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Kinetic Equation for Drifted Electron Plasma

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The kinetic equation for a plasma consisting of ions and drifted electrons under the influence of an electric field is derived, neglecting electron-electron correlations. The high-frequency resistivity is calculated and its dependence on the drift velocity v_D is obtained. Resistive instability was found to occur for $v_D \cong 1.4 v_{th}$, where v_{th} is the steady-state electron thermal velocity.

I. INTRODUCTION

In recent years, calculation of the conductivity for electron-ion plasmas has been performed by many authors. The low-frequency conductivity, i.e., the case $\omega \ll \nu$, where ν is the collision frequency, has been calculated using the Fokker-Planck equation.[1] For the high-frequency conductivity, i.e., $\omega \gg \nu$, an elementary model has been developed[2] which takes into account the self-consistent field of the electrons but neglects electron-electron correlations. A more rigorous treatment for calculating the high-frequency conductivity has been given using the BBKGY hierarchy[3] or Kubo's formulation and the Green's-function technique of the many-body theory.[4] It follows from these calculations that one should treat the correlation functions on the same time scale as the one-particle distribution function, and that for massive ions the electron-electron correlations do not affect the high-frequency, long-wavelength conductivity. For small but finite wave number, one cannot neglect the electron-electron correlation in calculating the conductivity. However, assuming that for small but finite wavelength the effects of electron-electron correlation as well as electron-ion collision are small, we may hopefully treat each of them separately.

We discuss here a model of an electron-ion system, following Berk,[5] in which electron-electron correlation has been omitted. However, we consider in the zeroth order an electron gas having a Boltzman distribution but drifting relatively to massive ions (randomly distributed) which are embedded in the system. The purpose of this work is to calculate the conductivity for a system in steady state but not necessarily in equilibrium; i.e., when the electrons are drifting relative to the ions. A treatment of this problem starting with the Fokker-Planck equation is described by Musha and Yoshida.[6] This treatment cannot in principle describe the conductivity for frequencies ω, in the vicinity of the plasma frequency ω_p. Our method of solution is similar to that given by Kohn and Luttinger.[7] We write coupled kinetic equations for the coherent (with the external field) and incoherent "one-particle distribution functions." These distribution functions are treated as changing on the same time scale. Following Ref. 5, both the coherent and incoherent distribution functions are driven by the external field. We assume[7] that the amplitude of the incoherent distribution function is small and can be solved in terms of the coherent distribution function and the field. Then we substitute it in the kinetic equation for the coherent distribution function, and therefore reduce the problem to an integral equation for the coherent distribution function.

II. DERIVATION OF THE TRANSPORT EQUATION

We consider a system of drifted electrons relative to fixed ions in the rest frame. The Vlasov equation in the rest frame reads

$$\frac{\partial F}{\partial t}+\mathbf{V}\cdot\frac{\partial F}{\partial \mathbf{x}}+\frac{e}{m}\nabla\Phi\cdot\frac{\partial F}{\partial \mathbf{v}}-\frac{e}{m}\mathbf{E}_{q\omega}e^{-i\omega t+i\mathbf{q}\cdot\mathbf{k}}\cdot\frac{\partial F}{\partial \mathbf{v}}=0\,, \quad (1)$$

where Φ, the potential arising from the charged particles themselves, obeys

$$\nabla^2\Phi=4\pi e\left[n_0\int F d\mathbf{v}-Z\sum_l \delta(\mathbf{x}-\mathbf{R}_l)\right]. \quad (2)$$

Here $E_{q\omega}$ is the external field of wave number q and frequency ω; $-e$, m, and n_0 are, respectively, the electron charge, mass, and density, Z is the ion valence and R_l is the position of the lth ion.

We assume that the electric field $E_{q\omega}$ is small in the sense that we are interested in the linear response to the field; i.e., the field-independent conductivity. The only small parameter in our system is $r_s=1/n\lambda_D{}^3$, the number of particles in the Debye sphere. We are essentially looking for a correction to order r_s in the conductivity. We next write the kinetic equation for f, which describes the linear response to the external field,

[1] Many references may be found in I. B. Bernstein and S. K. Trehan, Nucl. Fusion **1**, 3 (1960). See also H. Margeneau, Phys. Rev. **109**, 6 (1958).
[2] J. Dawson and C. Oberman, Phys. Fluids **5**, 517 (1962); **6**, 394 (1963).
[3] C. Oberman, A. Ron, and J. Dawson, Phys. Fluids **5**, 1517 (1962).
[4] V. I. Perel and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **41**, 886 (1961) [English transl.: Sov. Phys.—JETP **14**, 633 (1962)]; A. Ron and N. Tzoar, Phys. Rev. **131**, 12 (1963).
[5] H. L. Berk, Phys. Fluids **7**, 257 (1964).
[6] T. Musha and F. Yoshida, Phys. Rev. **133**, A1303 (1964).
[7] W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957); **109**, 1892 (1958).

of the unperturbed electron-ion system.

$$\frac{\partial f}{\partial t}+\mathbf{V}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{e}{m}\nabla\varphi\cdot\frac{\partial F^s}{\partial \mathbf{v}}+\frac{e}{m}\nabla\Phi^s\cdot\frac{\partial f}{\partial \mathbf{v}}+\frac{e}{m}\mathbf{E}_{q\omega}e^{-i\omega t+i\mathbf{q}\cdot\mathbf{x}}\cdot\frac{\partial F^s}{\partial \mathbf{v}}=0\,, \quad (3)$$

and

$$\nabla^2\varphi=4\pi e n_0\int f d\mathbf{v}. \quad (4)$$

Here F^s and Φ^s are, respectively, the system static distribution function and its potential, i.e., when the external electric field is zero. In order to be consistent with our assumption of $r_s \ll 1$, we solve for F^s to the lowest order in the electron-ion collision process.

We define

$$F^s=F^0+F'\,, \quad (5)$$

$$\Phi^s=0+\Phi'\,, \quad (6)$$

where F' and Φ' obey

$$\mathbf{V}\cdot\frac{\partial F'}{\partial \mathbf{x}}+\frac{e}{m}\nabla\Phi'\cdot\frac{\partial F^0}{\partial \mathbf{v}}=0\,, \quad (7)$$

and

$$\nabla^2\Phi'=4\pi e\left[n_0\int F' d\mathbf{v}-Z\sum_l \delta(\mathbf{x}-\mathbf{R}_l)\right], \quad (8)$$

where F^0 is the drifted Maxwell-Boltzmann distribution for the electrons given by

$$F^0=\frac{1}{(2\pi v_{\mathrm{th}}^2)^{3/2}}e^{-(\mathbf{v}-\mathbf{v}_0)^2/2v_{\mathrm{th}}^2}. \quad (9)$$

Here v_{th} is the thermal velocity of the electrons in the steady state. The solution of F' and Φ' in Eqs. (7) and (8) are straightforward and are given in terms of their Fourier components $F_{\mathbf{k}}'$ and $\Phi_{\mathbf{k}}'$ as

$$\Phi_{\mathbf{k}}'=\frac{4\pi e Z}{k^2 D_{\mathbf{k}}}\frac{1}{V}\sum_l e^{-i\mathbf{k}\cdot\mathbf{R}_l} \quad (k\neq 0)\,, \quad (10)$$

and

$$F_{\mathbf{k}}'=-\frac{e}{m}\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-i\eta}\Phi_{\mathbf{k}}' \quad (k\neq 0)\,. \quad (11)$$

Here $D_{\mathbf{k}}$ is the limit of the dielectric function $D_{\mathbf{k}\omega}$ when $\omega\to 0$. $D_{\mathbf{k}\omega}$ is defined by

$$D_{\mathbf{k}\omega}=1+\frac{\omega_p^2}{k^2}Q_{\mathbf{k}\omega}\,, \quad (12)$$

where

$$Q_{\mathbf{k}\omega}=\int d\mathbf{v}\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}. \quad (13)$$

We point out that the D_k is complex here because of the drift which imposes a Doppler frequency shift in the laboratory system. This should not disturb us since $\Phi_{\mathbf{k}}'$ is not a measurable quantity by itself. If we average over all possible ion positions in Eq. (10) we obtain $\langle\Phi_{\mathbf{k}}'\rangle\to 0$. However, upon developing our theory it will be obvious later that our result will depend on $|\Phi_{\mathbf{k}}'|^2$, which is a real quantity and represents the electrostatic screening of the ions by the electrons.

We next consider the Fourier transform of Eq. (3) and obtain the kinetic equation for the coherent and noncoherent distribution functions, $f_{\mathbf{q}}$ and $f_{\mathbf{k}}$ $(k\neq q)$, respectively:

$$(-i\omega+i\mathbf{q}\cdot\mathbf{v}+\eta)f_{\mathbf{q}}-\frac{e}{m}\boldsymbol{\varepsilon}_{\mathbf{q}\omega}\cdot\frac{\partial F^0}{\partial\mathbf{v}}=-i\frac{e}{m}\sum_{\mathbf{k}}\left[\varphi_{\mathbf{k}}\mathbf{k}\cdot\frac{\partial F'_{\mathbf{q}-\mathbf{k}}}{\partial\mathbf{v}}+\Phi'_{\mathbf{q}-\mathbf{k}}(\mathbf{q}-\mathbf{k})\frac{\partial f_{\mathbf{k}}}{\partial\mathbf{v}}\right], \quad (14)$$

where

$$\boldsymbol{\varepsilon}_{\mathbf{q}\omega}=\mathbf{E}_{\mathbf{q}\omega}-i\mathbf{q}\varphi_{\mathbf{q}}\,, \quad (15)$$

and

$$(-i\omega+i\mathbf{k}\cdot\mathbf{v}+\eta)f_{\mathbf{k}}+i\frac{e}{m}\varphi_{\mathbf{k}}\mathbf{k}\cdot\frac{\partial F^0}{\partial\mathbf{v}}-\frac{e}{m}\boldsymbol{\varepsilon}_{\mathbf{q}\omega}\cdot\frac{\partial F'_{k-q}}{\partial\mathbf{v}}+i\frac{e}{m}\Phi'_{\mathbf{k}-\mathbf{q}}(\mathbf{k}-\mathbf{q})\cdot\frac{\partial f_{\mathbf{q}}}{\partial\mathbf{v}}=-i\frac{e}{m}\sum_{\mathbf{k}'\neq\mathbf{q}}\left[\varphi_{\mathbf{k}}\mathbf{k}'\cdot\frac{\partial F'_{\mathbf{k}-\mathbf{k}'}}{\partial\mathbf{v}}+\Phi'_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_{\mathbf{k}'}}{\partial\mathbf{v}}\right]\approx 0. \quad (16)$$

Here, φ_q and φ_k represent the coherent and incoherent parts of the self-consistent potential, and are obtained from Eq. (4),

$$-q^2\varphi_{\mathbf{q}}=4\pi e n_0\int f_{\mathbf{q}}d\mathbf{v}\,,$$

$$-k^2\varphi_{\mathbf{k}}=4\pi e n_0\int f_{\mathbf{k}}d\mathbf{v}. \quad (17)$$

In Eq. (16) the right-hand side is of higher order in the plasma parameter r_s, since it formally includes two collisions with ions. In the zeroth order, the electrons oscillate coherently with the field with wave number q. One collision changes the wave number of the electron from q to k', and the second one brings it back to wave number k. It should be pointed out here that the right-hand side of Eq. (16) gives no contribution for $\mathbf{k}'=\mathbf{k}$, since $F'_{k=0}$ and $\Phi'_{k=0}$ are zero. This is obviously the effect of charge neutrality for our system.

We therefore obtain, consistent with our perturbation approach, the following equation governing the noncoherent distribution function

$$(-i\omega+i\mathbf{k}\cdot\mathbf{v}+\eta)f_{\mathbf{k}}+\frac{ie}{m}\Phi_{\mathbf{k}}\mathbf{k}\cdot\frac{\partial F^0}{\partial\mathbf{v}}-\frac{e}{m}\boldsymbol{\varepsilon}_{\mathbf{q}\omega}\frac{\partial F'_{\mathbf{k}-\mathbf{q}}}{\partial\mathbf{v}}+i\frac{e}{m}\Phi'_{\mathbf{k}-\mathbf{q}}(\mathbf{k}-\mathbf{q})\frac{\partial f_{\mathbf{q}}}{\partial\mathbf{v}}=0. \quad (18)$$

Equations (14), (17), and (18) now constitute a closed set which we solve in the following way: We first assume $f_\mathbf{q}$ to be larger than $f_\mathbf{k}$, in accord with our perturbation approach, and obtain $f_\mathbf{k}$ and $\Phi_\mathbf{k}$ in terms of the field and $f_\mathbf{q}$. In turn we substitute $f_\mathbf{k}$ and $\Phi_\mathbf{k}$ in Eq. (14) to obtain an integral equation for $f_\mathbf{q}$ which reads

$$
\begin{aligned}
(\omega-\mathbf{q}\cdot\mathbf{v}+i\eta)f_\mathbf{q}-\frac{ie}{m}\boldsymbol{\varepsilon}\cdot\frac{\partial F^0}{\partial\mathbf{v}}=\frac{4\pi e^2Z\omega_p{}^2}{m(2\pi)^3}\int d\mathbf{k}\Psi_{\mathbf{k}-\mathbf{q}}\times\Bigg\{(\mathbf{q}-\mathbf{k})\frac{\partial}{\partial\mathbf{v}}\left[\frac{(\mathbf{k}-\mathbf{q})\cdot\partial f_\mathbf{q}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right]-\left(\frac{\omega_p{}^2}{k^2D_{\mathbf{k}\omega}}\right) \\
\times\left[(\mathbf{q}-\mathbf{k})\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{k\cdot\partial F^0/\partial\mathbf{v}}{\omega-\mathbf{k}\mathbf{v}+i\eta}\right\}-\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-i\eta}\right\}\right]\int d\mathbf{v}\frac{(\mathbf{k}-\mathbf{q})\partial f_q/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}-i\frac{e}{m}(\mathbf{q}-\mathbf{k})\cdot\frac{\partial}{\partial\mathbf{v}} \\
\times\left[\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\left(\boldsymbol{\varepsilon}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-i\eta}\right)\right]+\left(\frac{ie}{m}\frac{\omega_p{}^2}{k^2D_{\mathbf{k}\omega}}\right)\left[(\mathbf{q}-\mathbf{k})\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right\}\right. \\
\left.-\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left\{\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-i\eta}\right\}\right]\int d\mathbf{v}\frac{\boldsymbol{\varepsilon}\cdot\frac{\partial}{d\mathbf{v}}\left[\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-in}\right]}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\Bigg\}. \quad (19)
\end{aligned}
$$

Here we have already averaged over the ion positions, and $\Psi_\mathbf{k}$ is given by

$$\Psi_\mathbf{k}=\left|\frac{1}{k^2D_\mathbf{k}}\right|^2\frac{1}{N}\sum_{ll'}e^{i\mathbf{k}\cdot(\mathbf{R}_l-\mathbf{R}_{l'})}=\left|\frac{1}{k^2D_\mathbf{k}}\right|^2N_\mathbf{k}. \quad (20)$$

For randomly distributed ions one obtains

$$N_k=N^{-1}\sum_{ll'}\exp i\mathbf{k}(\mathbf{R}_l-\mathbf{R}_{l'})=1. \quad (21)$$

The integral equation, given by Eq. (19), is exact to the first order in r_s, our expansion parameter. Although this equation is very complicated, we shall extract useful information concerning the high-frequency conductivity.[8]

III. THE HIGH-FREQUENCY LONG-WAVELENGTH CONDUCTIVITY

We shall consider long wavelength in the following sense: The wave number q is chosen to be small enough so that

$$\frac{qv_{\text{th}}}{\omega}\ll1\,;\quad\frac{qv_D}{\omega}<1 \quad (22)$$

for any frequency ω and drift velocity v_D in our problem. We therefore expand the right-hand side in Eq. (19) in a power series of q, and take only the dominant term when $q\to0$. It is clear that in the contribution of the electron-ion collision to the conductivity, we have neglected finite-wavelength effects.

This approximation will not change the qualitative behavior of the result. However, it will simplify the procedure. We therefore obtain, from Eq. (19),

$$
\begin{aligned}
f_\mathbf{q}-\frac{ie}{m}\frac{\boldsymbol{\varepsilon}\cdot\partial F^0/\partial\mathbf{v}}{\omega-\mathbf{q}\cdot\mathbf{v}+i\eta}=-\frac{4\pi e^2Z\omega_p{}^2}{m\omega(2\pi)^3}\int d\mathbf{k}\,\Psi_\mathbf{k}\times\Bigg\{\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\left[\frac{\mathbf{k}\cdot\partial f_q/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right]-\left(\frac{\omega_p{}^2}{k^2D_{\mathbf{k}\omega}}\right)\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left[\frac{1}{\mathbf{k}\cdot\mathbf{v}-i\eta}+\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right]\mathbf{k}\cdot\frac{\partial F^0}{\partial\mathbf{v}} \\
\times\int d\mathbf{v}\frac{\mathbf{k}\cdot\partial f_\mathbf{q}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}-i\frac{e}{m}\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left[\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\left(\boldsymbol{\varepsilon}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v}-i\eta}\right)\right]+\frac{ie}{m}\left(\frac{\omega_p{}^2}{k^2D_{\mathbf{k}\omega}}\right)\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right) \\
\times\left(\left[\frac{1}{\mathbf{k}\cdot\mathbf{v}-i\eta}+\frac{1}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right]\mathbf{k}\cdot\frac{\partial F^0}{\partial\mathbf{v}}\right)\int d\mathbf{v}\frac{\boldsymbol{\varepsilon}\cdot\partial/\partial\mathbf{v}[(\mathbf{k}\cdot\partial F^0/\partial\mathbf{v})/\mathbf{k}\cdot\mathbf{v}-i\eta]}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\Bigg\}. \quad (23)
\end{aligned}
$$

Now, to obtain the high-frequency conductivity, i.e., $\omega\gg\nu$, ν being the collision frequency, we assume that the zeroth-order solution for $f_\mathbf{q}$ (in power of r_s) can be treated as the dominant part of the coherent distribution function. We therefore substitute in the right-hand side of Eq. (23) the approximate $f_\mathbf{q}{}^{(0)}$ given by

$$f_\mathbf{q}{}^{(0)}\approx\frac{ie}{m}\frac{\mathbf{E}_{q\omega}\cdot\partial F^0/\partial\mathbf{v}}{\omega}$$

[8] It should be noted that, in the integral Eq. (23), the dominant (divergent) contribution from large k's is given by the first term on the right-hand side. For frequencies ω different from the plasma frequency ω_p, it is sufficient to retain only this term in the integral Eq. (23) in order to derive the dominant contribution to the collision term valid also at low frequencies ($\omega<\nu$) (5). However, the appearance of an additional vector $\mathbf{v}_D$ in the integrand makes the integration difficult.

in accord with our previous approximation. Then we operate on both sides of Eq. (23) with $-\int en_0\mathbf{v}d\mathbf{v}$ and obtain our solution for the current density:

$$\mathbf{j}=\mathbf{j}_0+\frac{ie^2Z\omega_p^4}{m\omega^3(2\pi)^3}\int d\mathbf{k}\,\mathbf{k}\Psi_\mathbf{k}\frac{D_\mathbf{k}}{D_{\mathbf{k}\omega}}\left\{\int d\mathbf{v}\frac{(\mathbf{k}\cdot\partial/\partial\mathbf{v})(\boldsymbol{\varepsilon}\cdot\partial F^0/\partial\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}-\int d\mathbf{v}\omega\frac{(\boldsymbol{\varepsilon}\cdot\partial/\partial\mathbf{v})[(\mathbf{k}\cdot\partial F^0/\partial\mathbf{v})/\mathbf{k}\cdot\mathbf{v}+i\eta]}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta}\right\}. \tag{24}$$

Here we have made use of Eqs. (12) and (13).

In Eq. (24) $\mathbf{j}_0$ represents the current due to the system of free electrons and is given by

$$\mathbf{j}_0=-i\frac{\mathbf{q}}{q}\frac{\omega_p^2}{4\pi}\frac{\omega}{q^2}\int d\mathbf{v}\frac{\mathbf{k}\cdot\partial F^0/\partial\mathbf{v}}{\omega-\mathbf{q}\cdot\mathbf{v}+i\eta}. \tag{25}$$

The velocity integration in the right-hand side of Eq. (24) is straightforward. We obtain after some algebra the longitudinal conductivity, using the fact that $\mathbf{j}$, $\mathbf{j}_0$, $\mathbf{E}_{q\omega}$, and $\mathbf{q}$ are all parallel to each other:

$$\sigma_{q\omega}=\frac{i\omega_p^2}{4\pi\omega}\left\{1+\frac{3q^2v_{\text{th}}^2}{\omega^2}-i\surd(\pi/2)\frac{\omega^2}{\omega_p^2}\frac{k_D^2}{q^2}\frac{\omega}{qv_{\text{th}}}\times\exp(-\omega^2/2q^2v_{\text{th}}^2)+\frac{4\pi e^2Z}{m\omega^2(2\pi)^3}\int d\mathbf{k}(\hat{k}\cdot\hat{q})^2\times\left(\frac{D_\mathbf{k}}{D_\mathbf{k}^*}\right)\left[\frac{1}{D_{\mathbf{k}\omega}}-\frac{1}{D_\mathbf{k}}\right]\right\}. \tag{26}$$

The conductivity given by Eq. (26) was calculated using the restriction given by Eq. (22). It represents the response of the free-electron gas and takes into account approximately the electron-ion contribution to the response, in the sense that the small but finite q effect in the collision integral can be neglected.

We next check the limit of $v_D=0$. In this case, using Eqs. (9), (12), and (13), we obtain $D_{k,\omega}\to\mathcal{E}_{k,\omega}$, where

$$\mathcal{E}_{k\omega}=1+\frac{\omega_p^2}{k^2}\int d\mathbf{v}\frac{\mathbf{k}\cdot\partial f^0/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}+i\eta} \tag{27}$$

and $f^0=(2\pi v^2_{\text{th}})^{-3/2}\exp(-v^2/2v^2_{\text{th}})$ is the Maxwell-Boltzmann distribution function. We also deduce by observation that

$$D_{\mathbf{k},\omega}=\mathcal{E}_{k,\omega-\mathbf{k}\cdot\mathbf{v}_D}. \tag{28}$$

It is therefore clear that for $v_D=0$ our result given by Eq. (26) reduces to the well-known expression for the conductivity.[2]

We next define the resistivity $R_{q\omega}$ as the real part of $(\sigma_{q\omega})^{-1}$ and obtain

$$R_{q\omega}=R_{q\omega}^{(1)}+R_{q\omega}^{(2)}, \tag{29}$$

where

$$R_{q\omega}^{(1)}=4\pi\surd(\pi/2)\frac{\omega^3}{\omega_p^4}\frac{k_D^2}{q^2}\left(\frac{\omega}{qv_{\text{th}}}\right)\exp(-\omega^2/2q^2v_{\text{th}}^2) \tag{30}$$

is the contribution of Landau damping to the resistivity and

$$R_{q\omega}^{(2)}=-\frac{16\pi^2e^2Z}{m\omega\omega_p^2(2\pi)^3}\int d\mathbf{k}(\hat{k}\cdot\hat{q})^2\times\text{Im}\left[\frac{\mathcal{E}_{k,-\mathbf{k}\cdot\mathbf{v}_D}}{\mathcal{E}^*_{k,-\mathbf{k}\cdot\mathbf{v}_D}}\left(\frac{1}{\mathcal{E}_{k,\omega-\mathbf{k}\cdot\mathbf{v}_D}}-\frac{1}{\mathcal{E}_{k,-\mathbf{k}\cdot\mathbf{v}_D}}\right)\right] \tag{31}$$

is the part of the resistivity that arises from the electron-ion collisions. Our calculations for the collisional resistivity, Eq. (31), are valid for $\omega>\nu$, where ν is the collision frequency.

The conductivity $R_{q\omega}^{(2)}$ is a tensor, depending on whether $\mathbf{q}$ and $\mathbf{E}_q$, respectively, are parallel or perpendicular to $\mathbf{v}_D$. In both cases using the relations $\text{Re}\mathcal{E}_{k,\omega}=\text{Re}\mathcal{E}_{-k,-\omega}$ and $\text{Im}\mathcal{E}_{k,\omega}=-\text{Im}\mathcal{E}_{-k,-\omega}$ it is easy to see that the second term in Eq. (31) vanishes after integration. Our result for the resistivity is therefore reduced to

$$R_{q\omega}^{(2)}=-\frac{16\pi^2e^2Z}{m\omega\omega_p^2(2\pi)^3}\int d\mathbf{k}(\hat{k}\cdot\hat{q})^2\times\text{Im}\left(\frac{\mathcal{E}_{k,-\mathbf{k}\cdot\mathbf{v}_D}}{\mathcal{E}^*_{k,-\mathbf{k}\cdot\mathbf{v}_D}}\frac{1}{\mathcal{E}_{k,\omega-\mathbf{k}\cdot\mathbf{v}_D}}\right). \tag{32}$$

To calculate exactly the integral in Eq. (32) a computer solution is required.

However, in order to obtain an analytical result we observe that the dominant contribution to the integral comes from large k's. We therefore limit the integration over k from k_D to $k_{\max}=(kT/e^2)$ (see a discussion of this point in Ref. 9). The cutoff at $k_{\max}$ was introduced to prevent the nonphysical divergence of the integral, since our theory does not properly treat large angle collisions. One can immediately observe that the singular term of the integrand comes from $\text{Im}\mathcal{E}_{k,\omega-\mathbf{k}\cdot\mathbf{v}_D}$. Therefore, we approximate our result for the resistivity by taking $\mathcal{E}_{k,-\mathbf{k}\cdot\mathbf{v}_D}\to 1$. which for $k\gg k_D$ is a good approximation. We also choose to calculate the case $\mathbf{E}\|\mathbf{q}\|\mathbf{v}_D$, in which the effect of the drift on the resistivity will be most noticeable, and obtain

$$R_{q\omega\|}^{(2)}\cong\frac{16\pi^2e^2z}{m\omega_p^2\omega(2\pi)^3}\int_{k_D}^{k_{\max}}k^2dk\int_{-1}^{+1}d\eta\eta^2\times\text{Im}\mathcal{E}_{k,\omega-kv_D\eta}. \tag{33}$$

[9] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 76.

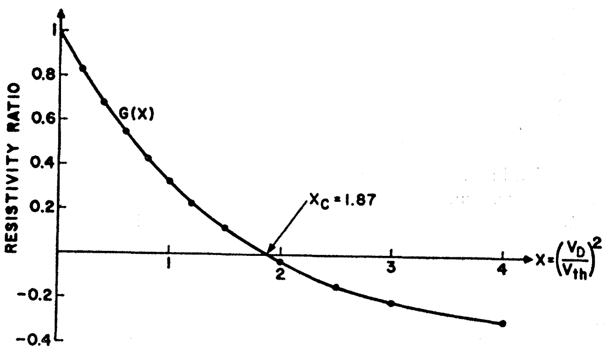

FIG. 1. Dependence of resistivity on drift velocity. $G(x)$ is the ratio of resistivity at finite drift velocity v_D to its value for $v_D=0$. The crossover from positive to negative resistivity occurs at $x_c=1.87$.

Here $\eta=\cos\theta$, and θ is the angle between $\hat{k}$ and $\hat{v}_D$. In Eq. (33) we have omitted completely the screening effect, which indicates that this expression is only valid for $\omega\gtrsim\omega_p$. We next substitute for $\mathrm{Im}\mathcal{E}_{k,\omega}$

$$\mathrm{Im}\epsilon_{k,\omega}=\sqrt{(\pi/2)}\frac{\omega_p{}^2\omega}{k^3v_{\mathrm{th}}{}^3}e^{-\omega^2/2k^2v_{\mathrm{th}}{}^2} \tag{34}$$

and we obtain after integration (retaining only dominant terms in the integral) the result

$$R_{q\omega\parallel}{}^{(2)}=(2\pi)^3\frac{r_s}{\omega_p}\frac{1}{2\sqrt{\pi}}F\left(\frac{v_D{}^2}{v_{\mathrm{th}}{}^2}\right)\ln\left(\frac{2v_{\mathrm{th}}{}^2k_{\max}{}^2}{\omega^2}\right), \tag{35}$$

where $r_s=(1/n\lambda_D{}^3)$ is the small parameter of the plasma and $F(x)$ is given by

$$F(x)=-4\left[\frac{\partial}{\partial x}+2x\frac{\partial^2}{\partial x^2}\right]\left(\frac{\mathrm{erf}[\sqrt{(x/2)}]}{\sqrt{(x/2)}}\right). \tag{36}$$

The function $F(x)$ is given by the uniformly convergent series,

$$F(x)=\tfrac{2}{3}\left[1+\tfrac{3}{2}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{2^{n-2}(n-1)!}\left(\frac{2n-1}{2n+1}\right)x^{n-1}\right]$$
$$=\tfrac{2}{3}G(x). \tag{37}$$

We immediately obtain the ratio of the resistivity with and without drift for the case of $\omega\gtrsim\omega_p$ to be

$$\rho=\frac{R_{q\omega\parallel}{}^{(2)}(v_D)}{R_{q\omega\parallel}{}^{(2)}(0)}=G(x). \tag{38}$$

In the limit of very small drift velocity we obtain

$$\rho=\left(1-\frac{9}{10}\frac{v_D{}^2}{v_{\mathrm{th}}{}^2}\right). \tag{39}$$

The numerical evaluation of $G(x)$ indicates the expected result, namely that the resistivity decreases when the drift velocity increases. Numerical evaluation of $G(x)$ shows (see Fig. 1) a negative resistance for $x>1.87$. This corresponds to a critical drift velocity $(v_D)_c\cong1.36v_{\mathrm{th}}$ which is approximately 20% higher than previous estimates.[6,10] It should be noted that this critical drift velocity can be of the same order as that for excitation of ion waves or phonons.

In conclusion, we have derived the kinetic equation for drifted electron-ion system, where the ions are assumed to be randomly distributed. We have utilized our equation to derive the high-frequency conductivity, which cannot be obtained using the diagrammatic technique as given in Ref. 4, since our system is not in thermal equilibrium. Finally, we have estimated the resistivity for $\omega\gtrsim\omega_p$ and did find a resistive instability of our system, however, for critical drift velocity higher than previously predicted.[6,10]

[10] D. E. McCumber and A. G. Chynoweth, Phys. Rev. Letters 22, 651 (1964).