emission vertex: $\Gamma_{0\mu\nu} \rightarrow \mathbf{p}_\mu \mathbf{p}_\nu$, as $q \rightarrow 0$. We shall now show how it comes about.

Let us use indices from the first part of the alphabet to distinguish the massless particles from the others. We shall continue to use the symbols m_a , m_b , etc. but with the understanding that these masses ultimately tend to zero. We may then write

$$\frac{1+v_{ab}^2}{(1-v_{ab}^2)^{1/2}} \frac{\eta_a \eta_b m_a m_b}{v_{ab}} \ln \frac{1+v_{ab}}{1-v_{ab}} \rightarrow -4\eta_a \eta_b \mathbf{p}_a \cdot \mathbf{p}_b \ln \left(-\frac{2\mathbf{p}_a \cdot \mathbf{p}_b}{m_a m_b} \right), \quad (8.6)$$

which permits (8.4) to be decomposed as follows:

$$\begin{aligned} B = & -(2G/\pi) \sum_{a,b} \eta_a \eta_b \mathbf{p}_a \cdot \mathbf{p}_b \ln(-2\mathbf{p}_a \cdot \mathbf{p}_b) \\ & - (4G/\pi) \sum_{a,m} \eta_a \eta_m \mathbf{p}_a \cdot \mathbf{p}_m \ln(-2\mathbf{p}_a \cdot \mathbf{p}_m / m_m) \\ & + \frac{G}{2\pi} \sum_{m,n} \frac{1+v_{nm}^2}{(1-v_{nm}^2)^{1/2}} \frac{\eta_n \eta_m m_n m_m}{v_{nm}} \ln \frac{1+v_{nm}}{1-v_{nm}} \\ & + (2G/\pi) \sum_{a,b} \eta_a \eta_b \mathbf{p}_a \cdot \mathbf{p}_b \ln(m_a m_b) \\ & + (4G/\pi) \sum_{a,m} \eta_a \eta_m \mathbf{p}_a \cdot \mathbf{p}_m \ln m_a. \quad (8.7) \end{aligned}$$

With the aid of the energy-momentum conservation law

$$\sum_a \eta_a \mathbf{p}_a + \sum_m \eta_m \mathbf{p}_m = 0, \quad (8.8)$$

the last two terms of (8.7) may be combined into

$$(2G/\pi) \sum \eta_a \eta_b \mathbf{p}_a \cdot \mathbf{p}_b (\ln m_b - \ln m_a),$$

which vanishes by symmetry. The masses m_a , m_b thus disappear from (8.7), and since $\mathbf{p}_a \cdot \mathbf{p}_b \ln(-2\mathbf{p}_a \cdot \mathbf{p}_b)$ vanishes when either $a=b$ or \mathbf{p}_a is parallel to \mathbf{p}_b , it is evident that B is completely free of divergences.

The only uncertainty which remains in Weinberg's analysis, and which he himself points out, concerns his use of the DeDonder gauge for the virtual gravitons. Except when the stress-energy tensor is conserved at each virtual graviton vertex it is not easy to see that the choice of gauge is immaterial. But stress-energy conservation of this simple type holds only when the particle lines on both sides of the vertex are on the mass shell. [See Eq. (3.8)]. Since only the external lines satisfy this condition Weinberg must appeal to the fact that the other lines are only *slightly off* the mass shell and hence violate the conservation conditions only minimally.³⁸

Weinberg's act of faith on this question can be rigorously justified within the framework of the complete theory developed in II. We know from this theory that the choice of gauge for internal lines is irrelevant provided: (a) it is applied consistently and (b) *all* diagrams contributing to a given process are included. Now Weinberg omits the diagrams which involve infrared *fictitious* quanta. But it is not hard to show that the contributions of these diagrams all vanish as the infrared momenta go to zero, and hence may be neglected. This is a consequence of the fact that the fictitious quanta always occur in closed loops containing uniformly oriented vertices $V_{(\alpha i)\beta}$. Because of the special

³⁸ In electrodynamics it is not difficult to show that gauge invariance holds when every vertex along each charged particle line is taken into account. In gravidynamics, however, every line is "charged," and the "charge" splits up or recombines at every vertex.

form (2.8) which these vertices possess, the uniform orientation guarantees that at least one of the vertices in each infrared loop is proportional to an infrared momentum.

We conclude this section by repeating Weinberg's calculation of B in the nonrelativistic limit and correcting a minor mistake in his result. The quantity v_{nm}^2 is first expanded in the form

$$v_{nm}^2 = (\mathbf{v}_n - \mathbf{v}_m)^2 - \mathbf{v}_n^2 \mathbf{v}_m^2 + 2(\mathbf{v}_n^2 + \mathbf{v}_m^2) \mathbf{v}_n \cdot \mathbf{v}_m - 3(\mathbf{v}_n \cdot \mathbf{v}_m)^2 + \dots, \quad (8.9)$$

where $\mathbf{v}_n = \mathbf{p}_n/E_n$. This expansion is then inserted into

$$\frac{1+v_{nm}^2}{(1-v_{nm}^2)^{1/2}} \frac{\eta_n \eta_m m_n m_m}{v_{nm}} \ln \frac{1+v_{nm}}{1-v_{nm}} = 2\eta_n \eta_m m_n m_m \left(1 + \frac{11}{6} v_{nm}^2 + \frac{63}{40} v_{nm}^4 + \dots \right) \quad (8.10)$$

to obtain a lengthy expression for B correct to the

fourth order in the velocities. This expression can be greatly simplified with the aid of the energy-momentum conservation laws

$$\sum_n \eta_n m_n (1 + \frac{1}{2} \mathbf{v}_n^2 + \frac{3}{8} \mathbf{v}_n^4 + \dots) = 0,$$

$$\sum_n \eta_n m_n \mathbf{v}_n (1 + \frac{1}{2} \mathbf{v}_n^2 + \dots) = 0,$$

and one finally obtains the compact formula

$$B = (4G/5\pi) \text{tr}(\Delta d^2 \mathbf{Q}/d\ell^2)^2, \quad (8.11)$$

where $\Delta d^2 \mathbf{Q}/d\ell^2$ is the dyadic previously defined by Eqs. (4.11) and (4.12), having the explicit traceless form³⁹

$$\Delta d^2 \mathbf{Q}/d\ell^2 = \sum_n \eta_n m_n (\mathbf{v}_n \mathbf{v}_n - \frac{1}{3} \mathbf{1} \mathbf{v}_n^2). \quad (8.12)$$

³⁹By inadvertently dropping a term Weinberg obtains a dyadic which is not traceless.

Quantum Statistics of Coupled Oscillator Systems

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The statistical properties of systems of coupled quantum-mechanical harmonic oscillators are analyzed. The Hamiltonian for the system is assumed to be an inhomogeneous quadratic form in the creation and annihilation operators, and is allowed to have an explicit time dependence. The relationship to classical theory is emphasized by expressing pure states in terms of the coherent-state vectors, and density operators by means of the P representation and an analogous representation involving the Wigner function. The state which evolves from an initially coherent state of the system is found, and equations governing the time evolution of the Wigner function and the weight function for the P representation are derived, in differential and integral form, for arbitrary initial states of the system. The results remain valid for couplings which do not preserve the vacuum state, and for cases in which the time dependence of the coupling parameters gives rise to large-scale amplification of the initial field intensities. The analysis is performed by first treating general linear inhomogeneous canonical transformations on the oscillator variables, and then specializing to the case in which these transformations represent the solutions for the Heisenberg operators in terms of their initial values. The results are illustrated within the context of a model of parametric amplification.

I. INTRODUCTION

IN a wide variety of physical processes, all of the dynamical elements which enter into the description of the state of the system may be treated formally as quantum-mechanical harmonic-oscillator modes. The coupling between the modes typically takes the form of a quadratic expression in the annihilation and creation operators $a_j(t)$ and $a_j^\dagger(t)$, in which the coupling parameters are time-dependent in the general case. In addition, driving terms linear in the oscillator variables may be present. The operators $a_j(t)$ and $a_j^\dagger(t)$ then obey linear inhomogeneous equations of motion, and the solutions to these equations take the same form as the solutions for the c -number complex amplitudes in the analogous

classical system. The time-dependent expectation values of dynamical operators for a given initial state of the system may be evaluated straightforwardly with the aid of the solutions to the Heisenberg equations of motion and the commutation relations for a_j and a_j^\dagger , and some indication is thereby provided of the way in which quantum fluctuations influence the time development of the oscillator system.¹⁻⁵

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